Algebra Colloquium **25** : 3 (2018) 387–398
DOI: 10.1142/S1005386718000287

Algebra Colloquium
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On Weakly δ-Semiprimary Ideals of Commutative Rings

Ayman Badawi
Department of Mathematics & Statistics, The American University of Sharjah
P.O. Box 26666, Sharjah, United Arab Emirates
E-mail: abadawi@aus.edu

Deniz Sonmez   Gursel Yesilot
Department of Mathematics, Yildiz Technical University
Davutpaşa-Istanbul, Turkey
E-mail: dnzguel@hotmail.com   gyesilot@yildiz.edu.tr

Received 4 July 2017
Revised 3 September 2017

Communicated by Nanqing Ding

Abstract. Let R be a commutative ring with $1 \neq 0$. A proper ideal I of R is a semiprimary ideal of R if whenever $a, b \in R$ and $ab \in I$, we have $a \in \sqrt{I}$ or $b \in \sqrt{I}$; and I is a weakly semiprimary ideal of R if whenever $a, b \in R$ and $0 \neq ab \in I$, we have $a \in \sqrt{I}$ or $b \in \sqrt{I}$. In this paper, we introduce a new class of ideals that is closely related to the class of (weakly) semiprimary ideals. Let $I(R)$ be the set of all ideals of R and let $\delta : I(R) \to I(R)$ be a function. Then δ is called an expansion function of ideals of R if whenever L, I, J are ideals of R with $J \subseteq I$, we have $L \subseteq \delta(L)$ and $\delta(J) \subseteq \delta(I)$. Let δ be an expansion function of ideals of R. Then a proper ideal I of R is called a *δ-semiprimary* (*weakly δ-semiprimary*) ideal of R if $ab \in I$ ($0 \neq ab \in I$) implies $a \in \delta(I)$ or $b \in \delta(I)$. A number of results concerning weakly δ-semiprimary ideals and examples of weakly δ-semiprimary ideals are given.

2010 Mathematics Subject Classification: 13A05, 13F05

Keywords: semiprimary ideal, weakly semiprimary ideal, weakly prime ideal, weakly primary ideal, δ-primary ideal, δ-2-absorbing ideal

1 Introduction

We assume throughout this paper that all rings are commutative with $1 \neq 0$. Let R be a commutative ring. An ideal I of R is said to be proper if $I \neq R$. When I is a proper ideal of R, then we use $\sqrt{I}$ to denote the radical ideal of I (that is, $\sqrt{I} = \{x \in R \mid x^n \in I$ for some positive integer $n \geq 1\}$). Note that $\sqrt{\{0\}}$ is the set (ideal) of all nilpotent elements of R.

Let I be a proper ideal of R. We recall from [1] and [6] that I is said to be *weakly semiprime* if $0 \neq x^2 \in I$ implies $x \in I$; recall from [1] (also, [4]) that a proper ideal

I of R is said to be *weakly prime* (*weakly primary*) if $0 \neq ab \in I$ implies $a \in I$ or $b \in I$ ($a \in I$ or $b \in \sqrt{I}$). Over the past several years, there has been considerable attention in the literature to prime ideals and their generalizations (for example, see [1]–[11], and [14]).

Recall that a proper ideal I of a ring R is called *semiprimary* if whenever $x, y \in R$ and $xy \in I$, we have $x \in \sqrt{I}$ or $y \in \sqrt{I}$. Gilmer [12] studied rings in which semiprimary ideals are primary. In this paper, we define a proper ideal I of R to be *weakly semiprimary* if whenever $x, y \in R$ and $0 \neq xy \in I$, we have $x \in \sqrt{I}$ or $y \in \sqrt{I}$. In fact, we will study a more general concept. Let $I(R)$ be the set of all ideals of R. Zhao [14] introduced the concept of expansion of ideals of R. We recall from [14] that a function $\delta : I(R) \to I(R)$ is called an expansion function of ideals of R if whenever L, I, J are ideals of R with $J \subseteq I$, we have $L \subseteq \delta(L)$ and $\delta(J) \subseteq \delta(I)$. In addition, recall from [14] that a proper ideal I of R is said to be a δ-primary ideal of R if whenever $a, b \in R$ with $ab \in I$, we have $a \in I$ or $b \in \delta(I)$, where δ is an expansion function of ideals of R. Let δ be an expansion function of ideals of R. In this paper, a proper ideal I of R is called a *δ-semiprimary* (*weakly δ-semiprimary*) ideal of R if $ab \in I$ ($0 \neq ab \in I$) implies $a \in \delta(I)$ or $b \in \delta(I)$. For example, let $\delta : I(R) \to I(R)$ such that $\delta(I) = \sqrt{I}$. Then δ is an expansion function of ideals of R, and hence a proper ideal I of R is a δ-semiprimary (weakly δ-semiprimary) ideal of R if and only if I is a semiprimary (weakly semiprimary ideal) of R. A number of results concerning weakly δ-semiprimary ideals and examples of weakly δ-semiprimary ideals are given.

Let δ be an expansion function of ideals of a ring R. Among many results in this paper, it is shown that if I is a weakly δ-semiprimary ideal of R that is not δ-semiprimary, then $I^2 = \{0\}$ and hence $I \subseteq \sqrt{\{0\}}$ (Theorem 2.10). If I is a proper ideal of R and $I^2 = \{0\}$, then I need not be a weakly δ-semiprimary ideal of R (Example 2.14). It is shown in Example 2.23 that if I, J are weakly δ-semiprimary ideals of R such that $\delta(I) = \delta(J)$ and $I + J \neq R$, then $I + J$ need not be a weakly δ-semiprimary ideal of R. We show that if R is a Boolean ring, then every weakly semiprimary ideal of R is weakly prime (Theorem 2.16); if S is a multiplicatively closed subset of R such that $S \cap Z(R) = \emptyset$ (where $Z(R)$ is the set of all zerodivisor elements of R) and I is a weakly semiprimary ideal of R such that $S \cap \sqrt{I} = \emptyset$, then I_S is a weakly semiprimary ideal of R_S (Theorem 3.1). We also show that if I is a weakly δ-semiprimary ideal of R and $\{0\} \neq AB \subseteq I$ for some ideals A, B of R, then $A \subseteq \delta(I)$ or $B \subseteq \delta(I)$ (Theorem 5.4).

2 Weakly δ-Semiprimary Ideals

Definition 2.1. [14] Let $I(R)$ be the set of all ideals of R. Then a function $\delta : I(R) \to I(R)$ is called an *expansion function of ideals of* R if whenever L, I, J are ideals of R with $J \subseteq I$, we have $L \subseteq \delta(L)$ and $\delta(J) \subseteq \delta(I)$.

In the following example, we give some expansion functions of ideals of a ring R.

Example 2.2. [8] Let $\delta : I(R) \to I(R)$ be a function. Then

(1) If $\delta(I) = I$ for every ideal I of R, then δ is an expansion function of ideals

of R.

(2) If $\delta(I) = \sqrt{I}$ (note that $\sqrt{R} = R$) for every ideal I of R, then δ is an expansion function of ideals of R.

(3) Suppose that R is a quasi-local ring (i.e., R has exactly one maximal ideal) with maximal ideal M. If $\delta(I) = M$ for every proper ideal I of R, then δ is an expansion function of ideals of R.

(4) Let I be a proper ideal of R. Recall from [13] that an element $r \in R$ is called integral over I if there is an integer $n \geq 1$ and $a_i \in I^i$, $i = 1, \ldots, n$, such that $r^n + a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_{n-1} r + a_n = 0$. Let $\overline{I} = \{r \in R \mid r \text{ is integral over } I\}$. Let $I \in I(R)$. It is known that $\overline{I}$ is an ideal of R and $I \subseteq \overline{I} \subseteq \sqrt{I}$, and if $J \subseteq I$, then $\overline{J} \subseteq \overline{I}$ (see [13]). If $\delta(I) = \overline{I}$ for every ideal I of R, then δ is an expansion function of ideals of R.

(5) Let J be a proper ideal of R. If $\delta(I) = I + J$ for every ideal I of R, then δ is an expansion function of ideals of R.

(6) Assume that δ_1 and δ_2 are expansion functions of ideals of R. We give $\delta : I(R) \to I(R)$ such that $\delta(I) = \delta_1(I) + \delta_2(I)$. Then δ is an expansion function of ideals of R.

(7) Assume that δ_1 and δ_2 are expansion functions of ideals of R. We give $\delta : I(R) \to I(R)$ such that $\delta(I) = \delta_1(I) \cap \delta_2(I)$. Then δ is an expansion function of ideals of R.

(8) Assume that δ_1 and δ_2 are expansion functions of ideals of R. We give $\delta : I(R) \to I(R)$ such that $\delta(I) = (\delta_1 \circ \delta_2)(I) = \delta_1(\delta_2(I))$. Then δ is an expansion function of ideals of R.

We recall the following definitions.

Definition 2.3. Let δ be an expansion function of ideals of a ring R.

(1) A proper ideal I of R is called a *δ-semiprimary* (*weakly δ-semiprimary*) ideal of R if whenever $a, b \in R$ and $ab \in I$ $(0 \neq ab \in I)$, we have $a \in \delta(I)$ or $b \in \delta(I)$.

(2) If $\delta : I(R) \to I(R)$ such that $\delta(I) = \sqrt{I}$ for every proper ideal I of R, then δ is an expansion function of ideals of R. In this case, a proper ideal I of R is called a *semiprimary* (*weakly semiprimary*) ideal of R if whenever $a, b \in R$ and $ab \in I$ $(0 \neq ab \in I)$, we have $a \in \sqrt{I}$ or $b \in \sqrt{I}$.

(3) A proper ideal I of R is called a *δ-primary* (*weakly δ-primary*) ideal of R if whenever $a, b \in R$ and $ab \in I$ $(0 \neq ab \in I)$, we have $a \in I$ or $b \in \delta(I)$.

(4) A proper ideal I of R is called a *weakly prime* ideal of R if whenever $a, b \in R$ and $0 \neq ab \in I$, we have $a \in I$ or $b \in I$.

(5) A proper ideal I of R is called a *weakly primary* ideal of R if whenever $a, b \in R$ and $0 \neq ab \in I$, we have $a \in I$ or $b \in \sqrt{I}$.

We have the following trivial result, whose proof we omit.

Theorem 2.4. *Let I be a proper ideal of R and let δ be an expansion function of ideals of R.*

1. *If I is a δ-primary ideal of R, then I is a weakly δ-semiprimary ideal of R. In particular, if I is a primary ideal of R, then I is a weakly semiprimary ideal of R.*

(2) *If I is a weakly δ-primary ideal of R, then I is a weakly δ-semiprimary ideal of R. In particular, if I is a weakly primary ideal of R, then I is a weakly semiprimary ideal of R.*
(3) *If I is a δ-semiprimary ideal of R, then I is a weakly δ-semiprimary ideal of R.*
(4) *$\sqrt{\{0\}}$ is a weakly prime ideal of R if and only if $\sqrt{\{0\}}$ is a weakly semiprimary ideal of R.*
(5) *If I is a weakly prime ideal of R, then I is a weakly semiprimary ideal of R.*

The following is an example of a proper ideal of a ring R that is a weakly semiprimary ideal of R, but neither weakly primary nor weakly prime.

Example 2.5. Let $A = Z_2[X, Y]$, where X and Y are indeterminates. Then

$$I = (Y^2, XY)A \quad \text{and} \quad J = (Y^2, X^2Y^2)A$$

are ideals of A. Set $R = A/J$. Hence, $L = I/J$ is an ideal of R and $\sqrt{L} = (Y, XY)A/J$. Since $0 \neq XY + J \in L$ and neither $X + J \in \sqrt{L}$ nor $Y + J \in L$, we conclude that L is not a weakly primary ideal of R. Since $0 + J \neq XY + J \in L$ but neither $X + J \in L$ nor $Y + J \in L$, we know that L is not a weakly prime ideal of R. It is easy to check that L is a weakly semiprimary ideal of R.

The next example is an ideal that is weakly semiprimary but not semiprimary.

Example 2.6. Let $R = Z_{36}$. Then $I = \{0\}$ is a weakly semiprimary ideal of R by definition. Note that $\sqrt{I} = 6R$. Since $0 = 4 \cdot 9 \in I$ but neither $4 \in \sqrt{I}$ nor $9 \in \sqrt{I}$, we conclude that I is not a semiprimary ideal of R.

Definition 2.7. Let δ be an expansion function of ideals of a ring R. Suppose that I is a weakly δ-semiprimary ideal of R and $x \in R$. Then x is called a dual-zero element of I if $xy = 0$ for some $y \in R$ and neither $x \in \delta(I)$ nor $y \in \delta(I)$. (Note that y is also a dual-zero element of I.)

Remark 2.8. Let δ be an expansion function of ideals of a ring R. If I is a weakly δ-semiprimary ideal of R which is not δ-semiprimary, then I must have a dual-zero element of R.

Theorem 2.9. *Let δ be an expansion function of ideals of a ring R and I be a weakly δ-semiprimary ideal of R. If $x \in R$ is a dual-zero element of I, then $xI = \{0\}$.*

Proof. Assume that $x \in R$ is a dual-zero element of I. Then $xy = 0$ for some $y \in R$ such that neither $x \in \delta(I)$ nor $y \in \delta(I)$. Let $i \in I$. Thus, $x(y+i) = 0 + xi = xi \in I$. Suppose that $xi \neq 0$. Since $0 \neq x(y + i) = xi \in I$ and I is a weakly δ-semiprimary ideal of R, we conclude that $x \in \delta(I)$ or $(y + i) \in \delta(I)$, and hence $x \in \delta(I)$ or $y \in \delta(I)$, a contradiction. Thus, $xi = 0$. □

Theorem 2.10. *Let δ be an expansion function of ideals of a ring R and I be a weakly δ-semiprimary ideal of R that is not δ-semiprimary. Then $I^2 = \{0\}$, and hence $I \subseteq \sqrt{\{0\}}$.*

Proof. Since I is a weakly δ-semiprimary ideal of R that is not δ-semiprimary, we conclude that I has a dual-zero element $x \in R$. Since $xy = 0$ and neither $x \in \delta(I)$ nor $y \in \delta(I)$, we conclude that y is a dual-zero element of I. Let $i, j \in I$. Then by Theorem 2.9, we have $(x + i)(y + j) = ij \in I$. Suppose that $ij \neq 0$. Since $0 \neq (x+i)(y+j) = ij \in I$ and I is a weakly δ-semiprimary ideal of R, we conclude that $x + i \in \delta(I)$ or $y + j \in \delta(I)$, and hence $x \in \delta(I)$ or $y \in \delta(I)$, a contradiction. Therefore $ij = 0$, and hence $I^2 = \{0\}$. □

In view of Theorem 2.10, we have the following result.

Corollary 2.11. *Let I be a weakly semiprimary ideal of R that is not semiprimary. Then $I^2 = \{0\}$, and hence $I \subseteq \sqrt{\{0\}}$.*

The following example shows that a proper ideal I of R with the property $I^2 = \{0\}$ need not be a weakly semiprimary ideal of R.

Example 2.12. Let $R = Z_{12}$. Then $I = \{0, 6\}$ is an ideal of R and $I^2 = \{0\}$. Note that $\sqrt{I} = I$. Since $0 \neq 2 \cdot 3 \in I$ and neither $2 \in \sqrt{I}$ nor $3 \in \sqrt{I}$, we conclude that I is not a weakly semiprimary ideal of R.

Theorem 2.13. *Let δ be an expansion function of ideals of a ring R and I be a proper ideal of R. If $\delta(I)$ is a weakly prime of R, then I is a weakly δ-semiprimary ideal of R. In particular, if $\sqrt{I}$ is a weakly prime of R, then I is a weakly semiprimary ideal of R.*

Proof. Suppose that $0 \neq xy \in I$ for some $x, y \in R$. Hence, $0 \neq xy \in \delta(I)$. Since $\delta(I)$ is weakly prime, we conclude that $x \in \delta(I)$ or $y \in \delta(I)$. Thus, I is a weakly δ-semiprimary ideal of R. □

Note that if I is a weakly semiprimary ideal of a ring R that is not semiprimary, then $\sqrt{I}$ need not be a weakly prime ideal of R. We have the following example.

Example 2.14. The ideal $I = \{0\}$ is a weakly semiprimary ideal of Z_{12}. However, $\sqrt{I} = \{0, 6\}$ is not a weakly prime ideal of Z_{12} since $0 \neq 2 \cdot 3 \in \sqrt{I}$, but neither $2 \in \sqrt{I}$ nor $3 \in \sqrt{I}$.

Remark 2.15. Note that a weakly prime ideal of a ring R is weakly semiprimary, but the converse is not true. Let $R = \frac{Z_4[X]}{(X^3)}$. Then $\frac{(X^2)}{(X^3)}$ is an ideal of R. Since $0 \neq (X + (X^3)) \cdot (X + (X^3)) = X^2 + (X^3) \in I$ but $X + (X^3) \notin I$, we conclude that I is not a weakly prime ideal of R. Since $\sqrt{I} = \frac{(2,X)}{(X^3)}$ is a prime ideal of R, I is a (weakly) semiprimary ideal of R.

Let R be a Boolean ring (i.e., $x^2 = x$ for every $x \in R$). Since $\sqrt{I} = I$ for every proper ideal I of R, we have the following result.

Theorem 2.16. *Let R be a Boolean ring and I be a proper ideal of R. Then the following statements are equivalent:*

(1) *I is a weakly semiprimary ideal of R.*

(2) *I is a weakly prime ideal of R.*

Theorem 2.17. *Let δ be an expansion function of ideals of a ring R and I be a weakly δ-semiprimary ideal of R. Suppose that $\delta(I) = \delta(\{0\})$. Then the following statements are equivalent:*
(1) *I is not δ-semiprimary.*
(2) *$\{0\}$ has a dual-zero element of R.*

Proof. (1)$\Rightarrow$(2) As I is a weakly δ-semiprimary ideal of R that is not δ-semiprimary, there are $x, y \in R$ such that $xy = 0$ and neither $x \in \delta(I)$ nor $y \in \delta(I)$. Since $\delta(I) = \delta(\{0\})$, we conclude that x is a dual-zero element of $\{0\}$.

(2)$\Rightarrow$(1) Suppose that x is a dual-zero element of $\{0\}$. Since $\delta(I) = \delta(\{0\})$, it is clear that x is a dual-zero element of I. $\square$

In view of Theorem 2.17, we have the following result.

Corollary 2.18. *Let $I \subseteq \sqrt{\{0\}}$ be a proper ideal of R such that I is a weakly semiprimary ideal of R. Then the following statements are equivalent:*
(1) *I is not semiprimary.*
(2) *$\{0\}$ has a dual-zero element of R.*

Proof. Since $\delta : I(R) \to I(R)$ such that $\delta(I) = \sqrt{I}$ for every proper ideal I of R is an expansion function of ideals of R, we have $\delta(I) = \delta(\{0\})$. Thus, the claim is clear by Theorem 2.17. $\square$

The hypothesis "$\delta(I) = \delta(\{0\})$" in Theorem 2.17 is crucial. To show this, we give an ideal I of a ring R in the following example, such that $I \subseteq \sqrt{\{0\}}$ and $\{0\}$ has a dual-zero element of R but I is a δ-semiprimary ideal of R for some expansion function δ of ideals of R.

Example 2.19. Let $R = Z_8$, $\delta : I(R) \to I(R)$ such that $\delta(I) = \sqrt{I}$ for every nonzero proper ideal I of R, and $\delta(\{0\}) = \{0\}$. Let $I = 4R$. Then $\delta(I) = \sqrt{I} = 2R$. It is clear that I is a δ-semiprimary ideal of R and 2 is a dual-zero element of $\{0\}$.

Theorem 2.20. *Let δ be an expansion function of ideals of a ring R and I be a weakly δ-semiprimary ideal of R. If $J \subseteq I$ and $\delta(J) = \delta(I)$, then J is a weakly δ-semiprimary ideal of R.*

Proof. Suppose that $0 \neq xy \in J$ for some $x, y \in R$. Since $J \subseteq I$, we have $0 \neq xy \in I$. Since I is a weakly δ-semiprimary ideal of R, we see that $x \in \delta(I)$ or $y \in \delta(I)$. Noticing $\delta(I) = \delta(J)$, we conclude that $x \in \delta(J)$ or $y \in \delta(J)$. Thus, J is a weakly δ-semiprimary ideal of R. $\square$

In view of Theorem 2.20, we have the following result.

Corollary 2.21. *Let I be a weakly semiprimary ideal of R such that $I \subseteq \sqrt{\{0\}}$. If $J \subseteq I$, then J is a weakly semiprimary ideal of R. In particular, if L is an ideal of R, then LI and $L \cap I$ are weakly semiprimary ideals of R. Furthermore, if $n \geq 1$ is a positive integer, then I^n is a weakly semiprimary ideal of R.*

Theorem 2.22. *Let $\{I_i\}_{i \in J}$ be a collection of weakly semiprimary ideals of a ring R that are not semiprimary. Then $I = \bigcap I_i$ is a weakly semiprimary ideal of R.*

Proof. Note that $\sqrt{I = \bigcap I_i} = \sqrt{I_1} = \sqrt{\{0\}}$ by Theorem 2.10. Hence, the result follows. □

If I, J are weakly semiprimary ideals of a ring R such that $\sqrt{I} = \sqrt{J}$ and $I + J \neq R$, then $I + J$ need not be a weakly semiprimary ideal of R.

Example 2.23. Let $A = \mathbb{Z}_2[T, U, X, Y]$,

$$H = (T^2,\ U^2,\ XY + T + U,\ TU,\ TX,\ TY,\ UX,\ UY)A$$

be an ideal of A, and $R = A/H$. Then by the construction of R, $I = (TA+H)/H = \{0, T + H\}$ and $J = (UA + H)/H = \{0, U + H\}$ are weakly semiprimary ideals of R such that $|I| = |J| = 2$ and $\sqrt{I} = \sqrt{J} = \sqrt{\{0\}}$ (in R) $= (T, U, XY)A/H$. Let $L = I + J = (H + (T, U)A)/H$. Thus, $\sqrt{L} = \sqrt{\{0\}}$ (in R) and L is not a weakly semiprimary ideal of R since $0 \neq X + H \cdot Y + H = XY + H$, $X + H \notin L$, and $Y + H \notin \sqrt{L}$.

Theorem 2.24. *Let δ be an expansion function of ideals of R such that $\delta(\{0\})$ is a δ-semiprimary ideal of R and $\delta(\delta(\{0\})) = \delta(\{0\})$. Then the following statements hold:*

(1) *$\delta(\{0\})$ is a prime ideal of R.*

(2) *Suppose that I is a weakly δ-semiprimary ideal of R. Then I is a δ-semiprimary ideal of R.*

Proof. (1) Let $ab \in \delta(\{0\})$ for some $a, b \in R$. Suppose that $a \notin \delta(\delta(\{0\})) = \delta(\{0\})$. Since $\delta(\{0\})$ is a δ-semiprimary ideal of R and $a \notin \delta(\delta(\{0\}))$, it follows that $b \in \delta(\delta(\{0\})) = \delta(\{0\})$. Thus, $\delta(\{0\})$ is a prime ideal of R.

(2) Suppose that I is not δ-semiprimary. Clearly, $\delta(\{0\}) \subseteq \delta(I)$. Since $I^2 = \{0\}$ by Theorem 2.10 and $\delta(\{0\})$ is a prime ideal of R, we have $I \subseteq \delta(\{0\})$. Noticing $\delta(\delta(\{0\})) = \delta(\{0\})$, we have $\delta(I) \subseteq \delta(\delta(\{0\})) = \delta(\{0\})$. Since $\delta(\{0\}) \subseteq \delta(I)$ and $\delta(I) \subseteq \delta(\{0\})$, it follows that $\delta(I) = \delta(\{0\})$ is a prime ideal of R. As $\delta(I)$ is prime, I is a δ-semiprimary ideal of R, which is a contradiction. □

Theorem 2.25. *Let δ be an expansion function of ideals of R such that $\delta(\{0\})$ is a weakly δ-semiprimary ideal of R, $\sqrt{\{0\}} \subseteq \delta(\{0\})$, and $\delta(\delta(\{0\})) = \delta(\{0\})$. Then the following statements hold:*

(1) *$\delta(\{0\})$ is a weakly prime ideal of R.*

(2) *Suppose that I is a weakly δ-semiprimary ideal of R that is not δ-semiprimary. Then $\delta(I) = \delta(\{0\}) = \delta(\sqrt{\{0\}})$ is a weakly prime ideal of R that is not prime. Furthermore, if $J \subseteq \sqrt{\{0\}}$, then J is a weakly δ-semiprimary ideal of R that is not δ-semiprimary and $\delta(J) = \delta(\{0\})$.*

Proof. (1) Let $0 \neq ab \in \delta(\{0\})$ for some $a, b \in R$. Now we can suppose that $a \notin \delta(\delta(\{0\})) = \delta(\{0\})$. Since $\delta(\{0\})$ is a weakly δ-semiprimary ideal of R and $a \notin \delta(\delta(\{0\}))$, we have $b \in \delta(\delta(\{0\})) = \delta(\{0\})$. Thus, $\delta(\{0\})$ is a weakly prime ideal of R.

(2) Suppose that I is not δ-semiprimary. Then $I^2 = \{0\}$ by Theorem 2.10, and hence $I \subseteq \sqrt{\{0\}}$. Let J be an ideal of R such that $J \subseteq \sqrt{\{0\}}$. Since $\sqrt{\{0\}} \subseteq \delta(\{0\})$,

we have $J \subseteq \delta(\{0\})$. Therefore, we obtain $\delta(J) \subseteq \delta(\delta(\{0\})) = \delta(\{0\})$. Since $\delta(\{0\}) \subseteq \delta(J)$ and $\delta(J) \subseteq \delta(\{0\})$, we conclude that $\delta(J) = \delta(\{0\})$. In particular, $\delta(I) = \delta(\{0\}) = \delta(\sqrt{\{0\}})$ is a weakly prime ideal of R. Noticing that $\delta(\{0\})$ is a weakly δ-semiprimary of R and $\delta(J) = \delta(\{0\})$, we conclude that J is a weakly δ-semiprimary ideal of R. As I is not δ-semiprimary, it follows that $\delta(I) = \delta(\{0\})$ is not a prime ideal of R. In addition, $\delta(J) = \delta(\{0\})$ is a weakly prime ideal of R that is not prime, so we conclude that J is a weakly δ-semiprimary ideal of R that is not δ-semiprimary. □

3 Weakly δ-Semiprimary Ideals Under Localization and Ring-homomorphism

For a ring R, let $Z(R)$ be the set of all zerodivisors of R.

Theorem 3.1. *Assume that S is a multiplicatively closed subset of R such that $S \cap Z(R) = \emptyset$. If I is a weakly semiprimary ideal of R and $S \cap \sqrt{I} = \emptyset$, then I_S is a weakly semiprimary ideal of R_S.*

Proof. Since $S \cap \sqrt{I} = \emptyset$, we conclude that $\sqrt{I_S} = (\sqrt{I})_S$. Let $a, b \in R$ and $s, t \in S$ such that $0 \neq \frac{a}{s}\frac{b}{t} \in I_S$. Then there exists $u \in S$ such that $0 \neq uab \in I$. Since $u \in S$ and $S \cap \sqrt{I} = \emptyset$, we conclude that $0 \neq ab \in \sqrt{I}$. Since I is a weakly semiprimary ideal of R, we see that $a \in \sqrt{I}$ or $b \in \sqrt{I}$. Thus, $\frac{a}{s} \in \sqrt{I_S}$ or $\frac{b}{t} \in \sqrt{I_S}$. Consequently, I_S is a weakly semiprimary ideal of R_S. □

Theorem 3.2. *Let γ be an expansion function of ideals of R and let I, J be proper ideals of R with $I \subseteq J$. Let $\delta : I\big(\frac{R}{I}\big) \to I\big(\frac{R}{I}\big)$ be an expansion function of ideals of $S = \frac{R}{I}$ such that $\delta\big(\frac{L+I}{I}\big) = \frac{\gamma(L+I)}{I}$ for every $L \in I(R)$. Then the followings statements hold:*

1. *(1) If J is a weakly γ-semiprimary ideal of R, then $\frac{J}{I}$ is a weakly δ-semiprimary ideal of S.*
2. *(2) If I is a weakly γ-semiprimary ideal of R and $\frac{J}{I}$ is a weakly δ-semiprimary ideal of S, then J is a a weakly γ-semiprimary ideal of R.*

Proof. First observe that since $I \subseteq J$, we have $I \subseteq J \subseteq \gamma(J)$ and $\delta(\frac{J}{I}) = \frac{\gamma(J)}{I}$.

(1) Assume that $ab \in J \backslash I$ for some $a, b \in R$. Then $0 \neq ab \in J$. Hence, $a \in \gamma(J)$ or $b \in \gamma(J)$. Thus, $a + I \in \frac{\gamma(J)}{I}$ or $b + I \in \frac{\gamma(J)}{I}$. It follows that $\frac{J}{I}$ is a weakly δ-semiprimary ideal of $S = \frac{R}{I}$.

(2) Since $I \subseteq J$, we have $\gamma(I) \subseteq \gamma(J)$. Assume that $0 \neq ab \in J$ for some $a, b \in R$. Let $ab \in I$. Since I is a weakly γ-semiprimary ideal of R, we have $a \in \gamma(I) \subseteq \gamma(J)$ or $b \in \gamma(I) \subseteq \gamma(J)$. Assume that $ab \in J \backslash I$. Thus, $I \neq ab + I \in \frac{J}{I}$. Since $\frac{J}{I}$ is a weakly δ-semiprimary ideal of S, we have $a + I \in \frac{\gamma(J)}{I}$ or $b + I \in \frac{\gamma(J)}{I}$. Hence, $a \in \gamma(J)$ or $b \in \gamma(J)$. Consequently, J is a a weakly γ-semiprimary ideal of R. □

In view of Theorem 3.2, we have the following result.

Corollary 3.3. *Let I and J be proper ideals of R with $I \subseteq J$. Then the following statements hold:*

(1) *If J is a weakly semiprimary ideal of R, then $\frac{J}{I}$ is a weakly semiprimary ideal of $\frac{R}{I}$.*
(2) *If I is a weakly semiprimary ideal of R and $\frac{J}{I}$ is a weakly semiprimary ideal of $\frac{R}{I}$, then J is a a weakly semiprimary ideal of R.*

Theorem 3.4. *Let R and S be rings and $f : R \to S$ be a surjective ring-homomorphism. Then the following statements hold:*
(1) *If I is a weakly semiprimary ideal of R and* $\text{kernel}(f) \subseteq I$*, then $f(I)$ is a weakly semiprimary ideal of S.*
(2) *If J is a weakly semiprimary ideal of S and* $\text{kernel}(f)$ *is a weakly semiprimary ideal of R, then $f^{-1}(J)$ is a weakly semiprimary ideal of R.*

Proof. (1) Since I is a weakly semiprimary ideal of R and $\text{kernel}(f) \subseteq I$, we conclude that $\frac{I}{\text{kernel}(f)}$ is a weakly semiprimary ideal of $\frac{R}{\text{kernel}(f)}$ by Corollary 3.3(1). Since $\frac{R}{\text{kernel}(f)}$ is ring-isomorphic to S, the result follows.

(2) Let $L = f^{-1}(J)$. Then $\text{kernel}(f) \subseteq L$. Since $\frac{R}{\text{kernel}(f)}$ is ring-isomorphic to S, we conclude that $\frac{L}{\text{kernel}(f)}$ is a weakly semiprimary ideal of $\frac{R}{\text{kernel}(f)}$. Since $\text{kernel}(f)$ is a weakly semiprimary ideal of R and $\frac{L}{\text{kernel}(f)}$ is a weakly semiprimary ideal of $\frac{R}{\text{kernel}(f)}$, we conclude that $L = f^{-1}(J)$ is a weakly semiprimary ideal of R by Corollary 3.3(2). □

4 Weakly δ-Semiprimary Ideals in Product of Rings

Let $R_1, \ldots, R_n$, where $n \geq 2$, be commutative rings with $1 \neq 0$. Assume that $\delta_1, \ldots, \delta_n$ are expansion functions of ideals of $R_1, \ldots, R_n$, respectively. Now we let $R = R_1 \times \cdots \times R_n$. Define a function $\delta_\times : I(R) \to I(R)$ such that

$$\delta_\times(I_1 \times \cdots \times I_n) = \delta_1(I_1) \times \cdots \times \delta_n(I_n)$$

for every $I_i \in I(R_i)$, where $1 \leq i \leq n$. Then it is clear that $\delta_\times$ is an expansion function of ideals of R. Note that every ideal of R is of the form $I_1 \times \cdots \times I_n$, where each I_i is an ideal of R_i for $1 \leq i \leq n$.

Theorem 4.1. *Let R_1 and R_2 be commutative rings with $1 \neq 0$, $R = R_1 \times R_2$, and δ_1, δ_2 be expansion functions of ideals of R_1, R_2, respectively. Let I be a proper ideal of R_1. Then the following statements are equivalent:*
(1) *$I \times R_2$ is a weakly $\delta_\times$-semiprimary ideal of R.*
(2) *$I \times R_2$ is a $\delta_\times$-semiprimary ideal of R.*
(3) *I is a δ_1-semiprimary ideal of R_1.*

Proof. (1)⇒(2) Let $J = I \times R_2$. Then $J^2 \neq \{(0,0)\}$. Hence, J is a $\delta_\times$-semiprimary ideal of R by Theorem 2.10.

(2)⇒(3) Suppose that I is not a δ_1-semiprimary ideal of R_1. Then there exist $a, b \in R_1$ such that $ab \in I$, but neither $a \in \delta_1(I)$ nor $b \in \delta_1(I)$. Since $(a,1)(b,1) = (ab,1) \in I \times R_2$, we have $(a,1) \in \delta_\times(I \times R_2)$ or $(b,1) \in \delta_\times(I \times R_2)$. It follows that $a \in \delta_1(I)$ or $b \in \delta_1(I)$, a contradiction. Thus, I is a δ_1-semiprimary ideal of R_1.

(3)⇒(1) Let I be a δ_1-semiprimary ideal of R_1. Then it is clear that $I \times R_2$ is a (weakly) $\delta_\times$-semiprimary ideal of R. □

Theorem 4.2. *Let R_1 and R_2 be commutative rings with $1 \neq 0$, $R = R_1 \times R_2$, and δ_1, δ_2 be expansion functions of ideals of R_1, R_2, respectively, such that $\delta_2(K) = R_2$ for some ideal K of R_2 if and only if $K = R_2$. Let $I = I_1 \times I_2$ be a proper ideal of R, where I_1 and I_2 are some ideals of R_1 and R_2, respectively. Suppose that $\delta_1(I_1) \neq R_1$. Then the following statements are equivalent:*

(1) *I is a weakly $\delta_\times$-semiprimary ideal of R.*

(2) *$I = \{(0,0)\}$ or $I = I_1 \times R_2$ is a $\delta_\times$-semiprimary ideal of R (and hence I_1 is a δ_1-semiprimary ideal of R_1).*

Proof. (1)⇒(2) Assume that $\{(0,0)\} \neq I = I_1 \times I_2$ is a weakly $\delta_\times$-semiprimary ideal of R. Then there exists $(0,0) \neq (x,y) \in I$ such that $x \in I_1$ and $y \in I_2$. Since I is a weakly $\delta_\times$-semiprimary ideal of R and $(0,0) \neq (x,1)(1,y) = (x,y) \in I$, we conclude that $(x,1) \in \delta_\times(I)$ or $(1,y) \in \delta_\times(I)$. As $\delta_1(I_1) \neq R_1$, we get $(1,y) \notin \delta_\times(I)$. Thus $(x,1) \in \delta_\times(I)$, and hence $1 \in \delta_2(I_2)$. Since $1 \in \delta_2(I_2)$, we see that $\delta_2(I_2) = R_2$, and hence $I_2 = R_2$ by hypothesis. Therefore, $I = I_1 \times R_2$ is a $\delta_\times$-semiprimary ideal of R by Theorem 4.1.

(2)⇒(1) Obvious. □

Corollary 4.3. *Let R_1 and R_2 be commutative rings with $1 \neq 0$ and $R = R_1 \times R_2$. Let I be a proper ideal of R. Then the following statements are equivalent:*

(1) *I is a weakly semiprimary ideal of R.*

(2) *$I = \{(0,0)\}$ or I is a semiprimary ideal of R.*

(3) *$I = \{(0,0)\}$ or $I = I_1 \times R_2$ for some semiprimary ideal I_1 of R_1 or $I = R_1 \times I_2$ for some semiprimary ideal I_2 of R_2.*

5 Strongly Weakly δ-Semiprimary Ideals

Definition 5.1. Let δ be an expansion function of ideals of a ring R. A proper ideal I of R is called a *strongly weakly δ-semiprimary ideal* of R if whenever $\{0\} \neq AB \subseteq I$ for some ideals A, B of R, we have $A \subseteq \delta(I)$ or $B \subseteq \delta(I)$. Hence, a proper ideal I of R is called a *strongly weakly semiprimary ideal* of R if whenever $\{0\} \neq AB \subseteq I$ for some ideals A, B of R, we have $A \subseteq \sqrt{\{0\}}$ or $B \subseteq \sqrt{\{0\}}$.

Remark 5.2. Let δ be an expansion function of ideals of a ring R. It is clear that a strongly weakly δ-semiprimary ideal of R is a weakly δ-semiprimary ideal of R. In this section, we show that a proper ideal I of R is a strongly weakly δ-semiprimary ideal of R if and only if I is a weakly δ-semiprimary ideal of R.

Theorem 5.3. *Let δ be an expansion function of ideals of a ring R and I be a weakly δ-semiprimary ideal of R. Suppose that $AB \subseteq I$ for some ideals A, B of R, and that $ab = 0$ for some $a \in A$ and $b \in B$ such that neither $a \in \delta(I)$ nor $b \in \delta(I)$. Then $AB = \{0\}$.*

Proof. First we will show $aB = bA = \{0\}$. Suppose that $aB \neq \{0\}$. Then $0 \neq ac \in I$ for some $c \in B$. Since I is a weakly δ-semiprimary ideal of R and

$a \notin \delta(I)$, we conclude that $c \in \delta(I)$. Hence, $0 \neq a(b+c) = ac \in I$. Thus, $a \in \delta(I)$ or $(b+c) \in \delta(I)$. Since $c \in \delta(I)$, we see that $a \in \delta(I)$ or $b \in \delta(I)$, a contradiction. Thus, $aB = \{0\}$. Similarly, $bA = \{0\}$.

Now suppose that $AB \neq \{0\}$. Then there is an element $d \in A$ and there is an element $e \in B$ such that $0 \neq de \in I$. Since I is a weakly δ-semiprimary ideal of R, we conclude that $d \in \delta(I)$ or $e \in \delta(I)$. We consider three cases:

Case I. Suppose that $d \in \delta(I)$ and $e \notin \delta(I)$. Since $aB = \{0\}$, we can obtain $0 \neq e(d+a) = ed \in I$, and thus we conclude that $e \in \delta(I)$ or $(d+a) \in \delta(I)$. Since $d \in \delta(I)$, we have $e \in \delta(I)$ or $a \in \delta(I)$, a contradiction.

Case II. Suppose that $d \notin \delta(I)$ and $e \in \delta(I)$. Since $bA = \{0\}$, we have $0 \neq d(e+b) = de \in I$, and hence we conclude that $d \in \delta(I)$ or $(e+b) \in \delta(I)$. As $e \in \delta(I)$, we have $d \in \delta(I)$ or $b \in \delta(I)$, a contradiction.

Case III. Suppose that $d, e \in \delta(I)$. Since $aB = bA = \{0\}$, we can obtain $0 \neq (b+e)(d+a) = ed \in I$, and hence $b+e \in \delta(I)$ or $d+a \in \delta(I)$. As $d, e \in \delta(I)$, we have $b \in \delta(I)$ or $a \in \delta(I)$, a contradiction.

Thus, $AB = \{0\}$. □

Theorem 5.4. *Let δ be an expansion function of ideals of a ring R and I be a weakly δ-semiprimary ideal of R. Suppose that $\{0\} \neq AB \subseteq I$ for some ideals A, B of R. Then $A \subseteq \delta(I)$ or $B \subseteq \delta(I)$ (i.e., I is a strongly weakly δ-semiprimary ideal of R).*

Proof. Since $AB \neq \{0\}$, by Theorem 5.3 we conclude that whenever $ab \in I$ for some $a \in A$ and $b \in B$, we obtain $a \in \delta(I)$ or $b \in \delta(I)$. Assume that $\{0\} \neq AB \subseteq I$ and $A \not\subseteq \delta(I)$. Then there is an $x \in A \backslash \delta(I)$. Let $y \in B$. Since $xy \in AB \subseteq I$, $\{0\} \neq AB$ and $x \notin \delta(I)$, we obtain $y \in \delta(I)$ by Theorem 5.3. Hence, $B \subseteq \delta(I)$. □

In view of Theorem 5.4, we have the following result.

Corollary 5.5. *Let I be a weakly semiprimary ideal of R. We suppose that $\{0\} \neq AB \subseteq I$ for some ideals A, B of R. Then $A \subseteq \sqrt{I}$ or $B \subseteq \sqrt{I}$ (i.e., I is a strongly weakly semiprimary ideal of R).*

Acknowledgement. The authors are grateful to the referee for the great effort in proofreading the manuscript.

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