Zero Divisor Graph of Integers Modulo n

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1 Abstract

Let $n \geq 2$ be a positive integer. Then the zero-divisor graph of Z_n (i.e., the integers module n), denoted by ZG_n, is undirected simple graph with vertex set $V_n = \{a \in Z_n \mid a \neq 0$ and $ab = 0$ in Z_n for some nonzero $b \in Z_n\}$ such that two distinct vertices, x, y, in V_n are adjacent (i.e., connected by an edge) if and only if $xy = 0$ in Z_n. Using some elementary techniques and concepts from graph theory and discrete mathematics, we tackle some properties of ZG_n. Specifically, properties of interest are values of n in which the graph ZG_n becomes complete bipartite. Other properties will be studied as well, for example: connectedness, diameter, and girth of such graphs. We show that ZG_n is connected for every $n \geq 2$. We show ZG_n is complete bipartite if and only $n = 8, 9$, or $n = pq$ for some distinct prime integers p, q. We show that ZG_n is complete if and only if $n = p^2$ for some odd prime positive integer p. For a given integer $n \geq$, we show the diameter of ZG_n is at most 3 while its girth is either $3, 4$ or ∞.

2 Introduction

Let $n \in \mathbb{N}$, $n > 1$, and $V_n = \{a \in Z_n \mid a \neq 0$ and $ab = 0$ in Z_n for some nonzero $b \in Z_n\}$. The zero-divisor graph of Z_n, denoted by ZG_n is undirected simple graph with vertex set V_n such that two distinct vertices, x, y, in V_n are adjacent (i.e., connected by an edge) if and only if $xy = 0$ in Z_n.
This thesis is inspired by Anderson and Livingston's work on the zero-divisor graph of commutative rings [**?**]. However, our investigation in this paper focuses only on Z_n (note that $(Z_n, +, .)$ is a commutative ring, where "+" denotes addition modulo n and "." is multiplication modulo n). We apply concepts from basic Number Theory and Graph Theory to arrive to similar results as in [1].
This paper will tackle the following graph properties of ZG_n.

- What values of n is ZG_n *complete bipartite*?
- Describe connectedness of ZG_n.
- Describe diameter of ZG_n.
- Describe girth of ZG_n.

We refer to graph theory concepts from Bondy and Murty's *Graph Theory* [**?**]. This paper will also provide figures of some graphs of interest

We recall some definitions. Let (G, V, E) be a graph with vertex set V and edge set E. Then G is called *complete* if any two distinct vertices are adjacent. The complete graph with n vertices will be denoted by K_n. A graph G is called *bipartite* if V can be partitioned into two disjoint nonempty vertex sets A and B such that there is no edge between every two distinct vertices in A and there is no edge between every two distinct vertices in B. A graph G is called *complete bipartite* if it is bipartite and two distinct vertices are adjacent if and only if they are in distinct vertex sets. If one of the vertex sets is a singleton, then G is called a *star* graph. We denote the complete bipartite graph by $K_{m,n}$, where $|A| = m$ and $|B| = n$; so a star graph is a $K_{1,n}$.

3 Results

3.1 When is ZG_n a complete graph?

Let (G, V, E) be a graph with vertex set V and edge set E. Recall that G is called *complete* if any two distinct vertices are adjacent. The complete graph with n vertices will be denoted by K_n.

Theorem 3.1.1. *Assume $|V_n| \geq 2$. Then ZG_n is complete if and only if $n = p^2$ for some odd prime positive integer p.*

Proof. Assume that ZG_n is complete. Assume that $p_1p_2 \mid n$ for some distinct prime positive integers $p_1 < p_2$. Then $p_1, 2p_1 \in V_n$ and there is no edge between p_1 and $2p_1$. Thus $n = p^m$ for some prime positive integer p and a positive integer $m \geq 2$. If $p = 2$, then it is clear that ZG_n is not complete. Assume that $p \neq 2$ and $m \geq 3$. Then $p, 2p \in V_n$ and there is no edge between p and $2p$. Hence $m = 2$. Now suppose that $n = p^2$ for some odd prime positive integer p. Let $x, y \in V_n$. Then $p \mid x$ and $p \mid y$. Hence $xy = 0$ in Z_n. Thus ZG_n is a complete graph. □

Example 3.1.1. *For $n = 361 = 19^2$, figure 1 is the graph of ZG_{361}. Note that that $ZG_{361} = K_{18}$.*

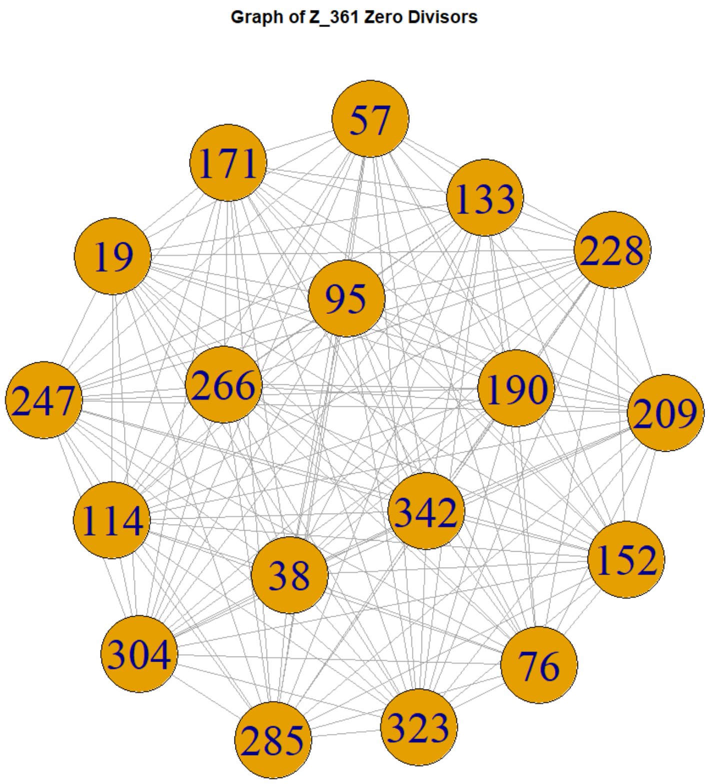


Figure 1: Graph of ZG_{361}

3.2 When is ZG_n a Complete Bipartite?

Let $n \geq 2$. Then using the concept of prime number decomposition, we have $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, where $p_1, ..., p_k$ are distinct prime numbers and $\alpha_1, ..., \alpha_k \in \mathbb{N}$.

We recall [**?**, Theorem 4.7] that states "A graph is bipartite if and only if it contains no odd cycle."

Proposition 3.2.1. *If n is a prime number, then $V_n = \emptyset$.*

Proof. We shall prove that $V_p = \emptyset$ by contradiction. Assume $\exists x, y \in \mathbb{Z}_p$ such that $xy = 0$ in $\mathbb{Z}_p$.

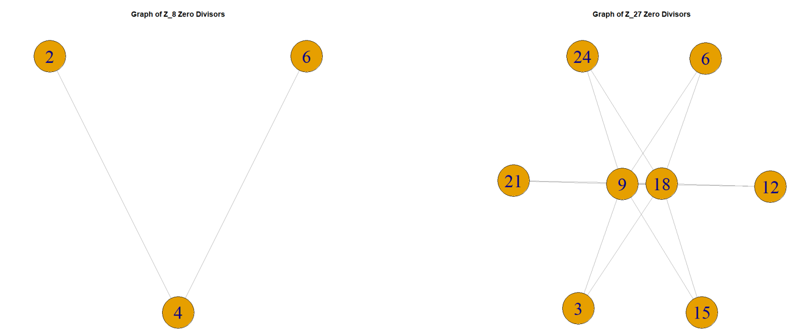
$$xy \equiv 0 \mod p \to xy = mp, m \in \mathbb{N}.$$

Since x, y are both nonzero integers, then at least one of them has to be divisible by p such that $\frac{xy}{p} = m$, but since $x < p$ and $y < p$ and products of primes less than n, then we have a contradiction and $V_p = \emptyset$. □

Proposition 3.2.2. *If $n = p^\alpha$, where $p \geq 2$ is a prime integer and $\alpha \geq 1$, then ZG_n is complete bipartite if and only if $n = 2^3 = 8$ or $n = 3^2 = 9$.*

Proof. If $n = 4$, then $V_4 = \{2\}$. Hence there is not much to say. If $n = 8$, then $V_8 = \{2, 4, 6\}$ and hence $ZG_8 = K_{1,2}$. If $n = 2^\alpha$, where $\alpha \geq 4$, then $4 - 3 \cdot 2^{\alpha-2} - 2^{\alpha-1} - 4$ is a cycle in ZG_n of length 3 (note that since $\alpha \geq 4$, we have $2^{\alpha-2} \cdot 2^{\alpha-1} = 2^{2\alpha-3} = 0$ in Z_n). Since ZG_n has an odd cycle, we conclude that ZG_n is not bipartite, and hence it is not complete bipartite. If $n = 3^2 = 9$, then $V_9 = \{3, 6\}$. It is clear that $ZG_9 = K_{1,1}$. Suppose that $n = 3^\alpha$, where $\alpha \geq 3$. Then $3 - 2 \cdot 3^{\alpha-1} - 3^{\alpha-1} - 3$ is a cycle in ZG_n of length 3 (note that since $\alpha \geq 3$, we have $3^{\alpha-1} \cdot 3^{\alpha-1} = 3^{2\alpha-2} = 0$ in Z_n). Since ZG_n has an odd cycle, we conclude that ZG_n is not bipartite, and hence it is not complete bipartite. Suppose that $n = p^\alpha$, where $p \neq 2$, $p \neq 3$, and $\alpha \geq 2$. Then $p - 2 \cdot p^{\alpha-1} - 3 \cdot p^{\alpha-1} - p$ is a cycle in ZG_n of length 3. Since ZG_n has an odd cycle, we conclude that ZG_n is not bipartite, and hence it is not complete bipartite. Thus ZG_n is complete bipartite if and only if $n = 2^3 = 8$ or $n = 3^2 = 9$. □

Example 3.2.1. *The graph of ZG_8 is given in figure 2a (note that $ZG_8 = K_{1,2}$). The graph of ZG_{27} is given in figure 2b (note that ZG_{27} is not complete bipartite; in fact, ZG_{27} is not a bipartite graph).*



(a) Graph of ZG_8

(b) Graph of ZG_{27}

Proposition 3.2.3. *Assume that $n = p_1^{m_1} \cdots p_k^{m_k}$, where $k \geq 3$, $p_1, ..., p_k$ are distinct prime positive integers, and $m_1, m_2, ..., m_k \geq 1$. Then ZG_n is not a complete bipartite graph.*

Proof. Let $v_1 = p_1^{m_1} p_2^{m_2}$, $v_2 = p_1^{m_1} p_3^{m_3} \cdots p_k^{m_k}$, and $v_3 = p_2^{m_2} p_3^{m_3} \cdots p_k^{m_k}$. Then $v_1, v_2, v_3 \in V_n$ and $v_1 - v_2 - v_3 - v_1$ is a cycle in ZG_n of length 3. Since ZG_n has an odd cycle, we conclude that ZG_n is not bipartite, and hence it is not a complete bipartite graph. □

Example 3.2.2. *For $n = 30 = 2 \cdot 3 \cdot 5$, figure 3 is the graph of ZG_{30}. Note that ZG_{30} is not bipartite and hence it is not complete bipartite.*

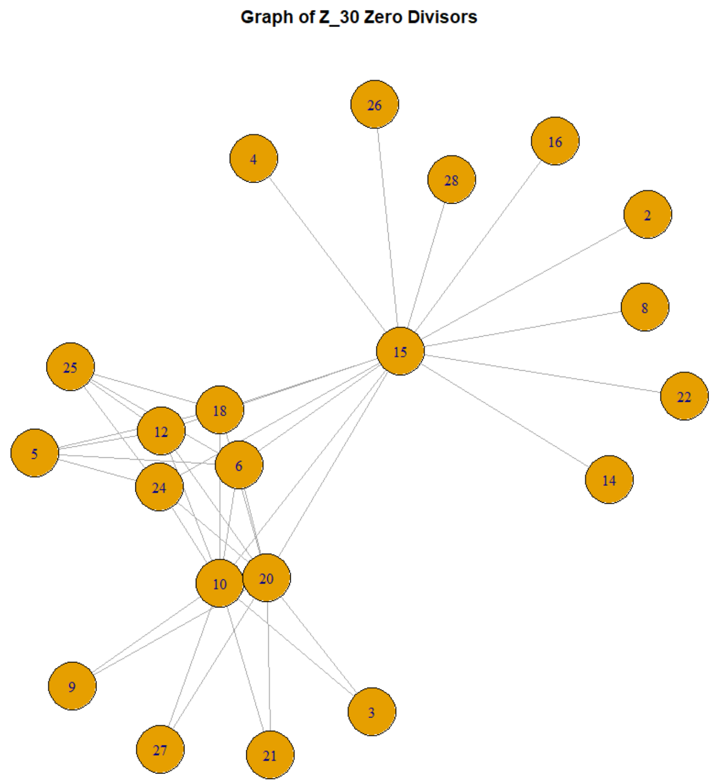


Figure 3: Graph of ZG_{30}

Proposition 3.2.4. *Assume that $n = p_1^{m_1} p_2^{m_2}$, where p_1, p_2 are distinct prime positive integers, $m_1 \geq 2$ and $m_2 \geq 1$. Then ZG_n is not a complete bipartite graph.*

Proof. Assume that ZG_n is a complete bipartite graph. Then V_n can be partitioned into two sets A, B such that two distinct vertices are adjacent if and only if they are in distinct vertex sets. We may assume that p_1 in A. Since $p_1 \cdot p_1^{m_1-1} p_2^{m_2} = 0$ in Z_n (note that $m_1 \geq 2$), we conclude that $p_1^{m_1-1} p_2^{m_2} \in B$. Since $p_1 p_2 \neq 0$ in Z_n and $p_1 \in A$, we conclude that $p_2 \in A$. Since $p_1^{m_1-1} p_2^{m_2} \in B$ and $p_2 \in A$, we have $p_2 \cdot p_1^{m_1-1} p_2^{m_2} = 0$ in Z_n, a contradiction. Thus ZG_n is not a complete bipartite graph. □

Example 3.2.3. *For $n = 12 = 2^2 \cdot 3$, figure 4 is the graph of ZG_{12}. Note that ZG_{12} is bipartite, but it is not complete bipartite.*

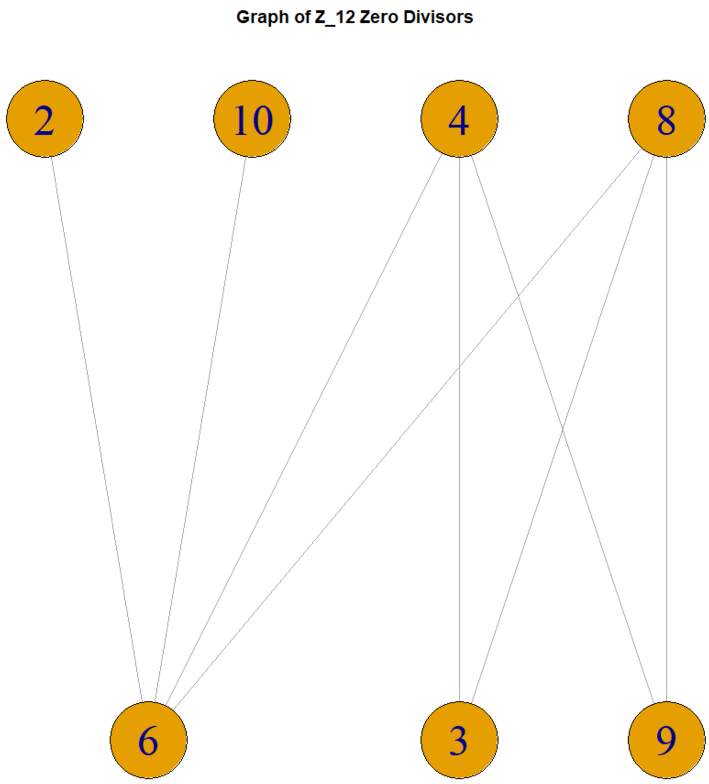


Figure 4: Graph of ZG_{12}

Example 3.2.4. *For* $n = 20 = 2^2 \cdot 5$*, figure 5 is the graph of* ZG_{20}*. Note that* ZG_{20} *is bipartite, but it is not complete bipartite.*

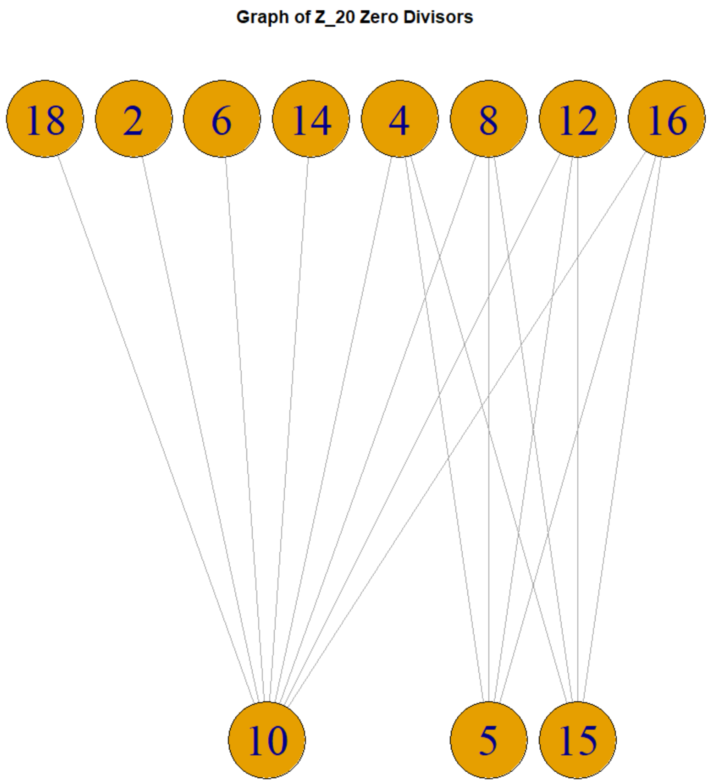


Figure 5: Graph of ZG_{20}

For $n = 28 = 2^2 \cdot 7$, figure 6 is the graph of ZG_{28}. Note that ZG_{28} is bipartite, but it is not complete bipartite.

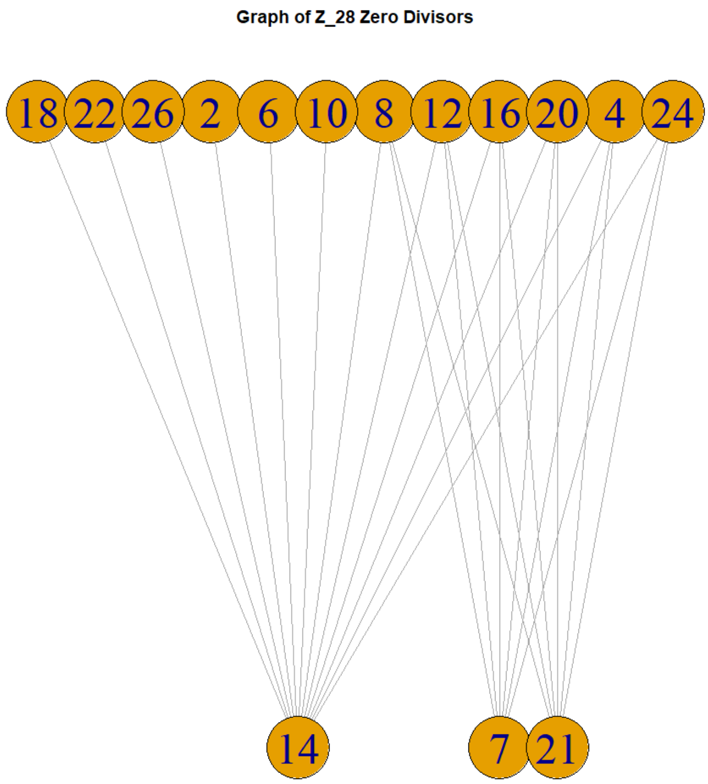


Figure 6: Graph of ZG_{28}

Example 3.2.5. *For $n = 24 = 2^3 \cdot 3$, figure 7 is the graph of ZG_{24}. Note that ZG_{24} is not bipartite and hence it is not complete bipartite.*

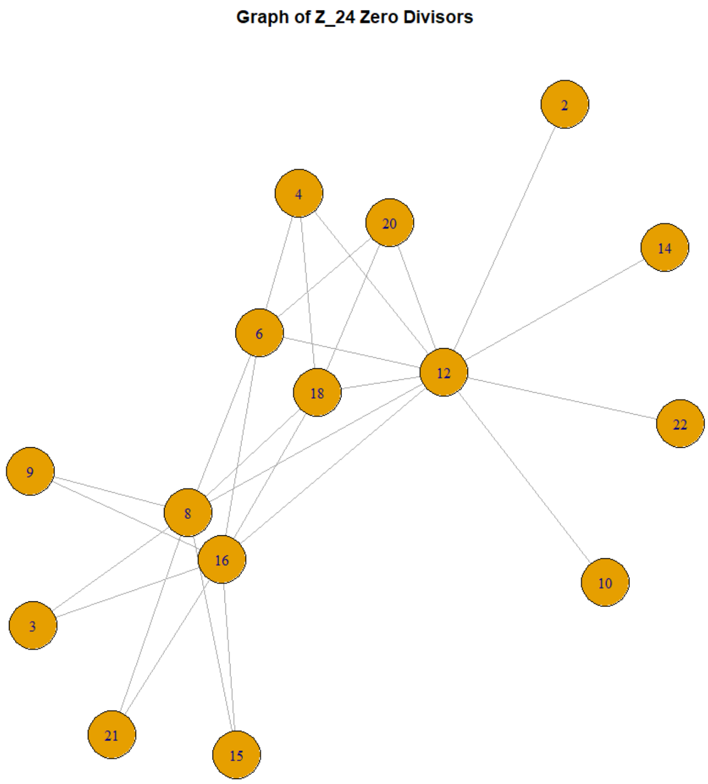


Figure 7: Graph of ZG_{24}

Example 3.2.6. *For $n = 18 = 3^2 \cdot 2$, figure 8 is the graph of ZG_{18}. Note that ZG_{18} is not bipartite and hence it is not complete bipartite.*

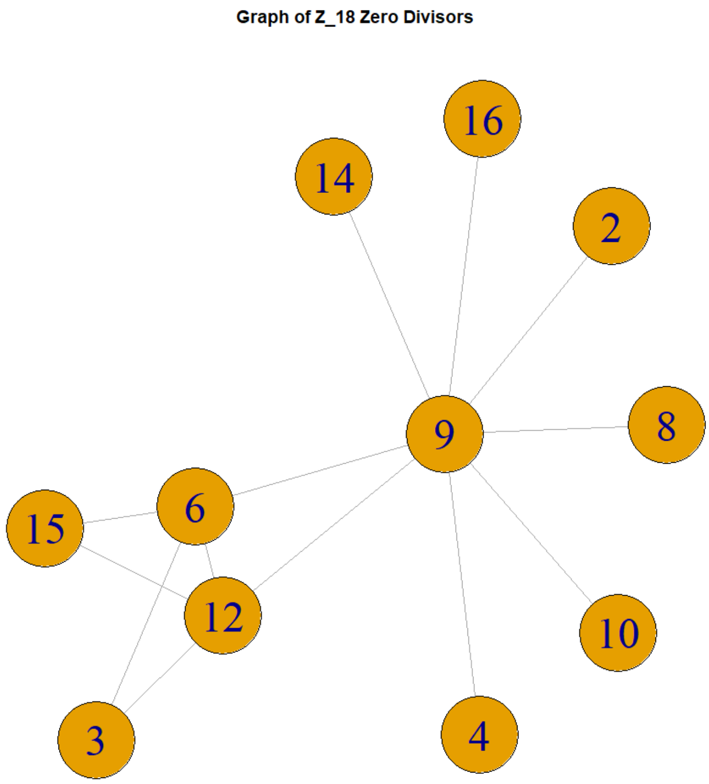


Figure 8: Graph of ZG_{18}

Example 3.2.7. *For $n = 45 = 3^2 \cdot 5$, figure 9 is the graph of ZG_{45}. Note that ZG_{45} is not bipartite and hence it is not complete bipartite.*

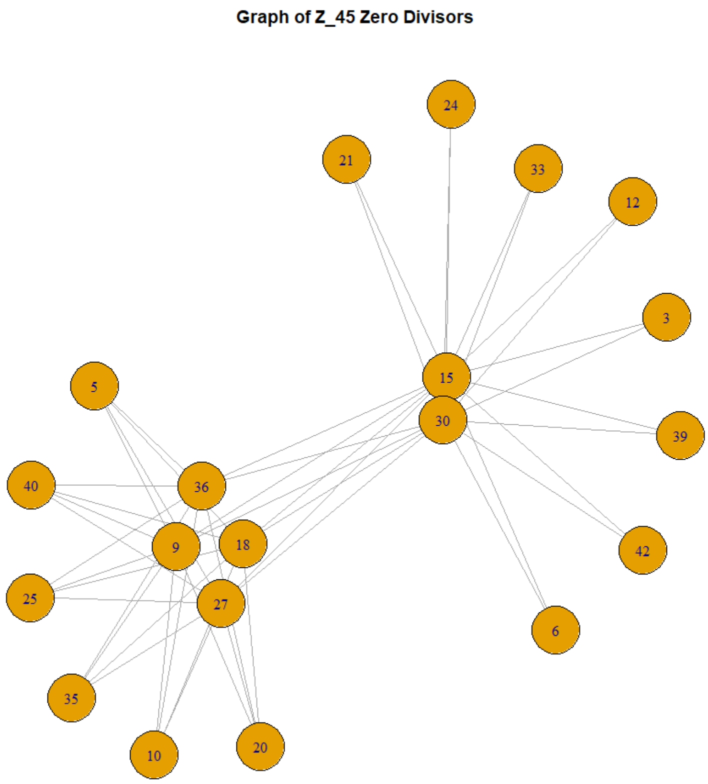


Figure 9: Graph of ZG_{45}

Example 3.2.8. *For $n = 36 = 2^2 \cdot 3^2$, figure 10 is the graph of ZG_{36}. Note that ZG_{36} is not bipartite and hence it is not complete bipartite.*

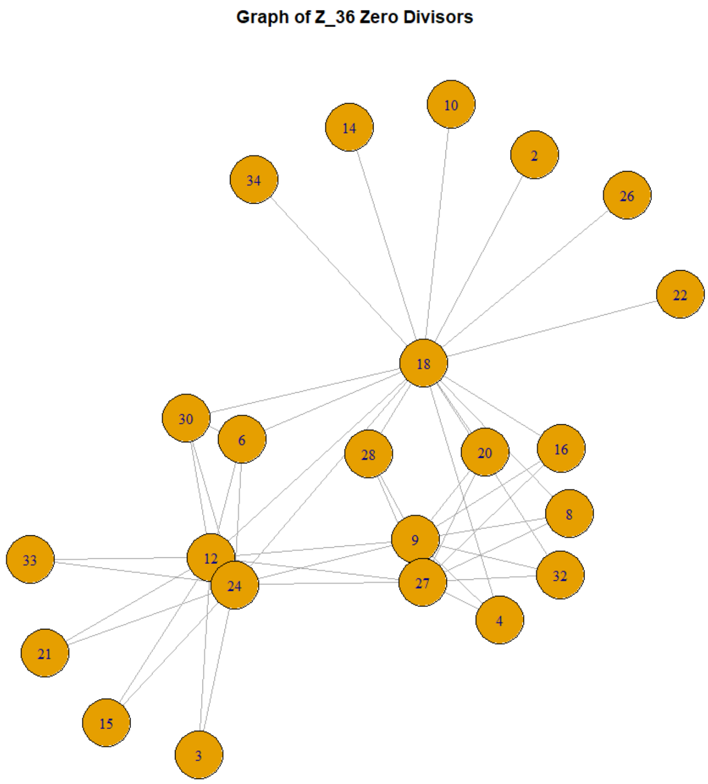


Figure 10: Graph of ZG_{36}

Proposition 3.2.5. *Assume that $n = p_1p_2$, where p_1, p_2 are distinct prime positive integers. Then ZG_n is complete bipartite and $ZG_n = K_{p_1-1,p_2-1}$.*

Proof. Let $A = \{a \in V_n \mid p_1 \mid a\} = \{aP_1 \mid 1 \leq a \leq p_2 - 1\}$ and $B = \{b \in V_n \mid p_2 \mid b\} = \{aP_2 \mid 1 \leq a \leq p_1 - 1\}$. It is clear that $|A| = p_2 - 1$, $|B| = p_1 - 1$, $A \cup B = V_n$ and $A \cap B = \emptyset$. Note that $p_1 \nmid b$ for every $b \in B$ and $p_2 \nmid a$ for every $a \in A$. It is clear that two distinct vertices of V_n are adjacent if and only if they are in distinct vertex sets. Thus ZG_n is a complete bipartite graph. Since $|B| = p_1 - 1$ and $|A| = p_2 - 1$, we conclude that $ZG_n = K_{p_1-1,p_2-1}$. □

Example 3.2.9. *The following (11) is the complete bipartite graph of ZG_{69}.*

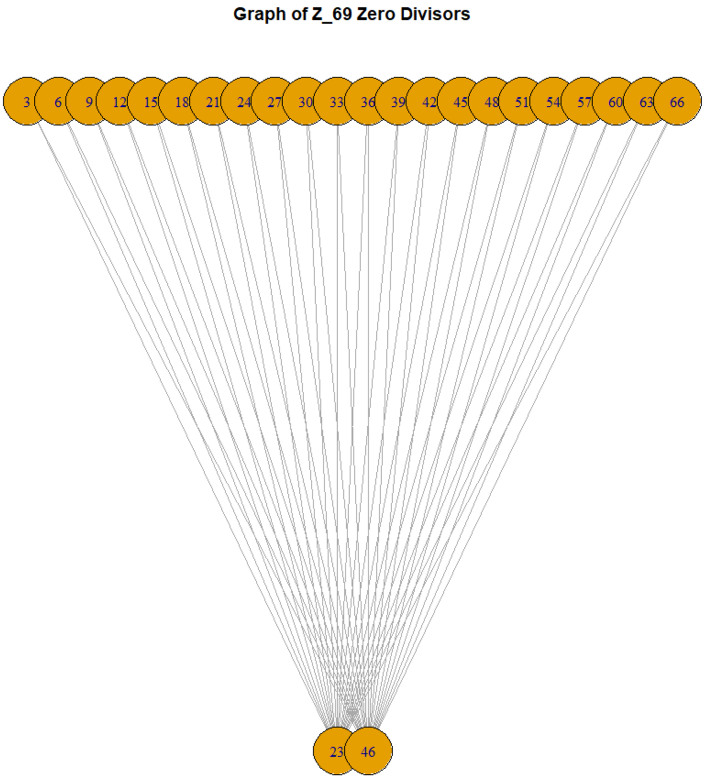


Figure 11: Graph of ZG_{69}

Example 3.2.10. *The following (12) is the complete bipartite graph of* ZG_{22}.

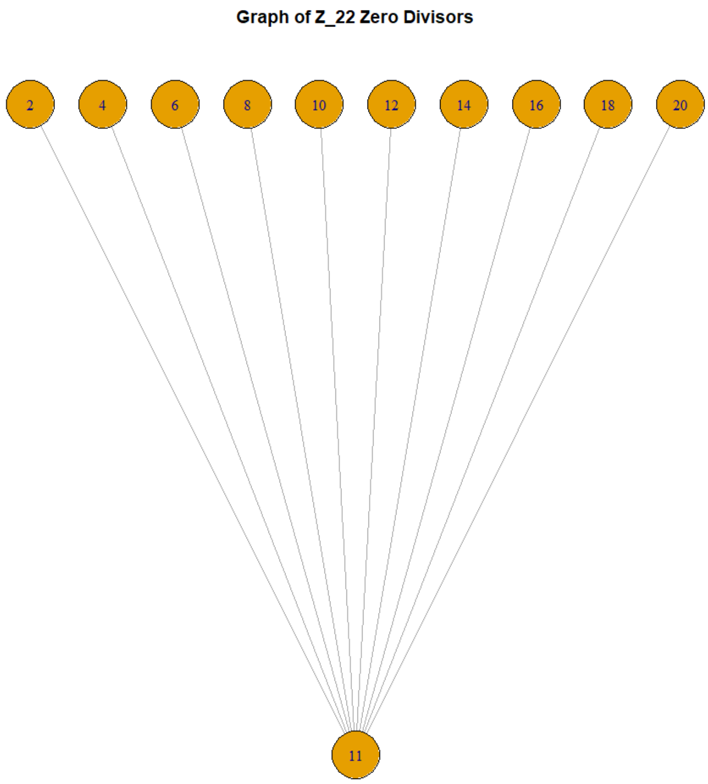


Figure 12: Graph of ZG_{22}

Combining Propositions 3.2.1, 3.2.2, 3.2.3, 3.2.4, and 3.2.5, we arrive at the following result

Theorem 3.2.6. *Assume* $|V_n| \geq 2$*. Then* ZG_n *is complete bipartite if and only if either* $n = p_1p_2$*, where* p_1, p_2 *are distinct prime positive integers or* $n = 8$ *or* $n = 9$*.*

3.3 Connectedness and diameter of ZG_n

Recall that a graph (G, V, E) is called *connected* if there exists a path between every two distinct vertices of G. If $P : v_1 - v_2 - \cdots - v_{m+1}$ is a path in G, where $v_1, ..., v_{m+1}$ are distinct vertices of G, then we say that "P" is a path of length m. If v_1, v_2 are two distinct vertices of G, then the *distance* between v_1, v_2, is denoted by $d(v_1, v_2)$ and it is the length of the shortest path between v_1 and v_2. The *diameter* of G, denoted by $diam(G)$, is defined as $sup\{d(v, w) \mid v, w$ are distinct vertices of $G\}$.

Theorem 3.3.1. *Assume* $|V_n| \geq 2$*. Then* ZG_n *is connected. Furthermore,* $1 \leq d(v, w) \leq 3$ *for every two distinct vertices* $v, w \in V_n$*, and hence* $1 \leq diam(ZG_n) \leq 3$.

Proof. Assume that $V_n \geq 2$. Let $x, y \in V_n$ be two distinct vertices. If $xy = 0$ in Z_n, then $d(x, y) = 1$. Suppose that $xy \neq 0$ in Z_n. Since $x, y \in V_n$, there exist $v, w \in V_n$ such that $vx = 0$ and $wy = 0$. We consider two cases.
Case 1. Suppose that $h = vw \neq 0$ in Z_n. Hence $hx = hy = 0$ in Z_n. Since $xy \neq 0$ in Z_n, $h = vw \neq x$ and $h = vw \neq y$. Thus $x - h - y$ is a path in ZG_n of length 2, and hence $d(x, y) = 2$.
Case 2. Suppose that $vw = 0$ in Z_n. If $vy = 0$ in Z_n, then $v \neq x$ (since $xy \neq 0$ and $vy = 0$ in Z_n), and hence $x - v - y$ is a path in ZG_n of length 2 (i.e, $d(x, y) = 2$). If $wx = 0$ in Z_n, then $w \neq y$ (since $xy \neq 0$ and $wx = 0$ in Z_n), and hence $x - w - y$ is a path in ZG_n of length 2 (i.e, $d(x, y) = 2$). Now, assume that $x, y, v, w \in V_n$ are distinct vertices. Since $xy \neq 0$ in Z_n and $vw = 0$ in Z_n, we conclude that $x - v - w - y$ is a path in ZG_n of length 3 (i.e, $d(x, y) = 3$).
Thus ZG_n is connected and $1 \leq diam(ZG_n) \leq 3$. □

3.4 Girth of ZG_n and bipartite ZG_n

Let (G, V, E) be a graph and $C : v_1 - v_2 - v_3 - \cdots - v_m - v_1$ be a path in G from v_1 to v_1, where $v_1, ..., v_m$ are distinct vertices of G. Then we say "C" is a *cycle* in G of length m. The *girth* of G, denoted by $gr(G)$, is the length of the shortest cycle in G and if G has no cycles, then we say $gr(G) = \infty$.

Proposition 3.4.1. *Assume that $n = 4p$ for some odd prime positive integer p. Then ZG_n is a bipartite graph that is not complete bipartite. Furthermore, $gr(ZG_n) = 4$.*

Proof. Let $A = \{a \in V_n \mid p \nmid a\}$ and $B = V_n - A$. Then $A \cup B = V_n$ and $A \cap B = \emptyset$. It is clear that every two distinct vertices in A are not connected by an edge. Let $x, y \in B$. Then $x = ap, y = bp$, where $1 \leq a, b \leq 2$. Hence $xy = abp^2 = 0$ in Z_n if and only if $a = b = 2$. Thus $xy = 0$ in Z_n if and only if $x = y = 2p$. Hence every two distinct vertices in B are not connected by an edge. Thus ZG_n is a bipartite graph. By Theorem 3.3.1, ZG_n is not complete bipartite. Since ZG_n is bipartite, $gr(ZG_n) \neq 3$. Since $n = 4p$ for some odd prime positive integer p, we conclude that $4, 8 \in V_n$. Hence $2p - 4 - p - 8 - 2p$ is a cycle in ZG_n of length 4. Thus $gr(ZG_n) = 4$. □

Proposition 3.4.2. *Assume that $n = p_1^{m_1} p_2^{m_2}$, where p_1, p_2 are distinct odd prime positive integers, $m_1 \geq 2$ and $m_2 \geq 1$. Then $gr(ZG_n) = 3$, and hence ZG_n is not a bipartite graph.*

Proof. Let $v_1 = p_1^{m_1}, v_2 = p_1 p_2^{m_2}$, and $v_3 = 2p_1^{m_1 - 1} p_2^{m_2}$. Since p_1, p_2 are odd prime integers and $m_1 \geq 2$, we conclude that v_1, v_2, v_3 are distinct vertices in V_n. Hence $v_1 - v_2 - v_3 - v_1$ is a cycle in ZG_n of length 3. Thus $gr(ZG_n) = 3$. □

Remark 3.4.3. *Observe that in the proofs of Propositions 3.2.1, 3.2.2, and 3.2.3, we constructed a cycle in ZG_n of length 3.*

In light of Remark 3.4.3, Theorem 3.3.1, Proposition 3.4.1, and Proposition 3.4.2, we arrive at the following result.

Theorem 3.4.4. *Assume $|V_n| \geq 2$. Then ZG_n is a bipartite graph if and only if either $n = 8$ or $n = 9$ or $n = p_1 p_2$ for some distinct prime positive integers p_1, p_2 or $n = 4p$ for some odd prime positive integer p.*

In view of Theorem 3.3.1 and Theorem 3.4.4, we have the following result.

Corollary 3.4.5. *Assume $|V_n| \geq 2$. Then ZG_n is a bipartite graph that is not complete bipartite if and only if $n = 4p$ for some odd prime positive integer p.*

Theorem 3.4.6. *Assume $|V_n| \geq 2$. Then $gr(ZG_n) \in \{\infty, 3, 4\}$. In particular:*

1. *$gr(ZG_n) = \infty$ if and only $n = 8$ or $n = 9$ or $n = 2p$ for some odd prime positive integer p.*
2. *$gr(ZG_n) = 4$ if and only if $n = 4p$ for some odd prime positive integer p or $n = p_1p_2$ for some odd prime positive integers p_1, p_2.*
3. *$gr(ZG_n) = 3$ if and only if neither $n = 8$ nor $n = 9$ nor $n = p_1p_2$ for some prime positive integers p_1, p_2 nor $n = 4p$ for some odd prime positive integer p.*

Proof. The proofs of (i), (ii), (iii) are now clear by Remark 3.4.3, Propositions 3.2.5, 3.4.1, 3.4.2 and Theorem 3.4.4. □

4 Conclusion

This thesis is inspired by Anderson and Livingston's work on the zero-divisor graph of commutative rings [**?**]. However, our investigation in this paper focused only on Z_n (note that $(Z_n, +, .)$ is a commutative ring, where "+" denotes addition modulo n and "." is multiplication modulo n). We applied concepts from basic Number Theory and Graph Theory to arrive to similar results as in [1].
Let $n \geq 2$ be a positive integer. Then the zero-divisor graph of Z_n (i.e., the integers module n), denoted by ZG_n, is undirected simple graph with vertex set $V_n = \{a \in Z_n \mid a \neq 0$ and $ab = 0$ in Z_n for some nonzero $b \in Z_n\}$ such that two distinct vertices, x, y, in V_n are adjacent (i.e., connected by an edge) if and only if $xy = 0$ in Z_n. We showed that ZG_n is complete bipartite if and only $n = 8, 9$, or $n = pq$ for some distinct prime integers p, q. We showed that ZG_n is complete if and only if $n = p^2$ for some odd prime positive integer p. For a given integer $n \geq$, we showed that the diameter of ZG_n is at most 3 while its girth is either $3, 4$ or ∞.

Further work related to this paper could include :

(1) For what values of n is ZG_n a polypartite ?

(2) For what values of n is ZG_n a planar?

(3) What is the relationship between Euler's Totient Function of n, denoted as $\varphi(n)$, and $|V_n|$ for ZG_n?