WEAKLY $(m, n)$-CLOSED IDEALS AND $(m, n)$-VON NEUMANN REGULAR RINGS

DAVID F. ANDERSON, AYMAN BADAWI, AND BRAHIM FAHID
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Abstract. Let \(R\) be a commutative ring with \(1 \neq 0\), \(I\) a proper ideal of \(R\), and \(m\) and \(n\) positive integers. In this paper, we define \(I\) to be a weakly \((m, n)\)-closed ideal if \(0 \neq x^m \in I\) for \(x \in R\) implies \(x^n \in I\), and \(R\) to be an \((m, n)\)-von Neumann regular ring if for every \(x \in R\), there is an \(r \in R\) such that \(x^m r = x^n\). A number of results concerning weakly \((m, n)\)-closed ideals and \((m, n)\)-von Neumann regular rings are given.

1. Introduction

Let \(R\) be a commutative ring with \(1 \neq 0\), \(I\) a proper ideal of \(R\), and \(n\) a positive integer. As in [2], \(I\) is an \(n\)-absorbing (resp., strongly \(n\)-absorbing) ideal of \(R\) if whenever \(x_1 \cdots x_{n+1} \in I\) for \(x_1, \ldots, x_{n+1} \in R\) (resp., \(I_1 \cdots I_{n+1} \subseteq I\) for ideals \(I_1, \ldots, I_{n+1}\) of \(R\)), then there are \(n\) of the \(x_i\)'s (resp., \(n\) of the \(I_i\)'s) whose product is in \(I\). As in [4], \(I\) is a semi-\(n\)-absorbing ideal of \(R\) if \(x^{n+1} \in I\) for \(x \in R\) implies \(x^n \in I\); and for positive integers \(m\) and \(n\), \(I\) is an \((m, n)\)-closed ideal of \(R\) if \(x^m \in I\) for \(x \in R\) implies \(x^n \in I\). And, as in [15], \(I\) is a weakly \(n\)-absorbing (resp., strongly weakly \(n\)-absorbing) ideal of \(R\) if whenever \(0 \neq x_1 \cdots x_{n+1} \in I\) for \(x_1, \ldots, x_{n+1} \in R\) (resp., \(0 \neq I_1 \cdots I_{n+1} \subseteq I\) for ideals \(I_1, \ldots, I_{n+1}\) of \(R\)), then there are \(n\) of the \(x_i\)'s (resp., \(n\) of the \(I_i\)'s) whose product is in \(I\).

In this paper, we define \(I\) to be a weakly semi-\(n\)-absorbing ideal of \(R\) if \(0 \neq x^{n+1} \in I\) for \(x \in R\) implies \(x^n \in I\). More generally, for positive integers \(m\) and \(n\), we define \(I\) to be a weakly \((m, n)\)-closed ideal of \(R\) if \(0 \neq x^m \in I\) for \(x \in R\) implies \(x^n \in I\). Thus \(I\) is a weakly semi-\(n\)-absorbing ideal if and only if \(I\) is a weakly \((n+1, n)\)-closed ideal. Moreover, an \((m, n)\)-closed ideal is a weakly \((m, n)\)-closed ideal, and the two concepts agree when \(R\) is reduced. Every proper ideal is weakly \((m, n)\)-closed for \(m \leq n\); so we usually assume that \(m > n\).

Received May 21, 2017; Accepted June 26, 2018.

2010 Mathematics Subject Classification. Primary 13A15; Secondary 13F05, 13G05.

Key words and phrases. prime ideal, radical ideal, 2-absorbing ideal, \(n\)-absorbing ideal, \((m, n)\)-closed ideal, weakly \((m, n)\)-closed ideal, \((m, n)\)-von Neumann regular.

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The above definitions all concern generalizations of prime ideals. A 1-absorbing ideal is just a prime ideal, and a weakly 1-absorbing ideal is just a weakly prime ideal (a proper ideal \( I \) of \( R \) is a weakly prime ideal if \( 0 \neq xy \in I \) for \( x, y \in R \) implies \( x \in I \) or \( y \in I \)). A proper ideal is a radical ideal if and only if it is (2, 1)-closed. However, a weakly (2, 1)-closed ideal need not be a weakly radical ideal (a proper ideal \( I \) of \( R \) is a weakly radical ideal if \( 0 \neq x^n \in I \) for \( x \in R \) and \( n \) a positive integer implies \( x \in I \)) (see Example 2.3(b)).

Weakly prime ideals and weakly radical ideals were studied in [1], and weakly radical (semiprime) ideals have been studied in more detail in [6]. The concept of 2-absorbing ideals was introduced in [5] and then extended to \( n \)-absorbing ideals in [2]. Related concepts include 2-absorbing primary ideals (see [9]), weakly 2-absorbing ideals (see [11]), weakly 2-absorbing primary ideals (see [10]), and \((m,n)\)-closed ideals (see [4]). Other generalizations and related concepts are investigated in [1], [6], [8], [11], [12], [13], and [15]. For a survey on \( n \)-absorbing ideals, see [7].

Let \( R \) be a commutative ring and \( m \) and \( n \) positive integers. We define \( R \) to be an \((m,n)\)-von Neumann regular ring if for every \( x \in R \), there is an \( r \in R \) such that \( x^m r = x^n \). Thus a \((2,1)\)-von Neumann regular ring is just a von Neumann regular ring. In this paper, we study weakly \((m,n)\)-closed ideals, \((m,n)\)-von Neumann regular rings, and the connections between the two concepts.

Let \( m \) and \( n \) be positive integers with \( m > n \). Among the many results in this paper, we show in Theorem 2.6 that if \( I \) is a weakly \((m,n)\)-closed, but not \((m,n)\)-closed, ideal of \( R \), then \( I \subseteq \text{Nil}(R) \). In Theorem 2.11, we determine when a proper ideal of \( R_1 \times R_2 \) is weakly \((m,n)\)-closed, but not \((m,n)\)-closed; and in Theorem 2.12, we investigate when a proper ideal of \( R(+)M \) is weakly \((m,n)\)-closed, but not \((m,n)\)-closed. In Section 3, we introduce and investigate \((m,n)\)-von Neumann regular elements and \((m,n)\)-von Neumann regular rings. It is shown in Theorem 3.5 that every proper ideal of \( R \) is weakly \((m,n)\)-closed if and only if every non-nilpotent element of \( R \) is \((m,n)\)-von Neumann regular and \( w^n = 0 \) for every \( w \in \text{Nil}(R) \). In Theorem 3.7, we show that every proper ideal of \( R \) is \((m,n)\)-closed if and only if \( R \) is \((m,n)\)-von Neumann regular. Finally, we define the concepts of \( n \)-regular and \( \omega \)-regular commutative rings as a way to measure how far a zero-dimensional commutative ring is from being von Neumann regular.

We assume throughout this paper that all rings are commutative with \( 1 \neq 0 \), all \( R \)-modules are unitary, and \( f(1) = 1 \) for all ring homomorphisms \( f : R \to T \). For such a ring \( R \), let \( \text{Nil}(R) \) be its ideal of nilpotent elements, \( \text{Z}(R) \) its set of zero-divisors, \( U(R) \) its group of units, \( \text{char}(R) \) its characteristic, and \( \text{dim}(R) \) its (Krull) dimension. Then \( R \) is reduced if \( \text{Nil}(R) = \{0\} \) and \( R \) is quasilocal if it has exactly one maximal ideal. As usual, \( \mathbb{N} \), \( \mathbb{Z} \), and \( \mathbb{Z}_n \) will denote the positive integers, integers, and integers modulo \( n \), respectively. Several of our results use the \( R(+)M \) construction as in [14]. Let \( R \) be a commutative ring and \( M \) an \( R \)-module. Then \( R(+)M = R \times M \) is a commutative ring with identity \((1,0)\).
under addition defined by \((r, m) + (s, n) = (r + s, m + n)\) and multiplication defined by \((r, m)(s, n) = (rs, rn + sm)\). Note that \((\{0\}(+)M)^2 = \{0\}\); so \(\{0\}(+)M \subseteq \text{Nil}(R(+)M)\).

2. Properties of weakly \((m, n)\)-closed ideals

In this section, we give some basic properties of weakly \((m, n)\)-closed ideals and investigate weakly \((m, n)\)-closed ideals in several classes of commutative rings. We start by recalling the definitions of weakly semi-\(n\)-absorbing and weakly \((m, n)\)-closed ideals.

**Definition 2.1.** Let \(R\) be a commutative ring, \(I\) a proper ideal of \(R\), and \(m\) and \(n\) positive integers.

1. \(I\) is a weakly semi-\(n\)-absorbing ideal of \(R\) if \(0 \neq x^{n+1} \in I\) for \(x \in R\) implies \(x^n \in I\).
2. \(I\) is a weakly \((m, n)\)-closed ideal of \(R\) if \(0 \neq x^m \in I\) for \(x \in R\) implies \(x^n \in I\).

The proof of the next result follows easily from the definitions, and thus will be omitted.

**Theorem 2.2.** Let \(R\) be a commutative ring and \(m\) and \(n\) positive integers.

1. If \(I\) is a weakly \(n\)-absorbing ideal of \(R\), then \(I\) is weakly semi-\(n\)-absorbing (i.e., weakly \((n+1, n)\)-closed).
2. If \(I\) is a weakly \((m, n)\)-closed ideal of \(R\), then \(I\) is weakly \((m, n')\)-closed for every positive integer \(n' \geq n\).
3. If \(I\) is a weakly \(n\)-absorbing ideal of \(R\), then \(I\) is weakly \((m, n)\)-closed for every positive integer \(m\).
4. An intersection of weakly \((m, n)\)-closed ideals of \(R\) is weakly \((m, n)\)-closed.

While an \((m, n)\)-closed ideal is always weakly \((m, n)\)-closed, the converse need not hold. If an ideal is \((m, n)\)-closed, then it is also \((m', n')\)-closed for all positive integers \(m' \leq m\) and \(n' \geq n\) [4, Theorem 2.1(3)]. However, a weakly \((m, n)\)-closed ideal need not be weakly \((m', n)\)-closed for \(m' < m\). We next give two examples to illustrate these differences.

**Example 2.3.** (a) Let \(R = \mathbb{Z}_8\) and \(I = \{0, 4\}\). Then \(I\) is weakly \((3, 1)\)-closed since \(x^3 = 0\) for every nonunit \(x\) in \(R\). However, \(I\) is not \((3, 1)\)-closed since \(2^3 = 0 \in I\) and \(2 \notin I\), and \(I\) is not weakly \((2, 1)\)-closed since \(0 \neq 2^2 = 4 \in I\) and \(2 \notin I\).

(b) Let \(R = \mathbb{Z}_{16}\) and \(I = \{0, 8\}\). Then \(I\) is weakly \((2, 1)\)-closed since \(8\) is not a square in \(\mathbb{Z}_{16}\). However, \(I\) is not \((2, 1)\)-closed since \(4^2 = 0 \in I\) and \(4 \notin I\), and \(I\) is not a weakly radical ideal (and thus not weakly prime) since \(0 \neq 2^4 = 8 \in I\) and \(2 \notin I\).

The following definition will be useful for studying weakly \((m, n)\)-closed ideals that are not \((m, n)\)-closed (cf. [6, Definition 2.2]).
Definition 2.4. Let $R$ be a commutative ring, $m$ and $n$ positive integers, and $I$ a weakly $(m,n)$-closed ideal of $R$. Then $a \in R$ is an $(m,n)$-unbreakable-zero element of $I$ if $a^m = 0$ and $a^n \notin I$. (Thus $I$ has an $(m,n)$-unbreakable-zero element if and only if $I$ is not $(m,n)$-closed.)

Theorem 2.5 (cf. [6, Theorem 2.3]). Let $R$ be a commutative ring, $m$ and $n$ positive integers, and $I$ a weakly $(m,n)$-closed ideal of $R$. If $a$ is an $(m,n)$-unbreakable-zero element of $I$, then $(a+i)^m = 0$ for every $i \in I$.

Proof. Let $i \in I$. Then

$$(a+i)^m = a^m + \sum_{k=1}^{m} \binom{m}{k} a^{m-k} i^k = 0 + \sum_{k=1}^{m} \binom{m}{k} a^{m-k} i^k \in I,$$

and similarly, $(a+i)^n \notin I$ since $a^n \notin I$. Thus $(a+i)^m = 0$ since $I$ is weakly $(m,n)$-closed. $\square$

Theorem 2.6 (cf. [1, p. 839] and [6, Theorems 2.4 and 2.5]). Let $R$ be a commutative ring, $m$ and $n$ positive integers, and $I$ a weakly $(m,n)$-closed ideal of $R$. If $I$ is not $(m,n)$-closed, then $I \subseteq \text{Nil}(R)$. Moreover, if $I$ is not $(m,n)$-closed and $\text{char}(R) = m$ is prime, then $i^m = 0$ for every $i \in I$.

Proof. Since $I$ is a weakly $(m,n)$-closed ideal of $R$ that is not $(m,n)$-closed, $I$ has an $(m,n)$-unbreakable-zero element $a$. Let $i \in I$. Then $a^m = 0$, and $(a+i)^m = 0$ by Theorem 2.5; so $a, a+i \in \text{Nil}(R)$. Thus $i = (a+i) - a \in \text{Nil}(R)$; so $I \subseteq \text{Nil}(R)$.

The “moreover” statement is clear since $0 = (a+i)^m = a^m + i^m = i^m$ when $\text{char}(R) = m$ is prime. $\square$

The next two theorems are the analogs of the results for $(m,n)$-closed ideals in [4, Theorem 2.8] and [4, Theorem 2.10], respectively. Their proofs are similar, and thus will be omitted.

Theorem 2.7. Let $R$ be a commutative ring, $I$ a proper ideal of $R$, $S \subseteq R \setminus \{0\}$ a multiplicative set, and $m$ and $n$ positive integers. If $I$ is a weakly $(m,n)$-closed ideal of $R$, then $I_S$ is a weakly $(m,n)$-closed ideal of $RS$.

Theorem 2.8. Let $f : R \to T$ be a homomorphism of commutative rings and $m$ and $n$ positive integers.

1. If $f$ is injective and $J$ is a weakly $(m,n)$-closed ideal of $T$, then $f^{-1}(J)$ is a weakly $(m,n)$-closed ideal of $R$. In particular, if $R$ is a subring of $T$ and $J$ is a weakly $(m,n)$-closed ideal of $T$, then $J \cap R$ is a weakly $(m,n)$-closed ideal of $R$.

2. If $f$ is surjective and $I$ is a weakly $(m,n)$-closed ideal of $R$ containing $\ker f$, then $f(I)$ is a weakly $(m,n)$-closed ideal of $T$. In particular, if $I$ is a weakly $(m,n)$-closed ideal of $R$ and $J \subseteq I$ is an ideal of $R$, then $I/J$ is a weakly $(m,n)$-closed ideal of $R/J$. 
In the following theorems, we determine when an ideal of \( R_1 \times R_2 \) is weakly \((m, n)\)-closed, but not \((m, n)\)-closed. (Recall that an ideal of \( R_1 \times R_2 \) has the form \( I_1 \times I_2 \) for ideals \( I_1 \) of \( R_1 \) and \( I_2 \) of \( R_2 \).) It is easy to determine when an ideal of \( R_1 \times R_2 \) is \((m, n)\)-closed.

**Theorem 2.9** (cf. [4, Theorem 2.12]). Let \( R = R_1 \times R_2 \), where \( R_1 \) and \( R_2 \) are commutative rings, \( J \) a proper ideal of \( R \), and \( m \) and \( n \) positive integers. Then the following statements are equivalent.

1. \( J \) is an \((m, n)\)-closed ideal of \( R \).
2. \( J = I_1 \times R_2 \), \( R_1 \times I_2 \), or \( I_1 \times I_2 \) for \((m, n)\)-closed ideals \( I_1 \) of \( R_1 \) and \( I_2 \) of \( R_2 \).

**Proof.** This follows directly from the definitions. \( \square \)

The analog of (1) \( \Rightarrow \) (2) of Theorem 2.9 clearly holds for weakly \((m, n)\)-closed ideals by Theorem 2.8(2), but our next theorem shows that the analog of (2) does not hold for weakly \((m, n)\)-closed ideals.

**Theorem 2.10.** Let \( R = R_1 \times R_2 \), where \( R_1 \) and \( R_2 \) are commutative rings, \( I_1 \) a proper ideal of \( R_1 \), and \( m \) and \( n \) positive integers. Then the following statements are equivalent.

1. \( I_1 \times R_2 \) is a weakly \((m, n)\)-closed ideal of \( R \).
2. \( I_1 \) is an \((m, n)\)-closed ideal of \( R_1 \).
3. \( I_1 \times R_2 \) is an \((m, n)\)-closed ideal of \( R \).

A similar result holds for \( R_1 \times I_2 \) when \( I_2 \) is a proper ideal of \( R_2 \).

**Proof.** (1) \( \Rightarrow \) (2) \( I_1 \) is a weakly \((m, n)\)-closed ideal of \( R_1 \) by Theorem 2.8(2). If \( I_1 \) is not an \((m, n)\)-closed ideal of \( R_1 \), then \( I_1 \) has an \((m, n)\)-unbreakable-zero element \( a \). Thus \((0, 0) \neq (a, 1)^m \in I_1 \times R_2 \), but \((a, 1)^n \notin I_1 \times R_2 \), a contradiction. Hence \( I_1 \) is an \((m, n)\)-closed ideal of \( R_1 \).

(2) \( \Rightarrow \) (3) This is clear (cf. [4, Theorem 2.12]).

(3) \( \Rightarrow \) (1) This is clear by definition. \( \square \)

**Theorem 2.11.** Let \( R = R_1 \times R_2 \), where \( R_1 \) and \( R_2 \) are commutative rings, \( J \) a proper ideal of \( R \), and \( m \) and \( n \) positive integers. Then the following statements are equivalent.

1. \( J \) is a weakly \((m, n)\)-closed ideal of \( R \) that is not \((m, n)\)-closed.
2. \( J = I_1 \times I_2 \) for proper ideals \( I_1 \) of \( R_1 \) and \( I_2 \) of \( R_2 \) such that either
   a. \( I_1 \) is a weakly \((m, n)\)-closed ideal of \( R_1 \) that is not \((m, n)\)-closed, \( y^m = 0 \) whenever \( y^m \in I_2 \) for \( y \in R_2 \) (in particular, \( x^n = 0 \) for every \( i \in I_2 \)), and if \( 0 \neq x^m \in I_1 \) for some \( x \in R_1 \), then \( I_2 \) is an \((m, n)\)-closed ideal of \( R_2 \), or
   b. \( I_2 \) is a weakly \((m, n)\)-closed ideal of \( R_2 \) that is not \((m, n)\)-closed, \( y^m = 0 \) whenever \( y^m \in I_1 \) for \( y \in R_1 \) (in particular, \( x^n = 0 \) for every \( i \in I_1 \)), and if \( 0 \neq x^m \in I_2 \) for some \( x \in R_2 \), then \( I_1 \) is an \((m, n)\)-closed ideal of \( R_1 \).
Proof. (1) ⇒ (2) Since \( J \) is not an \((m, n)\)-closed ideal of \( R \), by Theorem 2.10 we have \( J = I_1 \times I_2 \), where \( I_1 \) is a proper ideal of \( R_1 \) and \( I_2 \) is a proper ideal of \( R_2 \). Since \( J \) is not an \((m, n)\)-closed ideal of \( R \), either \( I_1 \) is a weakly \((m, n)\)-closed ideal of \( R_1 \) that is not \((m, n)\)-closed or \( I_2 \) is a weakly \((m, n)\)-closed ideal of \( R_2 \) that is not \((m, n)\)-closed. Assume that \( I_1 \) is a weakly \((m, n)\)-closed ideal of \( R_1 \) that is not \((m, n)\)-closed. Thus \( I_1 \) has an \((m, n)\)-unbreakable-zero element \( a \). Assume that \( y^m \in I_2 \) for \( y \in R_2 \). Since a is an \((m, n)\)-unbreakable-zero element of \( I_1 \) and \((a, y)^m = (0, 0) \). Hence \( y^m = 0 \) (in particular, \( i^m = 0 \) for every \( i \in I_2 \)). Now assume that \( 0 \neq x^m \in I_1 \) for some \( x \in R_1 \). Let \( y \in R_2 \) such that \( y^m \in I_2 \). Then \((0, 0) \neq (x, y)^m \in J \). Thus \( y^n \in I_2 \), and hence \( I_2 \) is an \((m, n)\)-closed ideal of \( R_2 \). Similarly, if \( I_2 \) is a weakly \((m, n)\)-closed ideal of \( R_2 \) that is not \((m, n)\)-closed, then \( y^m = 0 \) whenever \( y^m \in I_1 \) for \( y \in R_1 \) (in particular, \( i^m = 0 \) for every \( i \in I_1 \)), and if \( 0 \neq x^m \in I_2 \) for some \( x \in R_2 \), then \( I_1 \) is an \((m, n)\)-closed ideal of \( R_1 \).

(2) ⇒ (1) Suppose that \( I_1 \) is a weakly \((m, n)\)-closed proper ideal of \( R_1 \) that is not \((m, n)\)-closed, \( y^m = 0 \) whenever \( y^m \in I_2 \) for \( y \in R_2 \) (in particular, \( i^m = 0 \) for every \( i \in I_2 \)), and if \( 0 \neq x^m \in I_1 \) for some \( x \in R_1 \), then \( I_2 \) is an \((m, n)\)-closed ideal of \( R_2 \). Let \( a \) be an \((m, n)\)-unbreakable-zero element of \( I_1 \). Then \((a, 0) \) is an \((m, n)\)-unbreakable-zero element of \( J \). Thus \( J \) is not an \((m, n)\)-closed ideal of \( R \). Now assume that \( (0, 0) \neq (x, y)^m = (x^n, y^m) \in J \) for \( x \in R_1 \) and \( y \in R_2 \). Then \((0, 0) \neq (x, y)^m = (x^n, 0) \in J \) and \( 0 \neq x^n \in I_1 \).

Since \( I_1 \) is a weakly \((m, n)\)-closed ideal of \( R_1 \) and \( I_2 \) is an \((m, n)\)-closed ideal of \( R_2 \), we have \((x, y)^n \in J \). Similarly, assume that \( I_2 \) is a weakly \((m, n)\)-closed ideal of \( R_2 \) that is not \((m, n)\)-closed, \( y^m = 0 \) whenever \( y^m \in I_1 \) for \( y \in R_1 \) (in particular, \( i^m = 0 \) for every \( i \in I_1 \)), and if \( 0 \neq x^m \in I_2 \) for some \( x \in R_2 \), then \( I_1 \) is an \((m, n)\)-closed ideal of \( R_1 \). Then again, \( J \) is a weakly \((m, n)\)-closed ideal of \( R \) that is not \((m, n)\)-closed. ☐

We next consider when certain ideals of \( R(+)M \) are weakly \((m, n)\)-closed.

**Theorem 2.12.** Let \( R \) be a commutative ring, \( I \) a proper ideal of \( R \), \( M \) an \( R \)-module, and \( m \) and \( n \) positive integers. Then the following statements are equivalent.

1. \( I(+)M \) is a weakly \((m, n)\)-closed ideal of \( R(+)M \) that is not \((m, n)\)-closed.
2. \( I \) is a weakly \((m, n)\)-closed ideal of \( R \) that is not \((m, n)\)-closed and \( m(a^{m-1}M) = 0 \) for every \((m, n)\)-unbreakable-zero element \( a \) of \( I \).

**Proof.** (1) ⇒ (2) Let \( J = I(+)M \). Assume that \( 0 \neq r^m \in I \) for \( r \in R \). Thus \((0, 0) \neq (r, 0)^m = (r^m, 0) \in J \). Hence \((r, 0)^n = (r^n, 0) \in J \); so \( r^n \in I \). Thus \( I \) is a weakly \((m, n)\)-closed ideal of \( R \). Since \( J \) is not \((m, n)\)-closed, \( J \), and hence \( I \), has an \((m, n)\)-unbreakable-zero element; so \( I \) is not \((m, n)\)-closed. Let \( a \) be an \((m, n)\)-unbreakable-zero element of \( I \) and \( x \in M \). Then \((a, x)^m = (a^m, m(a^{m-1}x)) \in J \). Since \( a^n \notin I \), we have \((a, x)^m = (a^n, m(a^{m-1}x)) = (0, 0) \). Thus \( m(a^{m-1}M) = 0 \).
(2) ⇒ (1) Since I is a weakly (m, n)-closed ideal of R that is not (m, n)-closed, I has an (m, n)-unbreakable-zero element a. Hence (a, 0) is an (m, n)-unbreakable-zero element of $J = I(+)M$. Thus J is not an (m, n)-closed ideal of A. Suppose that $(0, 0) \neq (r, y)^m = (r^m, m(r^m-1)y)) \in J$. Then r is not an (m, n)-unbreakable-zero element of I by hypothesis. Hence $(r^n, n(r^n-1)y)) = (r, y)^n \in J$; so J is a weakly (m, n)-closed ideal of A that is not (m, n)-closed. □

We end this section with another way to construct weakly (m, n)-closed ideals that are not (m, n)-closed. See [4, Theorems 3.1 and 3.8] for similar results for (m, n)-closed ideals.

**Theorem 2.13.** Let R be an integral domain and $I = p^bR$ a principal ideal of R, where p is a prime element of R and k a positive integer. Let m be a positive integer such that $m < k$, and write $q = mq + r$ for integers $q, r$, where $q \geq 1$ and $0 \leq r < m$. Then $J = I/p^qR$ is a weakly (m, n)-closed ideal of $R/p^qR$ that is not (m, n)-closed for positive integers $n < m$ and $c \geq k + 1$ if and only if $r \neq 0$, $k + 1 \leq c \leq m(q + 1)$, and $n(q + 1) < k$.

**Proof.** Suppose that J is a weakly (m, n)-closed ideal of $R/p^qR$ that is not (m, n)-closed for positive integers $n < m$ and $c \geq k + 1$. It is clear that $r \neq 0$, for if $r = 0$, then $0 \neq (p^0)^m + p^qR \in J$, but $(p^0)^n + p^qR \notin J$. Since $q + 1$ is the smallest positive integer such that $(p^{q+1})^m + p^qR \in J$ and J is not (m, n) closed, we have $0 = (p^{q+1})^m + p^qR \in J$ and $(p^{q+1})^n + p^qR \notin J$. Thus $n(q + 1) < k$ and $k + 1 \leq c \leq (q + 1)m$.

Conversely, assume that $r \neq 0$, $k + 1 \leq c \leq m(q + 1)$, and $n(q + 1) < k$. Let $x \in R/p^qR$ such that $x^m \in J$. Then $x = p^{i+1}y + p^qR$ for some $y \in R$ such that $p^{i+1} \not| y$ in R. Since $x^m = (p^{i+1})^m + p^qR \in J$, we have $i \geq q + 1$. Thus by hypothesis, $x^m = 0$ in $R/p^qR$. Since $0 = (p^{q+1})^m + p^qR \in J$ and $n(q + 1) < k$, we have $(p^{q+1})^n + p^qR \notin J$. Hence J is not (m, n)-closed. □

**Example 2.14.** (a) Let $R = \mathbb{Z}$, $I = 2^{12}\mathbb{Z}$, and $J = I/2^{13}\mathbb{Z}$. Then by Theorem 2.13, J is a weakly (5, 3)-closed ideal of $\mathbb{Z}/2^{13}\mathbb{Z}$ that is not (5, 3)-closed.

(b) Let $R$, I, and J be as in part (a) above. Then J$(+)J$ is a weakly (5, 3)-closed ideal of $\mathbb{Z}/2^{13}\mathbb{Z}(+)J$ that is not (5, 3)-closed by Theorem 2.12.

3. (m, n)-von Neumann regular rings

In this section, we introduce the concepts of (m, n)-von Neumann regular elements and (m, n)-von Neumann regular rings and use them to determine when every proper ideal of R is (m, n)-closed or weakly (m, n)-closed. We also define the related concepts of n-regular and $\omega$-regular commutative rings. First, we handle the case for ideals contained in $\text{Nil}(R)$.

**Theorem 3.1.** Let R be a commutative ring and m and n positive integers with $m > n$. Then every ideal of R contained in $\text{Nil}(R)$ is weakly (m, n)-closed if and only if $w^m = 0$ for every $w \in \text{Nil}(R)$. 
Proof. Suppose that every ideal of $R$ contained in $\text{Nil}(R)$ is weakly $(m,n)$-closed, but $w^m \neq 0$ for some $w \in \text{Nil}(R)$. Let $J = w^mR \subseteq \text{Nil}(R)$. Then $J$ is weakly $(m,n)$-closed and $0 \neq w^m \in J$; so $w^n \in J$ and $w^n \neq 0$ since $n < m$. Thus $w^n = w^m a$ for some $a \in R$, and hence $w^n(1 - w^m a) = 0$. Then $1 - w^m a \in U(R)$ since $w^m a \in \text{Nil}(R)$; so $w^n = 0$, a contradiction. Thus $w^m = 0$ for every $w \in \text{Nil}(R)$.

Conversely, suppose that $w^m = 0$ for every $w \in \text{Nil}(R)$. Then every ideal of $R$ contained in $\text{Nil}(R)$ is weakly $(m,n)$-closed by definition. □

Recall that $x \in R$ is a von Neumann regular element of $R$ if $x^2 r = x$ for some $r \in R$. Similarly, $x \in R$ is a $\pi$-regular element of $R$ if $x^{2n} r = x^n$ for some $r \in R$ and positive integer $n$. Thus $R$ is a von Neumann regular ring (resp., $\pi$-regular ring) if and only if every element of $R$ is von Neumann regular (resp., $\pi$-regular). It is well known that $R$ is $\pi$-regular (resp., von Neumann regular) if and only if $\dim(R) = 0$ (resp., $R$ is reduced and $\dim(R) = 0$) [14, Theorem 3.1, p. 10]. A ring $R$ is a strongly $\pi$-regular ring if there is a positive integer $n$ such that for every $x \in R$, we have $x^{2n} r = x^n$ for some $r \in R$. For a recent article on von Neumann regular and related elements of a commutative ring, see [3]. These concepts are generalized in the next definition.

**Definition 3.2.** Let $R$ be a commutative ring and $m$ and $n$ positive integers. Then $x \in R$ is an $(m,n)$-von Neumann regular element of $R$ if $x^{2m} r = x^n$ for some $r \in R$. If every element of $R$ is $(m,n)$-vnr, then $R$ is an $(m,n)$-von Neumann regular ring.

Thus a commutative ring $R$ is von Neumann regular if and only if it is $(2,1)$-von Neumann regular, and $R$ is strongly $\pi$-regular if and only if it is $(2n,n)$-von Neumann regular for some positive integer $n$. The next theorem gives some basic facts about $(m,n)$-vnr elements.

**Theorem 3.3.** Let $R$ be a commutative ring, $x \in R$, and $m$ and $n$ positive integers.

1. $x$ is $(m,n)$-vnr for $m \leq n$ (so we usually assume that $m > n$).
2. If $x$ is $(m,n)$-vnr, then $x$ is $(m',n')$-vnr for all positive integers $m' \leq m$ and $n' \geq n$.
3. If $x \in U(R)$ or $x = 0$, then $x$ is $(m,n)$-vnr for all positive integers $m$ and $n$.
4. If $x \in R \setminus (Z(R) \cup U(R))$, then $x$ is $(m,n)$-vnr if and only if $m \leq n$.
5. If $x^m = 0$, then $x$ is $(m,n)$-vnr for every positive integer $m$.
6. If $x^k = 0$ and $x^{k-1} \neq 0$ for an integer $k \geq 2$, then $x$ is $(m,n)$-vnr if and only if $m \leq n$ or $n \geq k$.
7. If $x$ is $(m,n)$-vnr with $m > n$, then $x$ is $(m + 1,n)$-vnr. Moreover, in this case, $x$ is $(m',n')$-vnr for all positive integers $m'$ and $n' \geq n$. Thus $R$ is von Neumann regular if and only if $R$ is $(m,n)$-von Neumann regular for all positive integers $m$ and $n$. 
Proof. The proofs of (1)-(3) and (5) are clear.

(4) By (1), \( x \) is \( (m,n) \)-vnr for \( m \leq n \). If \( m > n \), then \( x^{m}r = x^{n} \) for \( r \in R \) implies \( x^{n-m}r = 1 \). Thus \( x \in U(R) \), a contradiction.

(6) Suppose that \( x^{m}r = x^{n} \) for \( r \in R \), but \( m > n \) and \( n < k \). Then \( x^{k-1} = x^{n}(x^{k-n}r) = (x^{m}r)(x^{k-n}r) = x^{k}(x^{m-n}r^{2}) = 0 \), a contradiction. Thus \( m \leq n \) or \( n \geq k \). The converse is clear.

(7) Let \( x \) be \( (m,n) \)-vnr with \( m > n \). Then \( x^{m}r = x^{n}r \) for \( r \in R \) implies \( x^{n} = x^{m}r = x^{n}(x^{m-n}r) = (x^{m}r)(x^{m-n}r) = x^{m+1}(x^{m-n}r^{2}) \) with \( x^{m-n}r^{2} \in R \). Thus \( x \) is \( (m+1,n) \)-vnr. The "moreover" statement follows by induction and (2).

\[ \square \]

Corollary 3.4. Let \( R \) be a commutative ring and \( m \) and \( n \) positive integers with \( m > n \). Then \( R \) is \( (m,n) \)-von Neumann regular if and only if \( R \) is \( (m',n') \)-von Neumann regular for all positive integers \( m' \) and \( n' \geq n \). In particular, if \( R \) is \( (m,n) \)-von Neumann regular, then \( R \) is strongly \( \pi \)-regular, and thus \( \dim(R) = 0 \).

We next determine when every proper ideal of \( R \) is weakly \( (m,n) \)-closed.

Theorem 3.5. Let \( R \) be a commutative ring and \( m \) and \( n \) positive integers with \( m > n \). Then the following statements are equivalent.

1. Every proper ideal of \( R \) is weakly \( (m,n) \)-closed.
2. Every non-nilpotent element of \( R \) is \( (m,n) \)-vnr and \( w^{m} = 0 \) for every \( w \in \text{Nil}(R) \).

Proof. (1) \( \Rightarrow \) (2) Since every ideal of \( R \) contained in \( \text{Nil}(R) \) is weakly \( (m,n) \)-closed, \( w^{m} = 0 \) for every \( w \in \text{Nil}(R) \) by Theorem 3.1. Let \( x \in R \setminus \text{Nil}(R) \). If \( x \in U(R) \), then \( x \) is \( (m,n) \)-vnr by Theorem 3.3(3). If \( x \notin U(R) \), then \( I = x^{n}R \) is weakly \( (m,n) \)-closed and \( 0 \neq x^{n} \in I \); so \( x^{n} \in I \). Thus \( x^{n} = x^{m}r \) for some \( r \in R \), and hence \( x \) is \( (m,n) \)-vnr.

(2) \( \Rightarrow \) (1) Let \( I \) be a proper ideal of \( R \) and \( 0 \neq x^{m} \in I \) for \( x \in R \). Then \( x \notin \text{Nil}(R) \); so \( x \) is \( (m,n) \)-vnr. Thus \( x^{m}r = x^{n} \) for some \( r \in R \); so \( x^{n} = x^{m}r \in I \). Hence \( I \) is weakly \( (m,n) \)-closed. \( \square \)

In view of Theorem 3.5, we have the following result.

Corollary 3.6. Let \( R \) be a reduced commutative ring and \( m \) and \( n \) positive integers. Then the following statements are equivalent.

1. Every proper ideal of \( R \) is weakly \( (m,n) \)-closed.
2. Every proper ideal of \( R \) is \( (m,n) \)-closed.
3. \( R \) is \( (m,n) \)-von Neumann regular.

The following result is the analog of Theorem 3.5 for \( (m,n) \)-closed ideals.

Theorem 3.7. Let \( R \) be a commutative ring and \( m \) and \( n \) positive integers. Then the following statements are equivalent.

1. Every proper ideal of \( R \) is \( (m,n) \)-closed.
(2) $R$ is $(m,n)$-von Neumann regular.

Proof. (1) $\Rightarrow$ (2) Let $x \in R$. If $x \in U(R)$, then $x$ is $(m,n)$-vnr by Theorem 3.3(3). If $x \notin U(R)$, then $I = x^mR$ is $(m,n)$-closed and $x^m \in I$. Thus $x^n \in I$; so $x^n = x^mr$ for some $r \in R$. Hence $x$ is $(m,n)$-vnr, and thus $R$ is $(m,n)$-von Neumann regular.

(2) $\Rightarrow$ (1) Let $I$ be a proper ideal of $R$ and $x^m \in I$ for $x \in R$. Since $x$ is $(m,n)$-vnr, $x^mr = x^n$ for some $r \in R$. Thus $x^n = x^mr \in I$; so $I$ is $(m,n)$-closed. □

Of course, we are mainly interested in the case when $m > n$. The next theorem incorporates Theorem 3.7 with another characterization ([4, Theorem 2.14]) of when every proper ideal is $(m,n)$-closed. Note that in Theorem 3.8(3) below, there are no conditions on $m$ other than $m > n$.

**Theorem 3.8.** Let $R$ be a commutative ring and $m$ and $n$ positive integers with $m > n$. Then the following statements are equivalent.

1. Every proper ideal of $R$ is $(m,n)$-closed.
2. $R$ is $(m,n)$-von Neumann regular.
3. $\dim(R) = 0$ and $w^n = 0$ for every $w \in \text{Nil}(R)$.

Proof. (1) $\Leftrightarrow$ (2) is Theorem 3.7 and (1) $\Leftrightarrow$ (3) is [4, Theorem 2.14]. □

Theorem 3.8 gives a nice ring-theoretic characterization of $(m,n)$-von Neumann regular rings (for $m > n$). This can now be used to give a characterization of strongly $\pi$-regular commutative rings which strengthens Corollary 3.4.

**Theorem 3.9.** Let $R$ be a commutative ring. Then the following statements are equivalent.

1. $R$ is strongly $\pi$-regular.
2. There are positive integers $m$ and $n$ with $m > n$ such that $R$ is $(m,n)$-von Neumann regular.
3. There is a positive integer $n$ such that $R$ is $(m,n)$-von Neumann regular for every positive integer $m$.
4. $\dim(R) = 0$ and there is a positive integer $n$ such that $w^n = 0$ for every $w \in \text{Nil}(R)$.

Proof. (1) $\Rightarrow$ (2) A strongly $\pi$-regular ring is $(2n,n)$-von Neuman regular for some positive integer $n$.

(2) $\Rightarrow$ (3) This follows from Corollary 3.4.

(3) $\Rightarrow$ (1) In particular, $R$ is $(2n,n)$-von Neumann regular, and thus strongly $\pi$-regular.

(2) $\Leftrightarrow$ (4) This is just (2) $\Leftrightarrow$ (3) of Theorem 3.8. □

We next investigate in more detail the pairs $(m,n)$ for which a commutative ring $R$ or an $x \in R$ is $(m,n)$-von Neumann regular.
Definition 3.10. Let $R$ be a commutative ring, $x \in R$, and $k$ a positive integer.

1. $V(R,x) = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid x \text{ is } (m,n)\text{-vnr}\}$.
2. $V(R) = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid R \text{ is } (m,n)\text{-von Neumann regular}\}$.
3. $B_k = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid m \leq n \text{ or } n \geq k\}$.
4. $B_\omega = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid m \leq n\}$.

Then $V(R) = \bigcap_{x \in R} V(R,x)$ and $\mathbb{N} \times \mathbb{N} = B_1 \supseteq B_2 \supseteq \cdots \supseteq B_\omega$.

Theorem 3.11. Let $R$ be a commutative ring and $x \in R$.

1. $V(R,x) = B_k$, where $k$ is the smallest positive integer such that $(i,k) \in V(R,x)$ for some $i > k$. (Thus $k$ is the smallest positive integer such that $x$ is $(m,k)$-vnr for every positive integer $m$.) If no such $k$ exists, then $V(R,x) = B_\omega$.
2. $V(R) = B_k$, where $k$ is the smallest positive integer such that $(i,k) \in V(R,x)$ for some $i > k$ and every $x \in R$. (Thus $k$ is the smallest positive integer such that $x$ is $(m,k)$-vnr for every $x \in R$ and positive integer $m$.) If no such $k$ exists, then $V(R) = B_\omega$.

Proof. (1) follows directly from Theorem 3.3(7). Thus (2) holds by definition.

These ideas can also be used to classify zero-dimensional commutative rings.

Definition 3.12. Let $R$ be a commutative ring and $n$ a positive integer.

1. $R$ is $n$-regular if $V(R) = B_n$, i.e., $n$ is the smallest positive integer such that for every $x \in R$ and positive integer $m$, $x^n = x^m r_m$ for some $r_m \in R$.
2. $R$ is $\omega$-regular if for every $x \in R$, $V(R,x) = B_{n_x}$ for some positive integer $n_x$, but $V(R) = B_\omega$.

A commutative ring $R$ is von Neumann regular if and only if it is 1-regular, and $R$ is strongly $\pi$-regular if and only if it is $n$-regular for some positive integer $n$. Note that $R$ is $\pi$-regular if and only if every $x \in R$ is $(m,n)$-vnr for some positive integers $m$ and $n$ with $m > n$, but a $\pi$-regular ring may be $\omega$-regular (see Example 3.13(d)). Thus $R$ is $\alpha$-regular for $\alpha$ a positive integer or $\omega$ if and only if $R$ is $\pi$-regular, if and only if $\dim(R) = 0$. So, in some sense, this concept measures how far a zero-dimensional commutative ring is from being von Neumann regular.

We next give several examples. In particular, we show that if $\alpha$ is any positive integer or $\omega$, there is a quasilocal commutative ring $R_\alpha$ that is $\alpha$-regular.

Example 3.13. Let $R$ be a commutative ring.
(a) Suppose that there is an \( x \in R \setminus (Z(R) \cup U(R)) \) (so \( \dim(R) > 0 \)). Then \( V(R) = V(R, x) = B_{\omega} \) by Theorem 3.3(4). Thus \( R \) is not \( \omega \)-regular or \( n \)-regular for any positive integer \( n \).

(b) Suppose that \( R \) is quasilocal with maximal ideal \( M = (x) \) with \( x^k = 0 \) and \( x^{k-1} \neq 0 \) for an integer \( k \geq 2 \). Then \( V(R) = B_k \) by Theorem 3.3(3),(6); so \( R \) is \( k \)-regular. This also holds for \( k = 1 \) since a field is von Neumann regular. In particular, for a prime \( p \) and any positive integer \( k \), \( V(\mathbb{Z}_{p^k}) = B_k \), and thus \( \mathbb{Z}_{p^k} \) is \( k \)-regular.

(c) Let \( R_1 \) and \( R_2 \) be commutative rings. Then \( x = (x_1, x_2) \in R_1 \times R_2 \) is \((m, n)\)-vnr if and only if \( x_1 \) and \( x_2 \) are \((m, n)\)-vnr in \( R_1 \) and \( R_2 \), respectively. Thus \( V(R_1 \times R_2) = B_{k_1} \), where \( V(R_1) = B_{k_1}, V(R_2) = B_{k_2} \), and \( k = \max\{k_1, k_2\} \); so \( R_1 \times R_2 \) is \( \max\{k_1, k_2\} \)-regular when \( R_1 \) and \( R_2 \) are \( k_1 \)-regular and \( k_2 \)-regular, respectively. In particular, for distinct primes \( p_1, \ldots, p_r \), positive integers \( k_1, \ldots, k_r \), and \( k = \max\{k_1, \ldots, k_r\} \), \( V(\mathbb{Z}_{p_1^{k_1}} \times \cdots \times \mathbb{Z}_{p_r^{k_r}}) = B_k \), and hence \( \mathbb{Z}_{p_1^{k_1}} \times \cdots \times \mathbb{Z}_{p_r^{k_r}} \) is \( k \)-regular.

(d) Let \( R = \mathbb{Z}_2[(X_n)_{n \in \mathbb{N}}]/(\{X_n^{n+1}\}_{n \in \mathbb{N}}) = \mathbb{Z}_2[(x_n)_{n \in \mathbb{N}}] \). Then \( R \) is a zero-dimensional quasilocal commutative ring with maximal ideal \( \Nil(R) = (\{x_n\}_{n \in \mathbb{N}}) \); so \( R \) is \( \pi \)-regular. Thus every \( x \in R \) has \( V(R, x) = B_k \) for some positive integer \( k \) and \( V(R, x_n) = B_{n+1} \) by Theorem 3.3(3),(6); so \( V(R) = B_{\omega} \). Hence \( R \) is \( \omega \)-regular.

References


David F. Anderson  
Department of Mathematics  
The University of Tennessee  
Knoxville, TN 37996-1320, USA  
Email address: anderson@math.utk.edu

Ayman Badawi  
Department of Mathematics & Statistics  
The American University of Sharjah  
P.O. Box 26666, Sharjah, United Arab Emirates  
Email address: abadawi@aus.edu

Brahim Fahid  
Department of Mathematics  
Faculty of Sciences, B.P. 1014  
Mohammed V University, Rabat, Morocco  
Email address: fahid.brahim@yahoo.fr