On Weakly $\delta$-Semiprimary Ideals of Commutative Rings

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Abstract. Let $R$ be a commutative ring with $1 \neq 0$. A proper ideal $I$ of $R$ is a semiprimary ideal of $R$ if whenever $a, b \in R$ and $ab \in I$, we have $a \in \sqrt{I}$ or $b \in \sqrt{I}$; and $I$ is a weakly semiprimary ideal of $R$ if whenever $a, b \in R$ and $0 \neq ab \in I$, we have $a \in \sqrt{I}$ or $b \in \sqrt{I}$. In this paper, we introduce a new class of ideals that is closely related to the class of (weakly) semiprimary ideals. Let $I(R)$ be the set of all ideals of $R$ and let $\delta : I(R) \to I(R)$ be a function. Then $\delta$ is called an expansion function of ideals of $R$ if whenever $L, I, J$ are ideals of $R$ with $J \subseteq I$, we have $L \subseteq \delta(J)$ and $\delta(J) \subseteq \delta(I)$. Let $\delta$ be an expansion function of ideals of $R$. Then a proper ideal $I$ of $R$ is called a $\delta$-semiprimary (weakly $\delta$-semiprimary) ideal of $R$ if $ab \in I$ ($0 \neq ab \in I$) implies $a \in \delta(I)$ or $b \in \delta(I)$. A number of results concerning weakly $\delta$-semiprimary ideals and examples of weakly $\delta$-semiprimary ideals are given.

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1 Introduction

We assume throughout this paper that all rings are commutative with $1 \neq 0$. Let $R$ be a commutative ring. An ideal $I$ of $R$ is said to be proper if $I \neq R$. When $I$ is a proper ideal of $R$, then we use $\sqrt{I}$ to denote the radical ideal of $I$ (that is, $\sqrt{I} = \{x \in R \mid x^n \in I$ for some positive integer $n \geq 1\}$). Note that $\sqrt{\{0\}}$ is the set (ideal) of all nilpotent elements of $R$.

Let $I$ be a proper ideal of $R$. We recall from [1] and [6] that $I$ is said to be weakly semiprime if $0 \neq x^2 \in I$ implies $x \in I$; recall from [1] (also, [4]) that a proper ideal
$I$ of $R$ is said to be weakly prime (weakly primary) if $0 \neq ab \in I$ implies $a \in I$ or $b \in I$ ($a \in I$ or $b \in \sqrt{I}$). Over the past several years, there has been considerable attention in the literature to prime ideals and their generalizations (for example, see [1]–[11], and [14]).

Recall that a proper ideal $I$ of a ring $R$ is called semiprimary if whenever $x, y \in R$ and $xy \in I$, we have $x \in \sqrt{I}$ or $y \in \sqrt{I}$. Gilmour [12] studied rings in which semiprimary ideals are primary. In this paper, we define a proper ideal $I$ of $R$ to be weakly semiprimary if whenever $x, y \in R$ and $0 \neq xy \in I$, we have $x \in \sqrt{I}$ or $y \in \sqrt{I}$. In fact, we will study a more general concept. Let $I(R)$ be the set of all ideals of $R$. Zhao [14] introduced the concept of expansion of ideals of $R$. We recall from [14] that a function $\delta : I(R) \to I(R)$ is called an expansion function of ideals of $R$ if whenever $L, I, J$ are ideals of $R$ with $J \subseteq I$, we have $L \subseteq \delta(L)$ and $\delta(J) \subseteq \delta(I)$. In addition, recall from [14] that a proper ideal $I$ of $R$ is said to be a $\delta$-primary ideal of $R$ if whenever $a, b \in R$ with $ab \in I$, we have $a \in I$ or $b \in \delta(I)$, where $\delta$ is an expansion function of ideals of $R$. Let $\delta$ be an expansion function of ideals of $R$. In this paper, a proper ideal $I$ of $R$ is called a $\delta$-semiprimary (weakly $\delta$-semiprimary) ideal of $R$ if $ab \in I$ ($0 \neq ab \in I$) implies $a \in \delta(I)$ or $b \in \delta(I)$.

Let $\delta$ be an expansion function of ideals of a ring $R$. Among many results in this paper, it is shown that if $I$ is a weakly $\delta$-semiprimary ideal of $R$ that is not $\delta$-semiprimary, then $I^2 = \{0\}$ and hence $I \subseteq \sqrt{\{0\}}$ (Theorem 2.10). If $I$ is a proper ideal of $R$ and $I^2 = \{0\}$, then $I$ need not be a weakly $\delta$-semiprimary ideal of $R$ (Example 2.14). It is shown in Example 2.23 that if $I, J$ are weakly $\delta$-semiprimary ideals of $R$ such that $\delta(I) = \delta(J)$ and $I + J \neq R$, then $I + J$ need not be a weakly $\delta$-semiprimary ideal of $R$. We show that if $R$ is a Boolean ring, then every weakly semiprimary ideal of $R$ is weakly prime (Theorem 2.16); if $S$ is a multiplicatively closed subset of $R$ such that $S \cap Z(R) = \emptyset$ (where $Z(R)$ is the set of all zero divisor elements of $R$) and $I$ is a weakly semiprimary ideal of $R$ such that $S \cap \sqrt{I} = \emptyset$, then $I_S$ is a weakly semiprimary ideal of $R_S$ (Theorem 3.1). We also show that if $I$ is a weakly $\delta$-primary ideal of $R$ and $\{0\} \neq AB \subseteq I$ for some ideals $A, B$ of $R$, then $A \subseteq \delta(I)$ or $B \subseteq \delta(I)$ (Theorem 5.4).

2 Weakly $\delta$-Semiprimary Ideals

Definition 2.1. [14] Let $I(R)$ be the set of all ideals of $R$. Then a function $\delta : I(R) \to I(R)$ is called an expansion function of ideals of $R$ if whenever $L, I, J$ are ideals of $R$ with $J \subseteq I$, we have $L \subseteq \delta(L)$ and $\delta(J) \subseteq \delta(I)$.

In the following example, we give some expansion functions of ideals of a ring $R$.

Example 2.2. [8] Let $\delta : I(R) \to I(R)$ be a function. Then

(1) If $\delta(I) = I$ for every ideal $I$ of $R$, then $\delta$ is an expansion function of ideals
of $R$.

(2) If $\delta(I) = \sqrt{T}$ (note that $\sqrt{R} = R$) for every ideal $I$ of $R$, then $\delta$ is an expansion function of ideals of $R$.

(3) Suppose that $R$ is a quasi-local ring (i.e., $R$ has exactly one maximal ideal) with maximal ideal $M$. If $\delta(I) = M$ for every proper ideal $I$ of $R$, then $\delta$ is an expansion function of ideals of $R$.

(4) Let $I$ be a proper ideal of $R$. Recall from [13] that an element $r \in R$ is called integral over $I$ if there is an integer $n \geq 1$ and $a_i \in I^i$, $i = 1, \ldots, n$, such that $r^n + a_1r^{n-1} + a_2r^{n-2} + \cdots + a_n = 0$. Let $\mathcal{T} = \{ r \in R \mid r \text{ is integral over } I \}$.

Let $I \in \mathcal{I}(R)$. It is known that $\mathcal{T}$ is an ideal of $R$ and $I \subseteq \mathcal{T} \subseteq \sqrt{\mathcal{T}}$, and if $J \subseteq I$, then $\mathcal{T} = \mathcal{T}$ (see [13]). If $\delta(I) = \mathcal{T}$ for every ideal $I$ of $R$, then $\delta$ is an expansion function of ideals of $R$.

(5) Let $J$ be a proper ideal of $R$. If $\delta(I) = I + J$ for every ideal $I$ of $R$, then $\delta$ is an expansion function of ideals of $R$.

(6) Assume that $\delta_1$ and $\delta_2$ are expansion functions of ideals of $R$. We give $\delta : \mathcal{I}(R) \to \mathcal{I}(R)$ such that $\delta(I) = \delta_1(I) + \delta_2(I)$. Then $\delta$ is an expansion function of ideals of $R$.

(7) Assume that $\delta_1$ and $\delta_2$ are expansion functions of ideals of $R$. We give $\delta : \mathcal{I}(R) \to \mathcal{I}(R)$ such that $\delta(I) = \delta_1(I) \cap \delta_2(I)$. Then $\delta$ is an expansion function of ideals of $R$.

(8) Assume that $\delta_1$ and $\delta_2$ are expansion functions of ideals of $R$. We give $\delta : \mathcal{I}(R) \to \mathcal{I}(R)$ such that $\delta(I) = (\delta_1 \circ \delta_2)(I) = \delta_1(\delta_2(I))$. Then $\delta$ is an expansion function of ideals of $R$.

We recall the following definitions.

**Definition 2.3.** Let $\delta$ be an expansion function of ideals of a ring $R$.

(1) A proper ideal $I$ of $R$ is called a $\delta$-semiprimary (weakly $\delta$-semiprimary) ideal of $R$ if whenever $a, b \in R$ and $ab \in I$ ($0 \neq ab \in I$), we have $a \in \delta(I)$ or $b \in \delta(I)$.

(2) If $\delta : \mathcal{I}(R) \to \mathcal{I}(R)$ such that $\delta(I) = \sqrt{T}$ for every proper ideal $I$ of $R$, then $\delta$ is an expansion function of ideals of $R$. In this case, a proper ideal $I$ of $R$ is called a semiprimary (weakly semiprimary) ideal of $R$ if whenever $a, b \in R$ and $ab \in I$ ($0 \neq ab \in I$), we have $a \in \sqrt{T}$ or $b \in \sqrt{T}$.

(3) A proper ideal $I$ of $R$ is called a $\delta$-primary (weakly $\delta$-primary) ideal of $R$ if whenever $a, b \in R$ and $ab \in I$ ($0 \neq ab \in I$), we have $a \in I$ or $b \in \delta(I)$.

(4) A proper ideal $I$ of $R$ is called a weakly prime ideal of $R$ if whenever $a, b \in R$ and $0 \neq ab \in I$, we have $a \in I$ or $b \in I$.

(5) A proper ideal $I$ of $R$ is called a weakly primary ideal of $R$ if whenever $a, b \in R$ and $0 \neq ab \in I$, we have $a \in I$ or $b \in \sqrt{T}$.

We have the following trivial result, whose proof we omit.

**Theorem 2.4.** Let $I$ be a proper ideal of $R$ and let $\delta$ be an expansion function of ideals of $R$.

(1) If $I$ is a $\delta$-primary ideal of $R$, then $I$ is a weakly $\delta$-semiprimary ideal of $R$. In particular, if $I$ is a primary ideal of $R$, then $I$ is a weakly semiprimary ideal of $R$. 
(2) If $I$ is a weakly $\delta$-primary ideal of $R$, then $I$ is a weakly $\delta$-semiprimary ideal of $R$. In particular, if $I$ is a weakly primary ideal of $R$, then $I$ is a weakly semiprimary ideal of $R$.

(3) If $I$ is a $\delta$-semiprimary ideal of $R$, then $I$ is a weakly $\delta$-semiprimary ideal of $R$.

(4) $\sqrt{(0)}$ is a weakly prime ideal of $R$ if and only if $\sqrt{(0)}$ is a weakly semiprimary ideal of $R$.

(5) If $I$ is a weakly prime ideal of $R$, then $I$ is a weakly semiprimary ideal of $R$.

The following is an example of a proper ideal of a ring $R$ that is a weakly semiprimary ideal of $R$, but neither weakly primary nor weakly prime.

**Example 2.5.** Let $A = \mathbb{Z}_2[X, Y]$, where $X$ and $Y$ are indeterminates. Then

$$I = (Y^2, XY)A \quad \text{and} \quad J = (Y^2, X^2Y^2)A$$

are ideals of $A$. Set $R = A/J$. Hence, $L = I/J$ is an ideal of $R$ and $\sqrt{L} = (Y, XY)A/J$. Since $0 \neq XY + J \in L$ and neither $X + J \in \sqrt{L}$ nor $Y + J \in L$, we conclude that $L$ is not a weakly primary ideal of $R$. Since $0 + J \neq XY + J \in L$ but neither $X + J \in L$ nor $Y + J \in L$, we know that $L$ is not a weakly prime ideal of $R$. It is easy to check that $L$ is a weakly semiprimary ideal of $R$.

The next example is an ideal that is weakly semiprimary but not semiprimary.

**Example 2.6.** Let $R = Z_{36}$. Then $I = \{0\}$ is a weakly semiprimary ideal of $R$ by definition. Note that $\sqrt{I} = 6R$. Since $0 = 4 \cdot 9 \in I$ but neither $4 \in \sqrt{I}$ nor $9 \in \sqrt{I}$, we conclude that $I$ is not a semiprimary ideal of $R$.

**Definition 2.7.** Let $\delta$ be an expansion function of ideals of a ring $R$. Suppose that $I$ is a weakly $\delta$-semiprimary ideal of $R$ and $x \in R$. Then $x$ is called a dual-zero element of $I$ if $xy = 0$ for some $y \in R$ and neither $x \in \delta(I)$ nor $y \in \delta(I)$. (Note that $y$ is also a dual-zero element of $I$.)

**Remark 2.8.** Let $\delta$ be an expansion function of ideals of a ring $R$. If $I$ is a weakly $\delta$-semiprimary ideal of $R$ which is not $\delta$-primary, then $I$ must have a dual-zero element of $R$.

**Theorem 2.9.** Let $\delta$ be an expansion function of ideals of a ring $R$ and $I$ be a weakly $\delta$-semiprimary ideal of $R$. If $x \in R$ is a dual-zero element of $I$, then $xI = \{0\}$.

**Proof.** Assume that $x \in R$ is a dual-zero element of $I$. Then $xy = 0$ for some $y \in R$ such that neither $x \in \delta(I)$ nor $y \in \delta(I)$. Let $i \in I$. Thus, $x(y + i) = 0 + xi = xi \in I$. Suppose that $xi \neq 0$. Since $0 \neq x(y + i) = xi \in I$ and $I$ is a weakly $\delta$-semiprimary ideal of $R$, we conclude that $x \in \delta(I)$ or $(y + i) \in \delta(I)$, and hence $x \in \delta(I)$ or $y \in \delta(I)$, a contradiction. Thus, $xi = 0$.

**Theorem 2.10.** Let $\delta$ be an expansion function of ideals of a ring $R$ and $I$ be a weakly $\delta$-semiprimary ideal of $R$ that is not $\delta$-primary. Then $I^2 = \{0\}$, and hence $I \subseteq \sqrt{(0)}$. 

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Proof. Since $I$ is a weakly $\delta$-semi-primary ideal of $R$ that is not $\delta$-semi-primary, we conclude that $I$ has a dual-zero element $x \in R$. Since $xy = 0$ and neither $x \in \delta(I)$ nor $y \in \delta(I)$, we conclude that $y$ is a dual-zero element of $I$. Let $i, j \in I$. Then by Theorem 2.9, we have $(x + i)(y + j) = ij \in I$. Suppose that $ij \neq 0$. Since $0 \neq (x + i)(y + j) = ij \in I$ and $I$ is a weakly $\delta$-semi-primary ideal of $R$, we conclude that $x + i \in \delta(I)$ or $y + j \in \delta(I)$, and hence $x \in \delta(I)$ or $y \in \delta(I)$, a contradiction. Therefore $ij = 0$, and hence $I^2 = \{0\}$. □

In view of Theorem 2.10, we have the following result.

Corollary 2.11. Let $I$ be a weakly semi-primary ideal of $R$ that is not semi-primary. Then $I^2 = \{0\}$, and hence $I \subseteq \sqrt{\{0\}}$.

The following example shows that a proper ideal $I$ of $R$ with the property $I^2 = \{0\}$ need not be a weakly semi-primary ideal of $R$.

Example 2.12. Let $R = \mathbb{Z}_{12}$. Then $I = \{0, 6\}$ is an ideal of $R$ and $I^2 = \{0\}$. Note that $\sqrt{I} = I$. Since $0 \neq 2 \cdot 3 \in I$ and neither $2 \in \sqrt{I}$ nor $3 \in \sqrt{I}$, we conclude that $I$ is not a weakly semi-primary ideal of $R$.

Theorem 2.13. Let $\delta$ be an expansion function of ideals of a ring $R$ and $I$ be a proper ideal of $R$. If $\delta(I)$ is a weakly prime of $R$, then $I$ is a weakly $\delta$-semi-primary ideal of $R$. In particular, if $\sqrt{I}$ is a weakly prime of $R$, then $I$ is a weakly semi-primary ideal of $R$.

Proof. Suppose that $0 \neq xy \in I$ for some $x, y \in R$. Hence, $0 \neq xy \in \delta(I)$. Since $\delta(I)$ is weakly prime, we conclude that $x \in \delta(I)$ or $y \in \delta(I)$. Thus, $I$ is a weakly $\delta$-semi-primary ideal of $R$. □

Note that if $I$ is a weakly semi-primary ideal of a ring $R$ that is not semi-primary, then $\sqrt{I}$ need not be a weakly prime ideal of $R$. We have the following example.

Example 2.14. The ideal $I = \{0\}$ is a weakly semi-primary ideal of $\mathbb{Z}_{12}$. However, $\sqrt{I} = \{0, 6\}$ is not a weakly prime ideal of $\mathbb{Z}_{12}$ since $0 \neq 2 \cdot 3 \in \sqrt{I}$, but neither $2 \in \sqrt{I}$ nor $3 \in \sqrt{I}$.

Remark 2.15. Note that a weakly prime ideal of a ring $R$ is weakly semi-primary, but the converse is not true. Let $R = \mathbb{Z}[(X)]$. Then $\frac{(X^2)}{(X^2)}$ is an ideal of $R$. Since $0 \neq (X + (X^3)) \cdot (X + (X^3)) = X^2 + (X^3) \in I$ but $X + (X^3) \notin I$, we conclude that $I$ is not a weakly prime ideal of $R$. Since $\sqrt{I} = \frac{(2, X)}{(X^2)}$ is a prime ideal of $R$, $I$ is a (weakly) semi-primary ideal of $R$.

Let $R$ be a Boolean ring (i.e., $x^2 = x$ for every $x \in R$). Since $\sqrt{I} = I$ for every proper ideal $I$ of $R$, we have the following result.

Theorem 2.16. Let $R$ be a Boolean ring and $I$ be a proper ideal of $R$. Then the following statements are equivalent:

1. $I$ is a weakly semi-primary ideal of $R$.
2. $I$ is a weakly prime ideal of $R$. 
Theorem 2.17. Let $\delta$ be an expansion function of ideals of a ring $R$ and $I$ be a weakly $\delta$-semiprimary ideal of $R$. Suppose that $\delta(I) = \delta(\{0\})$. Then the following statements are equivalent:

1. $I$ is not $\delta$-semiprimary.
2. $\{0\}$ has a dual-zero element of $R$.

Proof. (1)⇒(2) As $I$ is a weakly $\delta$-semiprimary ideal of $R$ that is not $\delta$-semiprimary, there are $x, y \in R$ such that $xy = 0$ and neither $x \in \delta(I)$ nor $y \in \delta(I)$. Since $\delta(I) = \delta(\{0\})$, we conclude that $x$ is a dual-zero element of $\{0\}$.

(2)⇒(1) Suppose that $x$ is a dual-zero element of $\{0\}$. Since $\delta(I) = \delta(\{0\})$, it is clear that $x$ is a dual-zero element of $I$. \qed

In view of Theorem 2.17, we have the following result.

Corollary 2.18. Let $I \subseteq \sqrt{\{0\}}$ be a proper ideal of $R$ such that $I$ is a weakly semiprimary ideal of $R$. Then the following statements are equivalent:

1. $I$ is not semiprimary.
2. $\{0\}$ has a dual-zero element of $R$.

Proof. Since $\delta : I(R) \to I(R)$ such that $\delta(I) = \sqrt{I}$ for every proper ideal $I$ of $R$ is an expansion function of ideals of $R$, we have $\delta(I) = \delta(\{0\})$. Thus, the claim is clear by Theorem 2.17. \qed

The hypothesis “$\delta(I) = \delta(\{0\})$” in Theorem 2.17 is crucial. To show this, we give an ideal $I$ of a ring $R$ in the following example, such that $I \subseteq \sqrt{\{0\}}$ and $\{0\}$ has a dual-zero element of $R$ but $I$ is a $\delta$-semiprimary ideal of $R$ for some expansion function $\delta$ of ideals of $R$.

Example 2.19. Let $R = \mathbb{Z}_8$, $\delta : I(R) \to I(R)$ such that $\delta(I) = \sqrt{I}$ for every nonzero proper ideal $I$ of $R$, and $\delta(\{0\}) = \{0\}$. Let $I = 4R$. Then $\delta(I) = \sqrt{I} = 2R$. It is clear that $I$ is a $\delta$-semiprimary ideal of $R$ and 2 is a dual-zero element of $\{0\}$.

Theorem 2.20. Let $\delta$ be an expansion function of ideals of a ring $R$ and $I$ be a weakly $\delta$-semiprimary ideal of $R$. If $J \subseteq I$ and $\delta(J) = \delta(I)$, then $J$ is a weakly $\delta$-semiprimary ideal of $R$.

Proof. Suppose that $0 \neq xy \in J$ for some $x, y \in R$. Since $J \subseteq I$, we have $0 \neq xy \in I$. Since $I$ is a weakly $\delta$-semiprimary ideal of $R$, we see that $x \in \delta(I)$ or $y \in \delta(I)$. Noticing $\delta(I) = \delta(J)$, we conclude that $x \in \delta(J)$ or $y \in \delta(J)$. Thus, $J$ is a weakly $\delta$-semiprimary ideal of $R$. \qed

In view of Theorem 2.20, we have the following result.

Corollary 2.21. Let $I$ be a weakly semiprimary ideal of $R$ such that $I \subseteq \sqrt{\{0\}}$. If $J \subseteq I$, then $J$ is a weakly semiprimary ideal of $R$. In particular, if $L$ is an ideal of $R$, then $LI$ and $L \cap I$ are weakly semiprimary ideals of $R$. Furthermore, if $n \geq 1$ is a positive integer, then $I^n$ is a weakly semiprimary ideal of $R$.

Theorem 2.22. Let $\{I_i\}_{i \in J}$ be a collection of weakly semiprimary ideals of a ring $R$ that are not semiprimary. Then $I = \bigcap_{i \in J} I_i$ is a weakly semiprimary ideal of $R$. 

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Proof. Note that \( \sqrt{I} = \bigcap I_1 = \sqrt{I_1} = \sqrt{\{0\}} \) by Theorem 2.10. Hence, the result follows. \( \Box \)

If \( I, J \) are weakly semiprimary ideals of a ring \( R \) such that \( \sqrt{I} = \sqrt{J} \) and \( I + J \neq R \), then \( I + J \) need not be a weakly semiprimary ideal of \( R \).

Example 2.23. Let \( A = \mathbb{Z}_2[T, U, X, Y] \),

\[ H = (T^2, U^2, XY + T + U, TU, TX, TY, UX, UY)A \]

be an ideal of \( A \), and \( R = A/H \). Then by the construction of \( R \), \( I = (TA + H)/H = \{0, T + H\} \) and \( J = (UA + H)/H = \{0, U + H\} \) are weakly semiprimary ideals of \( R \) such that \( |I| = |J| = 2 \) and \( \sqrt{I} = \sqrt{J} = \sqrt{\{0\}} \) (in \( R \)) = \((T, U, XY)A/H \). Let \( L = I + J = (H + (T, U)A)/H \). Thus, \( \sqrt{L} = \sqrt{\{0\}} \) (in \( R \)) and \( L \) is not a weakly semiprimary ideal of \( R \) since \( 0 \neq X + H \cdot Y + H = XY + H, X + H \notin L \), and \( Y + H \notin \sqrt{L} \).

Theorem 2.24. Let \( \delta \) be an expansion function of ideals of \( R \) such that \( \delta(\{0\}) \) is a weakly semiprimary ideal of \( R \) and \( \delta(\delta(\{0\})) = \delta(\{0\}) \). Then the following statements hold:

1. \( \delta(\{0\}) \) is a prime ideal of \( R \).
2. Suppose that \( I \) is a weakly \( \delta \)-semiprimary ideal of \( R \). Then \( I \) is a \( \delta \)-semiprimary ideal of \( R \).

Proof. (1) Let \( ab \in \delta(\{0\}) \) for some \( a, b \in R \). Suppose that \( a \notin \delta(\{0\}) = \delta(\{0\}) \). Since \( \delta(\{0\}) \) is a \( \delta \)-semiprimary ideal of \( R \) and \( a \notin \delta(\delta(\{0\})) \), it follows that \( b \in \delta(\{0\}) = \delta(\{0\}) \). Thus, \( \delta(\{0\}) \) is a prime ideal of \( R \).

(2) Suppose that \( I \) is not \( \delta \)-semiprimary. Clearly, \( \delta(\{0\}) \subseteq \delta(I) \). Since \( I^2 = \{0\} \) by Theorem 2.10 and \( \delta(\{0\}) \) is a prime ideal of \( R \), we have \( I \subseteq \delta(\{0\}) \). Noticing \( \delta(\delta(\{0\})) = \delta(\{0\}) \), we have \( \delta(I) \subseteq \delta(\delta(\{0\})) = \delta(\{0\}) \). Since \( \delta(\{0\}) \subseteq \delta(I) \) and \( \delta(I) \subseteq \delta(\delta(\{0\})) = \delta(\{0\}) \), it follows that \( \delta(I) = \delta(\{0\}) \) is a prime ideal of \( R \). As \( \delta(I) \) is prime, \( I \) is a \( \delta \)-semiprimary ideal of \( R \), which is a contradiction. \( \Box \)

Theorem 2.25. Let \( \delta \) be an expansion function of ideals of \( R \) such that \( \delta(\{0\}) \) is a weakly \( \delta \)-semiprimary ideal of \( R \), \( \sqrt{\{0\}} \subseteq \delta(\{0\}) \), and \( \delta(\delta(\{0\})) = \delta(\{0\}) \). Then the following statements hold:

1. \( \delta(\{0\}) \) is a weakly prime ideal of \( R \).
2. Suppose that \( I \) is a weakly \( \delta \)-semiprimary ideal of \( R \) that is not \( \delta \)-semiprimary. Then \( \delta(I) = \delta(\{0\}) = \delta(\sqrt{\{0\}}) \) is a weakly prime ideal of \( R \) that is not prime. Furthermore, if \( J \subseteq \sqrt{\{0\}} \), then \( J \) is a weakly \( \delta \)-semiprimary ideal of \( R \) that is not \( \delta \)-semiprimary and \( \delta(J) = \delta(\{0\}) \).

Proof. (1) Let \( 0 \neq ab \in \delta(\{0\}) \) for some \( a, b \in R \). Now we can suppose that \( a \notin \delta(\delta(\{0\})) = \delta(\{0\}) \). Since \( \delta(\{0\}) \) is a weakly \( \delta \)-semiprimary ideal of \( R \) and \( a \notin \delta(\delta(\{0\})) \), we have \( b \in \delta(\delta(\{0\})) = \delta(\{0\}) \). Thus, \( \delta(\{0\}) \) is a weakly prime ideal of \( R \).

(2) Suppose that \( I \) is not \( \delta \)-semiprimary. Then \( I^2 = \{0\} \) by Theorem 2.10, and hence \( I \subseteq \sqrt{\{0\}} \). Let \( J \) be an ideal of \( R \) such that \( J \subseteq \sqrt{\{0\}} \). Since \( \sqrt{\{0\}} \subseteq \delta(\{0\}) \),
we have $J \subseteq \delta\{0\}$. Therefore, we obtain $\delta(J) \subseteq \delta(\delta\{0\}) = \delta\{0\}$. Since $\delta\{0\} \subseteq \delta(J)$ and $\delta(J) \subseteq \delta\{0\}$, we conclude that $\delta(J) = \delta\{0\}$. In particular, $\delta(I) = \delta\{0\} = \delta(\sqrt{I})$ is a weakly prime ideal of $R$. Noticing that $\delta\{0\}$ is a weakly $\delta$-semiprimary of $R$ and $\delta(J) = \delta\{0\}$, we conclude that $J$ is a weakly $\delta$-semiprimary ideal of $R$. As $I$ is not $\delta$-semiprimary, it follows that $\delta(I) = \delta\{0\}$ is not a prime ideal of $R$. In addition, $\delta(J) = \delta\{0\}$ is a weakly prime ideal of $R$ that is not prime, so we conclude that $J$ is a weakly $\delta$-semiprimary ideal of $R$ that is not $\delta$-semiprimary.

\[ \Box \]

3 Weakly $\delta$-Semiprimary Ideals Under Localization and Ring-homomorphism

For a ring $R$, let $Z(R)$ be the set of all zerodivisors of $R$.

**Theorem 3.1.** Assume that $S$ is a multiplicatively closed subset of $R$ such that $S \cap Z(R) = \emptyset$. If $I$ is a weakly semiprimary ideal of $R$ and $S \cap \sqrt{I} = \emptyset$, then $I_S$ is a weakly semiprimary ideal of $R_S$.

**Proof.** Since $S \cap \sqrt{I} = \emptyset$, we conclude that $\sqrt{I_S} = (\sqrt{I})_S$. Let $a, b \in R$ and $s, t \in S$ such that $0 \neq \frac{a}{t} \in I_S$. Then there exists $u \in S$ such that $0 \neq uab \in I$. Since $u \in S$ and $S \cap \sqrt{I} = \emptyset$, we conclude that $0 \neq ab \in \sqrt{I}$. Since $I$ is a weakly semiprimary ideal of $R$, we see that $a \in \sqrt{I}$ or $b \in \sqrt{I}$. Thus, $\frac{a}{t} \in \sqrt{I_S}$ or $\frac{a}{t} \in I$. Consequently, $I_S$ is a weakly semiprimary ideal of $R_S$.

**Theorem 3.2.** Let $\gamma$ be an expansion function of ideals of $R$ and let $I, J$ be proper ideals of $R$ with $I \subseteq J$. Let $\delta : I\left(\frac{R}{I}\right) \to I\left(\frac{R}{J}\right)$ be an expansion function of ideals of $S = \frac{R}{J}$ such that $\delta\left(\frac{L+I}{J}\right) = \frac{\gamma(L) + I}{\gamma(J)}$ for every $L \in I(R)$. Then the following statements hold:

1. If $J$ is a weakly $\gamma$-semiprimary ideal of $R$, then $\frac{J}{J}$ is a weakly $\delta$-semiprimary ideal of $S$.
2. If $I$ is a weakly $\gamma$-semiprimary ideal of $R$ and $\frac{J}{J}$ is a weakly $\delta$-semiprimary ideal of $S$, then $J$ is a weakly $\gamma$-semiprimary ideal of $R$.

**Proof.** First observe that since $I \subseteq J$, we have $I \subseteq J \subseteq \gamma(J)$ and $\delta\left(\frac{J}{J}\right) = \frac{\gamma(J)}{J}$. (1) Assume that $ab \in J/I$ for some $a, b \in R$. Then $0 \neq ab \in J$. Hence, $a \in \gamma(J)$ or $b \in \gamma(J)$. Thus, $a + I \in \frac{2(J)}{J}$ or $b + I \in \frac{\gamma(J)}{J}$. It follows that $\frac{J}{J}$ is a weakly $\delta$-semiprimary ideal of $S = \frac{R}{J}$.

(2) Since $I \subseteq J$, we have $\gamma(I) \subseteq \gamma(J)$. Assume that $0 \neq ab \in J$ for some $a, b \in R$. Let $ab \in I$. Since $I$ is a weakly $\gamma$-semiprimary ideal of $R$, we have $a \in \gamma(I) \subseteq \gamma(J)$ or $b \in \gamma(I) \subseteq \gamma(J)$. Assume that $ab \in J \setminus I$. Thus, $I \neq ab + I \in \frac{J}{J}$. Since $\frac{J}{J}$ is a weakly $\delta$-semiprimary ideal of $S$, we have $a + I \in \frac{\gamma(J)}{J}$ or $b + I \in \frac{\gamma(J)}{J}$. Hence, $a \in \gamma(J)$ or $b \in \gamma(J)$. Consequently, $J$ is a weakly $\gamma$-semiprimary ideal of $R$.

In view of Theorem 3.2, we have the following result.

**Corollary 3.3.** Let $I$ and $J$ be proper ideals of $R$ with $I \subseteq J$. Then the following statements hold:
(1) If $J$ is a weakly semiprimary ideal of $R$, then $\frac{J}{I}$ is a weakly semiprimary ideal of $R$

(2) If $I$ is a weakly semiprimary ideal of $R$ and $\frac{J}{I}$ is a weakly semiprimary ideal of $R$

Theorem 3.4. Let $R$ and $S$ be rings and $f : R \to S$ be a surjective ring-homomorphism. Then the following statements hold:

(1) If $I$ is a weakly semiprimary ideal of $R$ and $\ker(f) \subseteq I$, then $f(I)$ is a weakly semiprimary ideal of $S$.

(2) If $J$ is a weakly semiprimary ideal of $S$ and $\ker(f)$ is a weakly semiprimary ideal of $R$, then $f^{-1}(J)$ is a weakly semiprimary ideal of $R$.

Proof. (1) Since $I$ is a weakly semiprimary ideal of $R$ and $\ker(f) \subseteq I$, we conclude that $\frac{I}{\ker(f)}$ is a weakly semiprimary ideal of $S$ by Corollary 3.3(1). Since $S$ is ring-isomorphic to $S$, the result follows.

(2) Let $L = f^{-1}(J)$. Then $\ker(f) \subseteq L$. Since $\frac{L}{\ker(f)}$ is ring-isomorphic to $S$, we conclude that $\frac{L}{\ker(f)}$ is a weakly semiprimary ideal of $S$ by Theorem 2.10. Hence, $L$ is a weakly semiprimary ideal of $R$ by Theorem 3.3(2).

4 Weakly $\delta$-Semiprimary Ideals in Product of Rings

Let $R_1, \ldots, R_n$, where $n \geq 2$, be commutative rings with $1 \neq 0$. Assume that $\delta_1, \ldots, \delta_n$ are expansion functions of ideals of $R_1, \ldots, R_n$, respectively. Now let $R = R_1 \times \cdots \times R_n$. Define a function $\delta_x : I(R) \to I(R)$ such that

$$\delta_x(I_1 \times \cdots \times I_n) = \delta_1(I_1) \times \cdots \times \delta_n(I_n)$$

for every $I_i \in I(R_i)$, where $1 \leq i \leq n$. Then it is clear that $\delta_x$ is an expansion function of ideals of $R$. Note that every ideal of $R$ is of the form $I_1 \times \cdots \times I_n$, where each $I_i$ is an ideal of $R_i$ for $1 \leq i \leq n$.

Theorem 4.1. Let $R_1$ and $R_2$ be commutative rings with $1 \neq 0$, $R = R_1 \times R_2$, and $\delta_1, \delta_2$ be expansion functions of ideals of $R_1, R_2$, respectively. Let $I$ be a proper ideal of $R_1$. Then the following statements are equivalent:

(1) $I \times R_2$ is a weakly $\delta_x$-semiprimary ideal of $R$.

(2) $I \times R_2$ is a $\delta_x$-semiprimary ideal of $R$.

(3) $I$ is a $\delta_1$-semiprimary ideal of $R_1$.

Proof. (1)$\Rightarrow$(2) Let $J = I \times R_2$. Then $J^2 \neq \{(0, 0)\}$. Hence, $J$ is a $\delta_x$-semiprimary ideal of $R$ by Theorem 2.10.

(2)$\Rightarrow$(3) Suppose that $I$ is not a $\delta_1$-semiprimary ideal of $R_1$. Then there exist $a, b \in R_1$ such that $ab \in I$, but neither $a \in \delta_1(I)$ nor $b \in \delta_1(I)$. Since $(a, 1)(b, 1) = (ab, 1) \in I \times R_2$, we have $(a, 1) \in \delta_x(I \times R_2)$ or $(b, 1) \in \delta_x(I \times R_2)$. It follows that $a \in \delta_1(I)$ or $b \in \delta_1(I)$, a contradiction. Thus, $I$ is a $\delta_1$-semiprimary ideal of $R_1$. 


(3)⇒(1) Let \( I \) be a \( \delta_1 \)-semiprimary ideal of \( R_1 \). Then it is clear that \( I \times R_2 \) is a (weakly) \( \delta_x \)-semiprimary ideal of \( R \). \hfill \( \square \)

**Theorem 4.2.** Let \( R_1 \) and \( R_2 \) be commutative rings with \( 1 \neq 0 \), \( R = R_1 \times R_2 \), and \( \delta_1, \delta_2 \) be expansion functions of ideals of \( R_1, R_2 \), respectively, such that \( \delta_2(K) = R_2 \) for some ideal \( K \) of \( R_2 \) if and only if \( K = R_2 \). Let \( I = I_1 \times I_2 \) be a proper ideal of \( R \), where \( I_1 \) and \( I_2 \) are some ideals of \( R_1 \) and \( R_2 \), respectively. Suppose that \( \delta_1(I_1) \neq R_1 \). Then the following statements are equivalent:

1. \( I \) is a weakly \( \delta_x \)-semiprimary ideal of \( R \).
2. \( I = \{(0,0)\} \) or \( I = I_1 \times R_2 \) is a \( \delta_x \)-semiprimary ideal of \( R \) (and hence \( I_1 \) is a \( \delta_1 \)-semiprimary ideal of \( R_1 \)).

**Proof.** (1)⇒(2) Assume that \( \{(0,0)\} \neq I = I_1 \times I_2 \) is a weakly \( \delta_x \)-semiprimary ideal of \( R \). Then there exists \( (0,0) \neq (x,y) \in I \) such that \( x \in I_1 \) and \( y \in I_2 \). Since \( I \) is a weakly \( \delta_x \)-semiprimary ideal of \( R \) and \( (0,0) \neq (x,1)(1,y) = (x,y) \in I \), we conclude that \( (x,1) \in \delta_x(I_1) \) or \( (1,y) \in \delta_x(I) \). As \( \delta_1(I_1) \neq R_1 \), we get \( (1,y) \notin \delta_x(I) \). Thus \( (x,1) \in \delta_x(I_1) \), and hence \( 1 \in \delta_2(I_2) \). Since \( 1 \in \delta_2(I_2) \), we see that \( \delta_2(I_2) = R_2 \), and hence \( I_2 = R_2 \). Therefore, \( I = I_1 \times R_2 \) is a \( \delta_x \)-semiprimary ideal of \( R \) by Theorem 4.1.

(2)⇒(1) Obvious. \hfill \( \square \)

**Corollary 4.3.** Let \( R_1 \) and \( R_2 \) be commutative rings with \( 1 \neq 0 \) and \( R = R_1 \times R_2 \). Let \( I \) be a proper ideal of \( R \). Then the following statements are equivalent:

1. \( I \) is a weakly semiprimary ideal of \( R \).
2. \( I = \{(0,0)\} \text{ or } I \text{ is a semiprimary ideal of } R \).
3. \( I = \{(0,0)\} \text{ or } I = I_1 \times R_2 \text{ for some semiprimary ideal } I_1 \text{ of } R_1 \) or \( I = R_1 \times I_2 \)
   for some semiprimary ideal \( I_2 \) of \( R_2 \).

5 Strongly Weakly \( \delta \)-Semiprimary Ideals

**Definition 5.1.** Let \( \delta \) be an expansion function of ideals of a ring \( R \). A proper ideal \( I \) of \( R \) is called a strongly weakly \( \delta \)-semiprimary ideal of \( R \) if whenever \( \{0\} \neq AB \subseteq I \) for some ideals \( A, B \) of \( R \), we have \( A \subseteq \delta(I) \) or \( B \subseteq \delta(I) \). Hence, a proper ideal \( I \) of \( R \) is called a strongly weakly semiprimary ideal of \( R \) if whenever \( \{0\} \neq AB \subseteq I \) for some ideals \( A, B \) of \( R \), we have \( A \subseteq \sqrt{\{0\}} \) or \( B \subseteq \sqrt{\{0\}} \).

**Remark 5.2.** Let \( \delta \) be an expansion function of ideals of a ring \( R \). It is clear that a strongly weakly \( \delta \)-semiprimary ideal of \( R \) is a weakly \( \delta \)-semiprimary ideal of \( R \). In this section, we show that a proper ideal \( I \) of \( R \) is a strongly weakly \( \delta \)-semiprimary ideal of \( R \) if and only if \( I \) is a weakly \( \delta \)-semiprimary ideal of \( R \).

**Theorem 5.3.** Let \( \delta \) be an expansion function of ideals of a ring \( R \) and \( I \) be a weakly \( \delta \)-semiprimary ideal of \( R \). Suppose that \( AB \subseteq I \) for some ideals \( A, B \) of \( R \), and that \( ab = 0 \) for some \( a \in A \) and \( b \in B \) such that neither \( a \in \delta(I) \) nor \( b \in \delta(I) \). Then \( AB = \{0\} \).

**Proof.** First we will show \( aB = bA = \{0\} \). Suppose that \( aB \neq \{0\} \). Then \( 0 \neq ac \in I \) for some \( c \in B \). Since \( I \) is a weakly \( \delta \)-semiprimary ideal of \( R \) and
We consider three cases:

**Case I.** Suppose that \( d \in \delta(I) \) and \( e \notin \delta(I) \). Since \( aB = \{0\} \), we can obtain \( 0 \neq e(d + a) = ed \in I \), and thus we conclude that \( e \in \delta(I) \) or \( d + a \in \delta(I) \). Since \( d \in \delta(I) \), we have \( e \in \delta(I) \) or \( a \in \delta(I) \), a contradiction.

**Case II.** Suppose that \( d \notin \delta(I) \) and \( e \in \delta(I) \). Since \( bA = \{0\} \), we have \( 0 \neq (e + b) = de \in I \), and hence we conclude that \( d \in \delta(I) \) or \( e + b \in \delta(I) \). As \( e \in \delta(I) \), we have \( d \in \delta(I) \) or \( e \in \delta(I) \), a contradiction.

**Case III.** Suppose that \( d, e \in \delta(I) \). Since \( AB = bA = \{0\} \), we can obtain \( 0 \neq (b + e)(d + a) = ed \in I \), and hence \( b + e \in \delta(I) \) or \( d + a \in \delta(I) \). As \( d, e \in \delta(I) \), we have \( b \in \delta(I) \) or \( a \in \delta(I) \), a contradiction.

Thus, \( AB = \{0\} \).

**Theorem 5.4.** Let \( \delta \) be an expansion function of ideals of a ring \( R \) and \( I \) be a weakly \( \delta \)-semiprimary ideal of \( R \). Suppose that \( \{0\} \neq AB \subseteq I \) for some ideals \( A, B \) of \( R \). Then \( A \subseteq \delta(I) \) or \( B \subseteq \delta(I) \) (i.e., \( I \) is a strongly weakly \( \delta \)-semiprimary ideal of \( R \)).

**Proof.** Since \( AB \neq \{0\} \), by Theorem 5.3 we conclude that whenever \( ab \in I \) for some \( a \in A \) and \( b \in B \), we obtain \( a \in \delta(I) \) or \( b \in \delta(I) \). Assume that \( \{0\} \neq AB \subseteq I \) and \( A \nsubseteq \delta(I) \). Then there is an \( x \in A \setminus \delta(I) \). Let \( y \in B \). Since \( xy \in AB \subseteq I \), \( \{0\} \neq AB \) and \( x \notin \delta(I) \), we obtain \( y \in \delta(I) \) by Theorem 5.3. Hence, \( B \subseteq \delta(I) \).

In view of Theorem 5.4, we have the following result.

**Corollary 5.5.** Let \( I \) be a weakly semiprimary ideal of \( R \). We suppose that \( \{0\} \neq AB \subseteq I \) for some ideals \( A, B \) of \( R \). Then \( A \subseteq \sqrt{I} \) or \( B \subseteq \sqrt{I} \) (i.e., \( I \) is a strongly weakly semiprimary ideal of \( R \)).

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**References**


