THE N-ZERO-DIVISOR GRAPH OF A COMMUTATIVE SEMIGROUP

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Abstract. Let S be a (multiplicative) commutative semigroup with 0, $Z(S)$ the set of zero-divisors of S, and n a positive integer. The *zero-divisor graph* of S is the (simple) graph $\Gamma(S)$ with vertices $Z(S)^* = Z(S) \setminus \{0\}$, and distinct vertices x and y are adjacent if and only if $xy = 0$. In this paper, we introduce and study the *n-zero-divisor graph* of S as the (simple) graph $\Gamma_n(S)$ with vertices $Z_n(S)^* = \{x^n \mid x \in Z(S)\} \setminus \{0\}$, and distinct vertices x and y are adjacent if and only if $xy = 0$. Thus each $\Gamma_n(S)$ is an induced subgraph of $\Gamma(S) = \Gamma_1(S)$. We pay particular attention to $diam(\Gamma_n(S))$, $gr(\Gamma_n(S))$, and the case when S is a commutative ring with $1 \neq 0$. We also consider several other types of "n-zero-divisor" graphs and commutative rings such that some power of every element (or zero-divisor) is idempotent.

1. Introduction

Let R be a commutative ring with $1 \neq 0$ and $Z(R)$ the set of zero-divisors of R. As in [9], the *zero-divisor graph* of R is the (simple) graph $\Gamma(R)$ with vertices $Z(R)^* = Z(R) \setminus \{0\}$, and distinct vertices x and y are adjacent if and only if $xy = 0$. In [19], DeMeyer, McKenzie, and Schneider extended this concept to commutative semigroups. Let S be a (multiplicative) commutative semigroup with 0 (i.e., $0x = 0$ for every $x \in S$) and $Z(S) = \{x \in S \mid xy = 0 \text{ for some } 0 \neq y \in S\}$ the set of zero-divisors of S. Then the *zero-divisor graph* of S is the (simple) graph $\Gamma(S)$ with vertices $Z(S)^* = Z(S) \setminus \{0\}$, and distinct vertices x and y are adjacent if and only if $xy = 0$. Moreover, $\Gamma(S)$ is connected with $diam(\Gamma(S)) \in \{0, 1, 2, 3\}$ and $gr(\Gamma(S)) \in \{3, 4, \infty\}$ ([19]). Note that $Z(S)$ is a subsemigroup of S with 0 (if $S \neq \{0\}$) and $\Gamma(S) = \Gamma(Z(S))$; and if R is a commutative ring, then $\Gamma(R) = \Gamma(S)$, where S is either R or $Z(R)$ considered as a multiplicative semigroup.

For a commutative semigroup S with 0 and positive integer n, let $Z_n(S) = \{x^n \mid x \in Z(S)\}$. Then $Z_n(S)$ is a commutative subsemigroup of $Z(S)$ with 0 (if $S \neq \{0\}$) and $Z_1(S) = Z(S)$. In this paper, we introduce the *n-zero-divisor graph* of S to be the (simple) graph $\Gamma_n(S)$ with vertices $Z_n(S)^* = Z_n(S) \setminus \{0\}$, and distinct vertices x and y are adjacent if and only if $xy = 0$. Thus $\Gamma_1(S) = \Gamma(S) = \Gamma(Z(S))$ is the connected classical zero-divisor graph of S, $\Gamma_n(S)$ is an induced subgraph of $\Gamma(S)$ for every positive integer n, and $\Gamma_n(R) = \Gamma_n(Z(R))$ for every positive integer n.

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However, $\Gamma_n(S)$ need not be connected for $n \geq 2$ (see Example 2.1, Theorem 2.2, Theorem 3.1, and Theorem 4.16).

In this paper, we study some graph-theoretic properties of $\Gamma_n(S)$. We pay particular attention to $diam(\Gamma_n(S))$, $gr(\Gamma_n(S))$, and the case when S is a commutative ring with $1 \neq 0$. In Section 2, we investigate the case when S is a reduced commutative semigroup with 0. In this case, $\Gamma_n(S) = \Gamma(Z_n(S))$, and thus $\Gamma_n(S)$ is connected, for every positive integer n (Theorem 2.4). We concentrate on the relationship between $diam(\Gamma_n(S))$ (resp., $gr(\Gamma_n(S))$) and $diam(\Gamma(S))$ (resp., $gr(\Gamma(S))$). In Section 3, we consider the case when S is not reduced. In this case, $\Gamma_n(S)$ need not be connected for $n \geq 2$, and several other results from Section 2 need not hold. However, $\Gamma_n(S)$ is connected for every positive integer n when $Z(S) = Nil(S)$ (Theorem 3.3). In Section 4, we study $\Gamma_n(R)$ when R is a π-regular (i.e., zero-dimensional) commutative ring, amd more specifically, when R is a von Neumann regular (i.e., reduced and zero-dimensional) commutative ring. In this case, $\Gamma_n(R)$ is connected for every positive integer n (Theorem 4.1). Moreover, in some cases the $\Gamma_n(R)$'s eventually repeat in blocks (Theorem 4.2, Theorem 4.9, Theorem 4.11, and Theorem 4.15). Along the way, we also investigate commutative rings such that some power of every element (or zero-divisor) is idempotent. In the final section, Section 5, we discuss the n-zero-divisor analog for several other types of zero-divisor graphs, namely, the extended zero-divisor graph $\overline{\Gamma}(S)$, the annihilator graph $AG(S)$, and the congruence-based zero-divisor graphs $\Gamma_\sim(R)$, $\overline{\Gamma}_\sim(R)$, and $AG_\sim(R)$. Many examples are given throughout to illustrate the results.

Let R be a commutative ring with $1 \neq 0$. Then $Z(R)$ is the set of zero-divisors of R, $Nil(R)$ the ideal of nilpotent elements of R, $U(R)$ the group of units of R, $Id(R)$ the set of idempotents of R, and $T(R) = R_{R \setminus Z(R)}$ the total quotient ring of R. In like manner, we have $Z(S)$, $Nil(S)$, $U(S)$, and $Id(S)$ for a commutative semigroup S with 0. The ring R (resp., semigroup S) is *reduced* if $Nil(R) = \{0\}$ (resp., $Nil(S) = \{0\}$), *zero-dimensional* if every prime ideal of R is maximal, and *local* if it has a unique maximal ideal. For $x \in Nil(S)$, let n_x (*index of nilpotency*) be the least positive integer m such that $x^m = 0$; for an ideal $I \subseteq Nil(R)$, let $n_I = sup\{n_x \mid x \in I\}$. An $r \in R \setminus Z(R)$ is called a *regular element*, and $Reg(R) = R \setminus Z(R)$. Note that $Nil(R) \cap Id(R) = \{0\}$, $Reg(R) \cap Id(R) = \{1\}$, and a local ring has only the trivial idempotents 0 and 1. If A is a set with $0 \in A$, then $A^* = A \setminus \{0\}$. Let $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{F}_{p^n}$ denote the ring of integers, integers modulo n, the fields of rational, real, and complex numbers, and the finite field with p^n elements, respectively. All rings are commutative with $1 \neq 0$, and subrings have the same identity element as the ring. All semigroups are commutative (usually with 0), and subsemigroups have the same 0 as the semigroup. For any undefined ring-theoretic concepts or notation, see [21] and [22].

For a graph G with vertices $V(G)$, we will often write $|G|$ rather than $|V(G)|$. As usual, K_m and $K_{m,n}$ denote the complete graph and complete bipartite graph on m and m, n vertices, respectively (here, m and n may be infinite cardinals). We will call $K_{1,n}$ a *star graph* and often just write $K_{1,n} = K_{1,\infty}$ and $K_{m,n} = K_{\infty,\infty}$ when m and n are infinite cardinals. The graph with no vertices is called the *empty graph* and is denoted by $\emptyset$, and the graph with n (≥ 2) vertices and no edges is called the *empty graph on n vertices* and is denoted by $\overline{K_n}$ (for graph complement). Note that $\Gamma(R) = \emptyset$ for a commutative ring R (resp., $\Gamma(S) = \emptyset$ for a commutative semigroup S with 0) if and only if R is an integral domain (resp.,

$Z(S) \subseteq \{0\}$, e.g., $Z(S) = \emptyset$ if $S = \{0\}$); so to avoid trivialities, we implicitly assume (when necessary) that R is not an integral domain (resp., $Z(S) \nsubseteq \{0\}$, e.g., $S \neq \{0\}$). For a positive integer n, let $d_n(x,y)$ be the distance between x and y in $\Gamma_n(S)$ ($d_n(x,x) = 0$ and $d_n(x,y) = \infty$ if there is no path from x to y), $diam(\Gamma_n(S)) = sup\{d_n(x,y) \mid x,y \in Z_n(S)^*\}$, and $gr(\Gamma_n(S))$ the length of a shortest cycle in $\Gamma_n(S)$, where $gr(\Gamma_n(S)) = \infty$ if $\Gamma_n(S)$ has no cycles. If $n = 1$, then we just use $d(x,y)$, $diam(\Gamma(S))$, and $gr(\Gamma(S))$. For any undefined graph-theoretic concepts or notation, see [17]. For additional information and references about the zero-divisor graph of a commutative semigroup with 0 or associating graphs to rings, see the survey article [5] or recent book [2]. We would like to thank the referee for some helpful comments.

2. The n-zero-divisor graph of a reduced commutative semigroup

In this section, we study $\Gamma_n(S)$ when S is a reduced commutative semigroup with 0. We are particularly interested in $diam(\Gamma_n(S))$ and $gr(\Gamma_n(S))$, and their relationship to $diam(\Gamma(S))$ and $gr(\Gamma(S))$, respectively.

For a commutative semigroup S with 0 and positive integer n, let $S_n = \{s^n \mid s \in S\}$. Then S_n is a commutative subsemigroup of S with 0. Thus $\Gamma(S_n)$ is connected, $diam(\Gamma(S_n)) \in \{0,1,2,3\}$, and $gr(\Gamma(S_n)) \in \{3,4,\infty\}$ by [19]. Note that for $x \in S$, $x^n \in Z_n(S) \Leftrightarrow x \in Z(S)$; $Z_n(S)$ is a subsemigroup of S_n with $Z(Z_n(S)) \subseteq Z(S_n) \subseteq Z_n(S)$ and $Z(S_n) = Z(Z_n(S)) \subseteq Z_n(S)$ if $Z_n(S) \neq \{0\}$; $Z_m(S)_n = Z_{mn}(S) = Z_n(S)_m$ for all positive integers m and n (in particular, $Z(S)_n = Z_n(S)$ and $Z_{mn}(S)$ is a subsemigroup of $Z_n(S)$ for all positive integers m and n); $\Gamma_{mn}(S)$ is an induced subgraph of $\Gamma_n(S)$ for all positive integers m and n; and $\Gamma(S_n) = \Gamma(Z_n(S))$ is an induced subgraph of $\Gamma_n(S)$. Hence $\Gamma(S_n) = \Gamma_n(S)$ (i.e., $\Gamma_n(S) = \Gamma(Z_n(S))$) if and only if $Z(S_n) = Z_n(S)$ or $Z_n(S) = \{0\}$. Note that if S is reduced, then $S_n, Z_n(S)$, and $Z(S_n)$ are also reduced for every positive integer n.

We next give several examples of $\Gamma_n(S)$. Parts (a) and (b) of Example 2.1 give commutative semigroups S with 0 such that $Z(S_n) = Z(Z_n(S)) \subsetneq Z_n(S)$, $\Gamma(S_n) = \Gamma(Z_n(S)) \subsetneq \Gamma_n(S)$, and thus $\Gamma_n(S)$ is not connected by Theorem 2.2, for every integer $n \geq 2$.

Example 2.1. (a) Let $R = \mathbb{Z}_2[X,Y]/(X^2, XY) = \mathbb{Z}_2[x,y] = \{a+bx+yf(y) \mid a,b \in \mathbb{Z}_2, f(T) \in \mathbb{Z}_2[T]\}$ and $S = Z(R) = \{bx + yf(y) \mid b \in \mathbb{Z}_2, f(T) \in \mathbb{Z}_2[T]\}$. Then $\Gamma(R) = \Gamma(S) = K_{1,\aleph_0}$ is a star graph with center x. Moreover, $S_n = \{y^n f(y)^n \mid f(T) \in \mathbb{Z}_2[T]\}$ for every integer $n \geq 2$; so $Z(S_n) = \{0\}$, while $Z_n(S) = \{y^n f(y)^n \mid f(T) \in \mathbb{Z}_2[T]\} = S_n$, for every integer $n \geq 2$. Thus the $\Gamma_n(S)$'s are all distinct and $\{0\} = Z(S_n) = Z(Z_n(S)) \subsetneq Z_n(S)$; so $\Gamma(S_n) \neq \Gamma_n(S)$ and $\Gamma_n(S)$ is not connected for every integer $n \geq 2$ by Theorem 2.2. Also, $Z(S_n) = Z(Z_n(S)) = \{0\}$ for every integer $n \geq 2$; so $\Gamma_n(R) = \Gamma_n(S) = \overline{K_{\aleph_0}}$ is not connected (in fact, totally disconnected) and $\Gamma(S_n) = \emptyset$ for every integer $n \geq 2$.

(b) Let $R = \mathbb{Z}_2[X,Y,V,W]/(X^2, XY, VW) = \mathbb{Z}_2[x,y,v,w]$ and $S = Z(R)$. Then $y^n \in Z_n(S)$, but $y^n \notin Z(S_n)$, for every integer $n \geq 2$. Thus $Z(S_n) = Z(Z_n(S)) \subsetneq Z_n(S)$; so $\Gamma(S_n) \neq \Gamma_n(S)$ and $\Gamma_n(S)$ is not connected for every integer $n \geq 2$ by Theorem 2.2. Note that $v^n, w^n \in Z_n(S)^*$ are distinct adjacent vertices in $\Gamma_n(S)$; so $\Gamma_n(S)$ is nonempty, not connected, but not totally disconnected, for every integer $n \geq 2$.

(c) Let R be a Boolean ring (i.e., $x^2 = x$ for every $x \in R$). For example, let $R = \mathbb{Z}_2^m$ for an integer $m \geq 2$. Then $Z_n(R)^* = R \setminus \{0,1\} = Id(R) \setminus \{0,1\}$ for every positive integer n, and thus $\Gamma_n(R) = \Gamma(R)$ for every positive integer n. We could also let S be any Boolean semigroup with 0. See [23] for some characterizations of $\Gamma(R)$ when R is a Boolean ring.

(d) Let $S = \{0, x, y, z\}$ be the commutative semigroup with 0 and multiplication given by $xz = yz = z^2 = 0$, $xy = y$, and $x^2 = y^2 = x$. Then $Z(S) = S$, $S_n = Z_n(S) = \{0, x\}$ for every even integer $n \geq 2$, and $S_n = Z_n(S) = \{0, x, y\}$ for every odd integer $n \geq 3$. Thus $\Gamma(S) = K_{1,2}$ is a star graph with center z, $\Gamma_n(S) = K_1$ is connected for every even integer $n \geq 2$, and $\Gamma_n(S) = \overline{K_2}$ is not connected for every odd integer $n \geq 3$. Moreover, $Z(S_n) = Z(Z_n(S)) = \{0\}$, and hence $\Gamma(S_n) = \emptyset$, for every integer $n \geq 2$.

(e) Let R be a commutative ring with $Z(R) = Nil(R)$ and m an integer with $m \geq n_x$ for every $x \in Nil(R)$ (e.g., $R = \mathbb{Z}_{p^m}$ for a prime p). Then $Z_n(R) = \{0\}$, and thus $\Gamma_n(R) = \emptyset$, for every integer $n \geq m$. In particular, this holds when R is an Artinian (e.g., finite) local commutative ring.

Let be S be a commutative semigroup with 0. We start with the following result which gives criteria for $\Gamma_n(S)$ to be connected when $|Z_n(S)^*| \geq 2$ (cf. Theorem 3.1 and Theorem 4.16). Note that for a commutative ring R, $Id(R) \setminus \{0,1\} \subseteq Z_n(R)^*$ for every positive integer n, and thus $|Z_n(R)^*| \geq 2$, and so $\Gamma_n(R) \neq \emptyset$, if R has nontrivial idempotents. In particular, $|Z_n(R)^*| \geq 2$ and $\Gamma_n(R) \neq \emptyset$ for every positive integer n when R is an Artinian (e.g., finite) nonlocal commutative ring.

If $|Z_n(S)^*| = 0$, then $Z_n(S) \subseteq \{0\}$ (so $Z(S_n) \subseteq \{0\}$), and hence $\Gamma(S_n) = \Gamma_n(S) = \emptyset$ is (vacuously) connected. If $|Z_n(S)^*| = 1$, say $Z_n(S) = \{0, x\}$, then $\Gamma_n(S) = K_1$ is connected, $diam(\Gamma_n(S)) = 0$, and $gr(\Gamma_n(S)) = \infty$. Note that $Z_n(S)^* = \{x\}$ with either $x^2 = 0$ or $x^2 = x$ since $Z_n(S) = \{0, x\}$ is a subsemigroup of S. If $x^2 = 0$, then $Z(Z_n(S)) = Z(S_n) = \{0, x\}$, and thus $\Gamma_n(S) = \Gamma(S_n) = \Gamma(Z_n(S))$. Moreover, if S is a commutative ring with $Z_n(S)^* = \{x\}$, then $x^2 = 0$ (if $x^2 = x$, then $1 - x \in Id(S)^* \subseteq Z_n(S)^*$ and $1 - x \neq x$, a contradiction). Example 2.1(d) shows that we may have $x^2 = x$, and hence $x \notin Z(Z_n(S))$, when S is not a commutative ring. In this case (i.e., when $x^2 = x$), $Z(Z_n(S)) = Z(S_n) = \{0\}$; so $\Gamma(S_n) = \emptyset$, and thus (1), but not (2) $-$ (4), of Theorem 2.2 hold.

Theorem 2.2. *Let S be a commutative semigroup with 0, n a positive integer, and $|Z_n(S)^*| \geq 2$. Then the following statements are equivalent.*

(1) *$\Gamma_n(S)$ is connected.*
(2) *For every $x \in Z_n(S)^*$, there is a $y \in Z_n(S)^*$ such that $xy = 0$, i.e., $Z(Z_n(S))^* = Z_n(S)^*$.*
(3) *$Z(S_n) = Z(Z_n(S)) = Z_n(S)$.*
(4) *$\Gamma(S_n) = \Gamma_n(S) = \Gamma(Z_n(S))$.*

Moreover, if $\Gamma_n(S)$ is connected, then $diam(\Gamma_n(S)) \in \{1, 2, 3\}$ and $gr(\Gamma_n(S)) \in \{3, 4, \infty\}$. If S is a commutative ring, then (1) $-$ (4) all hold when $|Z_n(S)^*| = 1$.

Proof. (1) ⇒ (2) Suppose that $\Gamma_n(S)$ is connected. Let $x, z \in Z_n(S)^*$ be distinct. Then there is a path $x - y - \cdots - z$ in $\Gamma_n(S)$. Thus $xy = 0$ and $x, y \in Z_n(S)^*$; so $x \in Z(Z_n(S))^*$. Hence $Z(Z_n(S))^* = Z_n(S)^*$.

(2) ⇒ (3) By definition of S_n and $Z_n(S)$, it is clear that $Z(S_n) = Z(Z_n(S)) = Z_n(S)$ when $Z(Z_n(S))^* = Z_n(S)^*$.

(3) ⇒ (4) This is clear.

(4) $\Rightarrow$ (1) This is clear since $\Gamma(S_n)$ is connected by [19].

For the "moreover" statement, suppose that $\Gamma_n(S)$ is connected. Then $\Gamma_n(S) = \Gamma(S_n) = \Gamma(Z(S_n))$ by (1) $\Rightarrow$ (4), where $Z(S_n) = Z_n(S)$ is a commutative semigroup with 0 and $|Z_n(S)^*| \geq 2$. Thus $diam(\Gamma_n(S)) \in \{1, 2, 3\}$ by [19, Theorem 1.2] and $gr(\Gamma_n(S)) \in \{3, 4, \infty\}$ by [19, Theorem 1.5]. The sentence about commutative rings follows from the comments before this theorem. □

We now investigate $diam(\Gamma_n(S))$ when S is a reduced commutative semigroup with 0. The following lemma will prove extremely useful.

Lemma 2.3. *Let S be a commutative semigroup with 0, and $x, y \in S$ such that $x \notin Nil(S)$ and $xy = 0$. Then $x^m \neq y^n$ for all positive integers m and n. In particular, if S is reduced, then x and y are distinct adjacent vertices in $\Gamma(S)$ if and only if x^n and y^n are distinct adjacent vertices in $\Gamma_n(S)$.*

Proof. Suppose that $x^m = y^n$ for positive integers m and n. Then $x^{m+1} = xx^m = xy^n = 0$ since $xy = 0$, a contradiction since $x \notin Nil(S)$.

The "in particular" statement is clear. □

Theorem 2.4. *Let S be a reduced commutative semigroup with 0 and n a positive integer. Then $\Gamma_n(S)$ is connected and $\Gamma_n(S) = \Gamma(S_n) = \Gamma(Z_n(S))$. Moreover, $d_n(x^n, y^n) = d(x^n, y^n) = d(x, y)$ for $x, y \in Z(S)^*$ with $x^n \neq y^n$. In particular, $diam(\Gamma_n(S)) \leq diam(\Gamma(S)) \leq 3$ for every positive integer n.*

Proof. We may assume that $|Z(S)^*| \geq 1$. Since S is reduced, $x^n \in Z_n(S)^*$ for every $x \in Z(S)^*$. Let $x^n \in Z_n(S)^*$ for $x \in Z(S)^*$. Then $xy = 0$ for some $y \in Z(S)^* \setminus \{x\}$; so $y^n \in Z_n(S)^* \setminus \{x^n\}$ by Lemma 2.3 and $x^n y^n = 0$. Thus $|Z_n(S)^*| \geq 2$, and so $\Gamma_n(S) = \Gamma(S_n) = \Gamma(Z_n(S))$ is connected by Theorem 2.2.

Let x^n, y^n be distinct vertices in $Z_n(S)^*$ for $x, y \in Z(S)^*$. Then $d(x, y) \in \{1, 2, 3\}$ by Theorem 2.2. First, suppose that $d(x, y) = 1$. Then $d_n(x^n, y^n) = 1 \Leftrightarrow d(x, y) = 1$ by Lemma 2.3. So in this case, $d_n(x^n, y^n) = d(x^n, y^n) = d(x, y) = 1$. (For this case, we do not need to assume that $x^n \neq y^n$.) Next, suppose that $d(x, y) = 2$. By Lemma 2.3, $x - z - y$ is a path of length 2 in $\Gamma(S) \Leftrightarrow x^n - z^n - y^n$ is a path of length 2 in $\Gamma_n(S)$. Hence $d_n(x^n, y^n) = 2 \Leftrightarrow d(x, y) = 2$. So in this case, $d_n(x^n, y^n) = d(x^n, y^n) = d(x, y) = 2$. Finally, let $d(x, y) = 3$. By the two previous cases, we have $d_n(x^n, y^n) = 3 \Leftrightarrow d(x, y) = 3$. If $x^n - z - y^n$ is a path of length 2 in $\Gamma(S)$, then $x^n - z^n - y^n$ is a path of length 2 is $\Gamma_n(S)$ by Lemma 2.3 again, a contradiction. Thus $d_n(x^n, y^n) = d(x^n, y^n) = d(x, y) = 3$ in this case.

The "in particular" statement is now clear. □

Remark 2.5. Let S be a reduced commutative semigroup with 0 and $|Z(S)^*| \geq 1$ (i.e., $S \neq \{0\}$ and $Z(S) \neq \{0\}$). Then $|Z(S)^*| \geq 2$, and thus $|Z_n(S)^*| \geq 2$ for every positive integer n by Lemma 2.3. Moreover, if $|Z(S)^*| = 2$, then $\Gamma_n(S) = K_2 = K_{1,1}$ for every positive integer n; and if $|Z(S)^*| = 3$, then either $\Gamma(S) = K_{1,2}$ or $\Gamma(S) = K_3$. If $\Gamma(S) = K_3$, then it is easily shown that $\Gamma_n(S) = K_3$ for every positive integer n. However, for $S = Z(\mathbb{Z}_6) = \{0, 2, 3, 4\}$, we have $\Gamma_n(S) = K_{1,2}$ for every odd positive integer n and $\Gamma_n(S) = K_2 = K_{1,1}$ for every even positive integer n. Thus, for $|Z(S)^*| \geq 3$, we may have $|Z_m(S)^*| \neq |Z_n(S)^*|$ for postive integers m and n, also see Example 2.9 and Example 4.13.

For a reduced commutative semigroup S with 0 and $x, y \in Z(S)^*$, we have $d(x, y) = 1 \Leftrightarrow d_n(x^n, y^n) = 1$ by Lemma 2.3, and we next show that $d(x, y) = 3 \Leftrightarrow$

$d_n(x^n, y^n) = 3$. However, Example 2.9 shows that we may have $d(x, y) = 2$ and $d_n(x^n, y^n) = 0$, i.e., $x^n = y^n$.

Theorem 2.6. *Let S be a reduced commutative semigroup with 0, n a positive integer, and $x, y \in Z(S)^*$ with $d(x, y) = 3$. Then $x^n, y^n \in Z_n(S)^*$ are distinct and $d_n(x^n, y^n) = d(x, y) = 3$. Moreover, $diam(\Gamma_n(S)) = diam(\Gamma(S)) = 3$ for every positive integer n.*

Proof. Since $d(x, y) = 3$, there is a path $x - z - w - y$ of length 3 in $\Gamma(S)$ from x to y. Since S is reduced, $x^n, z^n, w^n, y^n \in Z_n(S)^*$ for every positive integer n. Suppose that $x^n = y^n$ for some positive integer n. Then z^n and y^n are distinct adjacent vertices in $\Gamma_n(S)$ by Lemma 2.3, and thus z and y are also distinct and adjacent in $\Gamma(S)$ by Lemma 2.3 again, a contradiction since $d(x, y) = 3$. Thus $x^n \neq y^n$, and hence $d_n(x^n, y^n) = d(x, y) = 3$ by Theorem 2.4.

The "moreover" statement is clear. □

We now study the relationship between $diam(\Gamma(S))$ and $diam(\Gamma_n(S))$ when S is reduced. Example 2.1(c) and Example 2.9 show that both cases are possible in parts (2) and (3) of the following theorem.

Theorem 2.7. *Let S be a reduced commutative semigroup with 0.*

(a) *If $diam(\Gamma_m(S)) = 3$ for some integer $m \geq 2$, then $diam(\Gamma_n(S)) = diam(\Gamma(S)) = 3$ for every positive integer n.*

(b) *If $diam(\Gamma_m(S)) = 1$ for some integer $m \geq 2$, then $diam(\Gamma(S)) \in \{1, 2\}$. Moreover, $diam(\Gamma_n(S)) \in \{1, 2\}$ for every positive integer n.*

(c) *If $diam(\Gamma_m(S)) = 2$ for some integer $m \geq 2$, then $diam(\Gamma(S)) = 2$. Moreover, $diam(\Gamma_n(S)) \in \{1, 2\}$ for every positive integer n.*

(d) *$diam(\Gamma_m(S)) = 0$ for some integer $m \geq 2$ if and only if $Z(S) \subseteq \{0\}$ (i.e., $\Gamma(S) = \emptyset$), if and only if $diam(\Gamma_n(S)) = 0$ for every positive integer n.*

Proof. (a) Suppose that $diam(\Gamma_m(S)) = 3$ for some integer $m \geq 2$. Then $3 = diam(\Gamma_m(S)) \leq diam(\Gamma(S)) \leq 3$ by Theorem 2.4; so $diam(\Gamma_n(S)) = diam(\Gamma(S)) = 3$ for every positive integer n by Theorem 2.6.

(b) Suppose that $diam(\Gamma_m(S)) = 1$ for some integer $m \geq 2$. Then $diam(\Gamma(S)) \neq 3$ by Theorem 2.6, and $diam(\Gamma_n(S)) \neq 3$ for every integer $n \geq 2$ by (a). Thus $diam(\Gamma_n(S)) \in \{1, 2\}$ for every positive integer n. In particular, $diam(\Gamma(S)) \in \{1, 2\}$.

(c) Suppose that $diam(\Gamma_m(S)) = 2$ for some integer $m \geq 2$. Since $2 \leq diam(\Gamma_m(S)) \leq diam(\Gamma(S)) \leq 3$ by Theorem 2.4 and Theorem 2.2, we have $diam(\Gamma(S)) \in \{2, 3\}$. Since $diam(\Gamma_m(S)) = 2$ for some positive integer m, we have $diam(\Gamma(S)) \neq 3$ by Theorem 2.6; so $diam(\Gamma(S)) = 2$. Since $1 \leq diam(\Gamma_n(S)) \leq diam(\Gamma(S)) = 2$ for every positive integer n by Theorem 2.4, we have $diam(\Gamma_n(S)) \in \{1, 2\}$ for every positive integer n.

(d) This is clear by Remark 2.5. □

We next consider $gr(\Gamma_n(S))$ for a reduced commutative semigroup S with 0. We show that if $gr(\Gamma(S)) \in \{3, \infty\}$, then $gr(\Gamma_n(S)) = gr(\Gamma(S))$ for every positive integer n. We first do the $gr(\Gamma(S)) = 3$ case, and then the $gr(\Gamma(S)) = \infty$ case in Theorem 2.10.

Theorem 2.8. *Let S be a reduced commutative semigroup with 0. Then the following statements are equivalent.*

(1) $gr(\Gamma(S)) = 3$.
(2) $gr(\Gamma_n(S)) = 3$ *for every positive integer* n.
(3) $gr(\Gamma_n(S)) = 3$ *for some positive integer* n.

Proof. (1) $\Rightarrow$ (2) Suppose that $gr(\Gamma(S)) = 3$. Let $x - y - z - x$ be a cycle of length 3 in $\Gamma(S)$. Then $x^n - y^n - z^n - x^n$ is a cycle of length 3 in $\Gamma_n(S)$ for every positive integer n by Lemma 2.3; so $gr(\Gamma_n(S)) = 3$ for every positive integer n.

(2) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (1) Suppose that $gr(\Gamma_n(S)) = 3$ for some positive integer n. Let $x^n - y^n - z^n - x^n$ be a cycle of length 3 in $\Gamma_n(S)$. Then $x - y - z - x$ is a cycle of length 3 in $\Gamma(S)$ by Lemma 2.3; so $gr(\Gamma(S)) = 3$. □

The following is an example of a reduced commutative semigroup (ring) S with 0 where $diam(\Gamma_2(S)) < diam(\Gamma(S))$ and $gr(\Gamma_2(S)) \neq gr(\Gamma(S))$. Thus the hypotheses "$d(x, y) = 3$" and "$gr(\Gamma(S)) = 3$" are crucial in Theorem 2.6 and Theorem 2.8, respectively.

Example 2.9. Let $R = \mathbb{Z}_3 \times \mathbb{Z}_3$; so $S = Z(R) = \{(0, 0), (1, 0), (2, 0), (0, 1), (0, 2)\}$ is a reduced commutative semigroup with 0, and $\Gamma_n(R) = \Gamma_n(S)$ for every positive integer n. Then $\Gamma(R) = K_{2,2}$; so $diam(\Gamma(R)) = 2$ and $gr(\Gamma(R)) = 4$. Note that $Z_n(R)^* = \{(1, 0), (0, 1)\}$ for every even positive integer n; so $\Gamma_n(R) = K_2 = K_{1,1}$, and hence $diam(\Gamma_n(R)) = 1$ and $gr(\Gamma_n(R)) = \infty$, for every even positive integer n. However, $Z_n(R)^* = Z(R)^*$ for every odd positive integer n, and thus $\Gamma_n(R) = \Gamma(R) = K_{2,2}$ for every odd positive integer n. For $x = (1, 0), y = (2, 0)$, we have $d(x, y) = 2$, but $x^n = y^n = (1, 0)$; so $d_n(x^n, y^n) = 0$ for n an even positive integer.

Next, we consider the case when $gr(\Gamma(S)) \in \{4, \infty\}$. Example 2.9 shows that both cases may occur in Theorem 2.10 (1) and (4) below.

Theorem 2.10. *Let* S *be a reduced commutative semigroup with* 0.
(a) *If* $gr(\Gamma(S)) = 4$, *then* $gr(\Gamma_n(S)) \in \{4, \infty\}$ *for every positive integer* n.
(b) *If* $gr(\Gamma(S)) = \infty$, *then* $gr(\Gamma_n(S)) = \infty$ *for every positive integer* n.
(c) *If* $gr(\Gamma_m(S)) = 4$ *for some integer* $m \geq 2$, *then* $gr(\Gamma(S)) = 4$.
(d) *If* $gr(\Gamma_m(S)) = \infty$ *for some integer* $m \geq 2$, *then* $gr(\Gamma_n(S)) \in \{4, \infty\}$ *for every positive integer* n.

Proof. (a) Assume that $gr(\Gamma(S)) = 4$. Since $gr(\Gamma_n(S)) \in \{3, 4, \infty\}$ for every positive integer n by Theorem 2.2 and $gr(\Gamma_n(S)) \neq 3$ for every positive integer n by Theorem 2.8, we have $gr(\Gamma_n(S)) \in \{4, \infty\}$ for every positive integer n.

(b) Assume that $gr(\Gamma(S)) = \infty$. Since $\Gamma_n(S)$ is a subgraph of $\Gamma(S)$ for every positive integer n, we have $gr(\Gamma_n(S)) = \infty$ for every positive integer n.

(c) Assume that $gr(\Gamma_m(S)) = 4$ for some integer $m \geq 2$. Then $gr(\Gamma(S)) \neq 3$ by Theorem 2.8; so $gr(\Gamma((S)) \in \{4, \infty\}$. If $gr(\Gamma(S)) = \infty$, then $gr(\Gamma_n(S)) = \infty$ for every positive integer n by (2), a contradiction. Thus $gr(\Gamma(S)) = 4$.

(d) Assume that $gr(\Gamma_m(S)) = \infty$ for some integer $m \geq 2$. Then $gr(\Gamma(S)) \neq 3$ by Theorem 2.8; so $gr(\Gamma(S)) \in \{4, \infty\}$. If $gr(\Gamma(S)) = \infty$, then $gr(\Gamma_n(S)) = \infty$ for every positive integer n by (2). If $gr(\Gamma(S)) = 4$, then $gr(\Gamma_n(S)) \in \{4, \infty\}$ for every positive integer n by (a). □

We have $\Gamma(T(R)) \cong \Gamma(R)$ for every commutative ring R by [10, Theorem 2.2]; so $diam(\Gamma(T(R))) = diam(\Gamma(R))$ and $gr(\Gamma(T(R))) = gr(\Gamma(R))$. We next show that these two equalities also hold for every $\Gamma_n(R)$.

Theorem 2.11. *Let R be a commutative ring and n a positive integer. Then $\Gamma_n(T(R))$ is connected if and only if $\Gamma_n(R)$ is connected. Moreover, $diam(\Gamma_n(T(R))) = diam(\Gamma_n(R))$ and $gr(\Gamma_n(T(R))) = gr(\Gamma_n(R))$.*

Proof. Let $S = R \setminus Z(R)$. Then $T(R) = R_S$ and $Z(T(R)) = Z(R)_S$. Note that $Z_n(T(R)) = \{0\} \Leftrightarrow Z_n(R) = \{0\}$; so we may assume that $|Z_n(T(R))^*|, |Z_n(R)^*| \geq 1$. Suppose that $\Gamma_n(T(R))$ is connected. Let $y \in Z_n(R)^* \subseteq Z_n(T(R))^*$. Then $yz = 0$ for some $z \in Z_n(T(R))^*$, where $z = b^n/t^n$ with $b \in Z(R)^*$ and $t \in S$, by Theorem 2.2. Thus $b^n \in Z_n(R)^*$ and $yb^n = 0$; so $\Gamma_n(R)$ is connected by Theorem 2.2.

Conversely, suppose that $\Gamma_n(R)$ is connected. Let $x \in Z_n(T(R))^*$. Then $x = a^n/s^n$ for some $a \in Z(R)^*$ and $s \in S$; so $a^n \in Z_n(R)^*$. Since $\Gamma_n(R)$ is connected and $a^n \in Z_n(R)^*$, there is a $b \in Z_n(R)^* \subseteq Z_n(T(R))^*$ with $ba^n = 0$ by Theorem 2.2. Hence $bx = 0$; so $\Gamma_n(T(R))$ is connected by Theorem 2.2.

For the "moreover" statement, let $x_1, \ldots, x_k$ be distinct vertices in $Z_n(R)^*$ for some integer $k \geq 2$ ($k \geq 3$ for the "cycle" case). Then $x_1 - \cdots - x_k$ (resp., $x_1 - \cdots - x_k - x_1$) is a path (resp., cycle) of length k in $\Gamma_n(R)$ if and only if $x_1/s^n - \cdots - x_k/s^n$ (resp., $x_1/s^n - \cdots - x_k/s^n - x_1/s^n$) is a path (resp., cycle) of length k in $\Gamma_n(T(R))$ for every $s \in S$, and every path (resp., cycle) of length k in $\Gamma_n(T(R))$ is of the form $y_1/t^n - \cdots - y_k/t^n$ (resp., $y_1/t^n - \cdots - y_k/t^n - y_1/t^n$) for distinct $y_1, \ldots, y_k \in Z_n(R)^*$ and $t \in S$. Thus $diam(\Gamma_n(T(R))) = diam(\Gamma_n(R))$ and $gr(\Gamma_n(T(R))) = gr(\Gamma_n(R))$. ☐

We recall the following two results which characterize the reduced commutative rings R with $gr(\Gamma(R)) \in \{4, \infty\}$ in terms of $T(R)$.

Theorem 2.12. ([12, Theorem 2.2], [26, Theorem 2.3]) *Let R be a reduced commutative ring. Then the following statements are equivalent.*

1. $gr(\Gamma(R)) = 4$.
2. $T(R) = K_1 \times K_2$, *where each K_i is a field with* $|K_i| \geq 3$.
3. $gr(\Gamma(R)) \neq \infty$ *and R is a subring of the product of two integral domains.*
4. $\Gamma(R) = K_{m,n}$ *with* $m, n \geq 2$.

Theorem 2.13. ([12, Theorem 2.4]) *Let R be a reduced commutative ring. Then the following statements are equivalent.*

1. $\Gamma(R)$ *is nonempty with* $gr(\Gamma(R)) = \infty$.
2. $T(R) = \mathbb{Z}_2 \times K$, *where K is a field.*
3. $\Gamma(R) = K_{1,n}$ *for some* $n \geq 1$.

We now specialize to the case $gr(\Gamma(R)) \in \{4, \infty\}$ when R is a reduced commutative ring. We will need the following lemma.

Lemma 2.14. *Let R be a commutative ring and ex_1, ex_2 distinct elements of R, where $e \in Id(R)^*$ and $x_1 \in R \setminus Z(R)$. If $ex_1^n = ex_2^n$ for some integer $n \geq 2$, then $ex_1^{kn+1} \neq ex_2^{kn+1}$ for every positive integer k.*

Proof. Suppose that $ex_1^n = ex_2^n$ for some integer $n \geq 2$, where $e \in Id(R)^*$ and $x_1 \in R \setminus Z(R)$. Then $ex_1^{kn} = ex_2^{kn}$ for every positive integer k. Now, suppose that $ex_1^{kn+1} = ex_2^{kn+1}$ for some positive integer k. Then $(ex_1)x_1^{kn} = ex_1^{kn+1} = ex_2^{kn+1} = (ex_2)(ex_2^{kn}) = (ex_2)(ex_1^{kn}) = (ex_2)x_1^{kn}$, and thus $ex_1 = ex_2$ since $x_1^{kn} \in R \setminus Z(R)$, a contradiction. Hence $ex_1^{kn+1} \neq ex_2^{kn+1}$ for every positive integer k. ☐

We can improve Theorem 2.10 for reduced commutative rings.

Theorem 2.15. *Let R be a reduced commutative ring. Then $gr(\Gamma(R)) = 4$ if and only if $gr(\Gamma_n(R)) = 4$ for some integer $n \geq 2$. Moreover, if $gr(\Gamma(R)) = 4$ and $gr(\Gamma_n(R)) = \infty$ for some integer $n \geq 2$, then either $gr(\Gamma_{n+1}(R)) = 4$ or $gr(\Gamma_{n(n+1)+1}(R)) = 4$.*

Proof. Suppose that $gr(\Gamma_n(R)) = 4$ for some integer $n \geq 2$. Then $gr(\Gamma(R)) = 4$ by Theorem 2.10(3).

Conversely, assume that $gr(\Gamma(R)) = 4$. Then R is a subring of $D_1 \times D_2$, where each D_i is an integral domain, by Theorem 2.12. Thus $\Gamma(R)$ has a cycle of length 4; say $(x_1, 0) - (0, x_2) - (x_3, 0) - (0, x_4) - (x_1, 0)$ is a cycle of length 4 in $\Gamma(R)$, where $x_1, x_3 \in D_1^*$ and $x_2, x_4 \in D_2^*$. Assume that $gr(\Gamma_n(R)) \neq 4$ for some integer $n \geq 2$. Then $x_1^n = x_3^n$ or $x_2^n = x_4^n$. Without loss of generality, assume that $x_1^n = x_3^n$. If $x_2^{n+1} \neq x_4^{n+1}$, then $(x_1^{n+1}, 0) - (0, x_2^{n+1}) - (x_3^{n+1}, 0) - (0, x_4^{n+1}) - (x_1^{n+1}, 0)$ is a cycle of length 4 in $\Gamma_{n+1}(R)$ by Lemma 2.14. If $x_2^{n+1} = x_4^{n+1}$, let $m = n(n+1)$. Then $(x_1^{m+1}, 0) - (0, x_2^{m+1}) - (x_3^{m+1}, 0) - (0, x_4^{m+1}) - (x_1^{m+1}, 0)$ is a cycle of length 4 in $\Gamma_{m+1}(R)$ by Lemma 2.14.

The "moreover" statement is now clear. □

For a reduced commutative ring R that is not an integral domain, it is well known that $\Gamma(R)$ is a complete bipartite graph if and only if $gr(\Gamma(R)) \in \{4, \infty\}$ (Theorem 2.12 and Theorem 2.13). We next show that this also holds for every $\Gamma_n(R)$.

Theorem 2.16. *Let R be a reduced commutative ring that is not an integral domain and n a positive integer. Then $gr(\Gamma_n(R)) \in \{4, \infty\}$ if and only if $\Gamma_n(R)$ is a complete bipartite graph.*

Proof. If $\Gamma_n(R)$ is a complete bipartite graph for some integer $n \geq 2$, then $gr(\Gamma_n(R)) \in \{4, \infty\}$. Conversely, assume that $gr(\Gamma_n(R)) \in \{4, \infty\}$. Thus $gr(\Gamma(R)) \neq 3$ by Theorem 2.8; so $gr(\Gamma(R)) \in \{4, \infty\}$. Hence R is a subring of $D_1 \times D_2$, where each D_i is an integral domain, by Theorem 2.12 and Theorem 2.13. Let $A = \{(x^n, 0) \mid (x, 0) \in R^*\}$ and $B = \{(0, y^n) \mid (0, y) \in R^*\}$. Then $Z_n(R)^* = A \cup B$ with $A, B \neq \emptyset$; so $\Gamma_n(R) = K_{|A|,|B|}$ is a complete bipartite graph. □

In view of Theorem 2.12, Theorem 2.15, and Theorem 2.16, we have the following result. The proof is left to the reader.

Corollary 2.17. *Let R be a reduced commutative ring. Then the following statements are equivalent.*

1. *There is an integer $k \geq 2$ such that $\Gamma_k(R) = K_{m,n}$ with $m, n \geq 2$.*
2. *$gr(\Gamma_k(R)) = 4$ for some integer $k \geq 2$.*
3. *$gr(\Gamma(R)) = 4$.*
4. *$T(R) = K_1 \times K_2$, where each K_i is a field with $|K_i| \geq 3$.*
5. *$gr(\Gamma(R)) \neq \infty$ and R is a subring of the product of two integral domains.*
6. *$\Gamma(R) = K_{m,n}$ with $m, n \geq 2$.*

Next, we consider the case when both $gr(\Gamma_m(R)) = \infty$ and $gr(\Gamma_n(R)) = 4$.

Theorem 2.18. *Let R be a reduced commutative ring. Then the following statements are equivalent.*

1. *There are integers $m, n \geq 2$ such that $gr(\Gamma_m(R)) = \infty$ and $gr(\Gamma_n(R)) = 4$.*

(2) $T(R) = K_1 \times K_2$, *where each* K_i *is a field with* $|K_i| \geq 3$ *and either* K_1 *or* K_2 *is finite.*

Proof. (1) $\Rightarrow$ (2) Assume there are integers $m, n \geq 2$ such that $gr(\Gamma_m(R)) = \infty$ and $gr(\Gamma_n(R)) = 4$. Then $gr(\Gamma(R)) = 4$ and $T(R) = K_1 \times K_2$, where each K_i is a field with $|K_i| \geq 3$, by Corollary 2.17. We may assume that K_2 is infinite. We show that K_1 is finite. Assume, by way of contradiction, that K_1 is infinite. Let $x \in K_1^*$ and $w \in K_2^*$. For every integer $n \geq 2$, let $A_n(x) = \{y \in K_1 \mid y^n = x^n$, i.e., $(yx^{-1})^n = 1\}$ and $B_n(w) = \{a \in K_2 \mid a^n = w^n$, i.e., $(aw^{-1})^n = 1\}$. Since the equation $h^n - 1 = 0$ has at most n solutions in K_1, K_2, we have $1 \leq |A_n(x)|, |B_n(w)| \leq n$. Since K_1 and K_2 are infinite fields, there are $c \in K_1^* \setminus A_n(x)$ and $d \in K_2^* \setminus B_n(w)$. Thus $(x^n, 0) - (0, w^n) - (c^n, 0) - (0, d^n) - (x^n, 0)$ is a cycle of length 4 in $\Gamma_n(T(R))$; so $gr(\Gamma_n(R))) = gr(\Gamma_n(T(R))) = 4$ for every positive integer n by Theorem 2.11, a contradiction. Hence either K_1 or K_2 is finite.

(2) $\Rightarrow$ (1) Assume that $T(R) = K_1 \times K_2$, where each K_i is a field with $|K_i| \geq 3$ and either K_1 or K_2 is finite. Then $gr(\Gamma_m(R)) = 4$ for some integer $m \geq 2$ by Corollary 2.17. We may assume that $|K_1| = n+1 < \infty$, where $n \geq 2$ by hypothesis. Thus $Z_n(T(R))^* = \{(1,0)\} \cup \{(0, y^n) \mid y \in K_2^*\}$; so $\Gamma_n(T(R))$ is a star graph with center $(1,0)$. Hence $gr(\Gamma_n(R)) = gr(\Gamma_n(T(R))) = \infty$ by Theorem 2.11. □

In light of the proof of Theorem 2.18, we have the following result. Its proof is left to the reader.

Corollary 2.19. *Let* R *be a reduced commutative ring. Then the following statements are equivalent.*

(1) $gr(\Gamma_n(R)) = 4$ *for every positive integer* n.
(2) $T(R) = K_1 \times K_2$, *where each* K_i *is an infinite field.*

In view of Theorem 2.18 and Corollary 2.17, we have the following result. Its proof is left to the reader.

Corollary 2.20. *Let* R *be a reduced commutative ring. Then the following statements are equivalent.*

(1) *There are integers* $m, n \geq 2$ *such that* $gr(\Gamma_m(R)) = \infty$ *and* $gr(\Gamma_n(R)) = 4$.
(2) *There are integers* $m, n \geq 2$ *such that* $\Gamma_m(R) = K_{1,a}$ *with* $a \geq 1$ *and* $\Gamma_n(R) = K_{b,c}$ *with* $b, c \geq 2$ *and* $b < \infty$ *or* $c < \infty$.
(3) $gr(\Gamma(R)) = 4$ *and* $gr(\Gamma_n(R)) = \infty$ *for some integer* $n \geq 2$.
(4) $T(R) = K_1 \times K_2$, *where each* K_i *is a field with* $|K_i| \geq 3$ *and either* K_1 *or* K_2 *is finite.*
(5) $gr(\Gamma(R)) \neq \infty$ *and* R *is a subring of the product of two integral domains* D_1 *and* D_2 *such that* D_1 *or* D_2 *is a finite field.*
(6) $\Gamma(R) = K_{b,c}$ *with* $b, c \geq 2$ *and* $b < \infty$ *or* $c < \infty$.

In light of Theorem 2.18 and Corollary 2.19, we have the following result. Its proof is left to the reader.

Corollary 2.21. *Let* R *be a reduced commutative ring. Then the following statements are equivalent.*

(1) $gr(\Gamma_n(R)) = 4$ *for every positive integer* n. *In particular,* $gr(\Gamma(R)) = 4$.
(2) $\Gamma_n(R) = K_{\infty,\infty}$ *for every positive integer* n.
(3) $T(R) = K_1 \times K_2$, *where each* K_i *is an infinite field.*

(4) *$gr(\Gamma(R)) \neq \infty$ and R is a subring of the product of two infinite integral domains.*
(5) *$\Gamma(R) = K_{\infty,\infty}$.*

In view of Theorem 2.13 and Theorem 2.10(2), we have the following result. Its proof is left to the reader.

Corollary 2.22. *Let R be a reduced commutative ring that is not an integral domain. Then the following statements are equivalent.*

(1) *$gr(\Gamma_n(R)) = \infty$ for every positive integer n. In particular, $gr(\Gamma(R)) = \infty$.*
(2) *$\Gamma_n(R) = K_{1,\infty}$ for every positive integer n.*
(3) *$T(R) = \mathbb{Z}_2 \times K$, where K is an infinite field.*
(4) *R is a subring of $\mathbb{Z}_2 \times D$ for an infinite integral domain D.*
(5) *$\Gamma(R) = K_{1,\infty}$.*

3. The n-zero-divisor graph of a nonreduced commutative semigroup

In this section, we study $\Gamma_n(S)$ when the commutative semigroup S is not reduced. In this case, $\Gamma_n(S)$ need not be connected for $n \geq 2$, i.e., $\Gamma_n(S)$ is a proper subgraph of $\Gamma(Z_n(S))$ (see Example 2.1). First, we give another criterion for $\Gamma_n(S)$ to be connected (cf. Theorem 2.2).

For a commutative semigroup S with 0, $x \in Z(S)^*$, and n a positive integer, let $Nil_n(S) = \{y \in S \mid y^n = 0\} \subseteq Nil(S)$ and $nil_n(x) = \{y \in S \mid (xy)^n = 0\}$.

Theorem 3.1. *Let S be a (nonreduced) commutative semigroup S with 0 and $n \geq 2$ an integer such that $|Z_n(S)^*| \geq 2$. Then $\Gamma_n(S)$ is not connected if and only if there is an $x \in Z(S)^*$ such that $x^n \in Z_n(S)^*$ and $nil_n(x) \subseteq Nil_n(S)$.*

Proof. Assume that $\Gamma_n(S)$ is not connected. Then there is an $x \in Z(S)^*$ such that $x^n \in Z_n(S)^*$ and $x^n z \neq 0$ for every $z \in Z_n(S)^*$ by Theorem 2.2. Let $y \in nil_n(x)$. Then $x^n y^n = (xy)^n = 0$; so $y^n \notin Z_n(S)^*$. Thus $y^n = 0$; so $y \in Nil_n(S)$. Hence $nil_n(x) \subseteq Nil_n(S)$.

Conversely, assume there is an $x \in Z(S)^*$ such that $x^n \in Z_n(S)^*$ and $nil_n(x) \subseteq Nil_n(S)$. We show that $yx^n \neq 0$ for every $y \in Z_n(S)^*$. Assume that $yx^n = 0$ for some $y \in Z_n(S)^*$. Then $y = z^n$ for some $z \in Z(S)^*$ and $(zx)^n = z^n x^n = yx^n = 0$; so $z \in nil_n(x) \subseteq Nil_n(S)$. Thus $y = z^n = 0$, and hence $y \notin Z_n(S)^*$, a contradiction. Thus $yx^n \neq 0$ for every $y \in Z_n(S)^*$, and hence $\Gamma_n(S)$ is not connected by Theorem 2.2. □

Although $\Gamma_n(S)$ need not be connected when the commutative semigroup S is not reduced, we next show that $\Gamma_n(S)$ is connected in the "extreme" nonreduced case, i.e., when $Z(S) = Nil(S)$. Note that $diam(\Gamma(S)) \in \{0, 1, 2\}$ when $Z(S) = Nil(S)$ ([18, Theorem 5]), and $gr(\Gamma(R)) \in \{3, \infty\}$ when $Z(R) = Nil(R)$ for a commutative ring R ([3, Theorem 2.11]). First, a lemma.

Lemma 3.2. *Let S be a commutative semigroup with 0, $x \in Nil(S)$, and n a positive integer. If $x^n \neq 0$, then $x^n \in Z(Z_n(S))$.*

Proof. Let $y = x^n \neq 0$ and $m = n_y - 1$ ($m \geq 1$ since $y \neq 0$). Then $y \in Z_n(S)$ since $x \in Nil(S) \subseteq Z(S)$, and $0 \neq y^m \in Z_n(S)$ since $Z_n(S)$ is a subsemigroup of S. Thus $yy^m = y^{n_y} = 0$; so $x^n = y \in Z(Z_n(S))$. □

Theorem 3.3. *Let S be a commutative semigroup with 0, $Z(S) = Nil(S) \neq \{0\}$, and m a positive integer. Then $Z(Z_m(S)) = Z_m(S)$ if $Z_m(S) \neq \{0\}$, and thus $\Gamma_n(S) = \Gamma(S_n) = \Gamma(Z_n(S))$ is connected for every positive integer n. Moreover, let $N = sup\{\, n_x \mid x \in Nil(S)\}$. If $N < \infty$, then $\Gamma_n(S) = \emptyset$ for every integer $n \geq N$. Otherwise, $\Gamma_n(S) \neq \emptyset$ for every positive integer n.*

Proof. Let $0 \neq x \in Z(S) = Nil(S)$. If $m \geq n_x$, then $x^m = 0$. If $m < n_x$, then $0 \neq x^m \in Z(Z_m(S))$ by Lemma 3.2. Thus $Z(Z_m(S)) = Z_m(S)$ if $Z_m(S) \neq \{0\}$; so $\Gamma_m(S) = \Gamma(S_m) = \Gamma(Z_m(S))$ is connected by Theorem 2.2 and the comments before that theorem. If $Z_m(S) = \{0\}$, then $\Gamma_m(S) = \Gamma(S_m) = \Gamma(Z_m(S)) = \emptyset$ is (vacuously) connected. Hence $\Gamma_n(S) = \Gamma(S_n) = \Gamma(Z_n(S))$ is connected for every positive integer n.

The "moreover" statement is clear. □

The following is an example of a commutative semigroup (ring) S with 0 such that $Z(S) = Nil(S)$ and all the $\Gamma_n(S)$'s are distinct.

Example 3.4. Let $R = \mathbb{Z}_2[\{X_n\}_{n=1}^{\infty}]/(\{X_n^{n+1}\}_{n=1}^{\infty}) = \mathbb{Z}_2[\{x_n\}_{n=1}^{\infty}]$ and $S = Z(R) = Nil(R) = (\{x_n\}_{n=1}^{\infty})$. Then $S = Z(S) = Nil(S)$ and $x_n^n \in Z_n(S)^* \setminus Z_{n+1}(S)^*$ for every positive integer n. Thus the $\Gamma_n(S)$'s are all distinct and nonempty, and every $\Gamma_n(S)$ is connected by Theorem 3.3. Moreover, it is easily checked that $diam(\Gamma_n(S)) = 2$ and $gr(\Gamma_n(S)) = 3$ for every positive integer n.

The following is an example of a nonreduced semigroup (ring) S with 0 such that $diam(\Gamma_2(S)) = diam(\Gamma(S)) = 2$, $gr(\Gamma_2(S)) = \infty$, and $gr(\Gamma(S)) = 3$. Thus the "reduced" hypothesis in Theorem 2.8 is crucial. Also, see Example 3.11(d).

Example 3.5. Let $R = \mathbb{Z}_2[X]/(X^6) = \mathbb{Z}_2[x]$ and $S = Z(R) = Nil(R)$. Then R and S are not reduced, and $x^3 - x^4 - x^5 - x^3$ is a cycle of length 3 in $\Gamma(S)$; so $gr(\Gamma(S)) = 3$. Since x^5 is adjacent to every $y \in Z(S)^* = Z(R)^* = S^*$ and $\Gamma(S)$ is not a complete graph, we have $diam(\Gamma(S)) = 2$. Note that $Z_2(S)^* = \{x^2, x^4, x^2 + x^4\}$ and $\Gamma_2(S) = K_{1,2}$ is a star graph with center x^4; so $gr(\Gamma_2(S)) = \infty$ and $diam(\Gamma_2(S)) = 2$. Moreover, $\Gamma_n(S) = \Gamma(S_n)$ is connected for every positive integer n and $\Gamma_n(S) = \emptyset$ for every integer $n \geq 6$.

The following is an example of a nonreduced commutative semigroup (ring) S with 0 such that $diam(\Gamma_2(S)) = 3$, $diam(\Gamma(S)) = 2$, and $gr(\Gamma_2(S)) = gr(\Gamma(S)) = 3$. Thus the "reduced" hypothesis in Theorem 2.4 and Theorem 2.7(1) is crucial.

Example 3.6. Let $R = \mathbb{Z}_2[X, Y, Z, W, V]/(X^2, XY, XZ, XW, XV, WY, VZ, WV) = \mathbb{Z}_2[x, y, z, w, v]$ and $S = Z(R)$. Then R and S are not reduced, and $x - w - v - x$ is a cycle of length 3 in $\Gamma(S)$; so $gr(\Gamma(S)) = 3$. Since x is adjacent to every vertex in $Z(S)^* = Z(R)^* = S^*$ and $\Gamma(S)$ is not a complete graph, we have $diam(\Gamma(S)) = 2$. Note that $x^2 \notin Z_2(S)^*$. Since $nil_2(d) \not\subseteq Nil_2(S)$ for every $d \in Z(S)^*$ with $d^2 \in Z_2(S)^*$, we have $\Gamma_2(S)$ is connected by Theorem 3.1. Since $w^2 - v^2 - y^2z^2 - w^2$ is a cycle of length 3 in $\Gamma_2(S)$, we have $gr(\Gamma_2(S)) = 3$. Since $y^2 - w^2 - v^2 - z^2$ is a shortest path in $\Gamma_2(S)$ from y^2 to z^2, we have $d_2(y^2, z^2) = 3$. Thus $diam(\Gamma_2(S)) = 3$.

Let S be as in Example 3.6. Then $diam(\Gamma_2(S)) = 3$, $gr(\Gamma_2(S)) = 3$, $diam(\Gamma(S)) = 2$, and $gr(\Gamma(S)) = 3$. In view of Example 3.6, we have the following result.

Theorem 3.7. *Let S be a commutative semigroup with 0. Assume that $\Gamma_n(S)$ is connected for a positive integer n. If $diam(\Gamma_n(S)) = 3$ and $x - y - z - w$ is a* *shortest path in* $\Gamma_n(S)$ *from* x *to* w *with* $y^2 \neq 0$ *and* $z^2 \neq 0$ *(e.g., if* S *is reduced), then* $gr(\Gamma_n(S)) = 3$.

Proof. Since $y(xw) = z(xw) = 0$, $y^2 \neq 0$, and $z^2 \neq 0$, we have $y \neq xw$, $z \neq xw$, and $xw \neq 0$. Thus $xw - y - z - xw$ is a cycle of length 3 in $\Gamma_n(S)$; so $gr(\Gamma_n(S)) = 3$. □

We next give the analog of Theorem 2.10 for nonreduced commutative rings.

Theorem 3.8. *Let* R *be a nonreduced commutative ring with* $gr(\Gamma(R)) = 4$. *Then* $\Gamma_n(R)$ *is connected and* $gr(\Gamma_n(R)) \in \{4, \infty\}$ *for every integer* $n \geq 2$. *Moreover, there are integers* $m, n \geq 2$ *such that* $gr(\Gamma_m(R)) = 4$ *and* $gr(\Gamma_n(R)) = \infty$.

Proof. Suppose that R is not reduced and $gr(\Gamma(R)) = 4$. Then $R \cong D \times B$, where D is an integral domain with $|D| \geq 3$ and $B = \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2) = \mathbb{Z}_2[x]$ by [12, Theorem 2.3]; so assume that $R = D \times B$. It is easily checked that $Z_n(R)^* = \{(d^n, 0), (0, 1) \mid d \in D^*\}$ for n an even positive integer and $Z_n(R)^* = \{(d^n, 0), (0, 1), (0, b) \mid d \in D^*\}$ for $n \geq 3$ an odd integer (here, $b = 3$ if $B = \mathbb{Z}_4$, and $b = 1 + x$ if $B = \mathbb{Z}_2[X]/(X^2)$). Let $n \geq 2$. Then for every $z \in Z_n(R)^*$, there is a $y \in Z_n(R)^*$ such that $zy = 0$. Thus $\Gamma_n(R)$ is connected by Theorem 2.2.

Let $|\{(d^n, 0) \mid d \in D^*\}| = \alpha$. Then $\Gamma_n(R) = K_{1,\alpha}$ has girth ∞ for n even, and $\Gamma_n(R) = K_{2,\alpha}$ has girth 4 or ∞ for $n \geq 3$ odd. Since $|D| \geq 3$, we have $\alpha \geq 2$ for some odd integer $n \geq 3$. Hence there are integers $m, n \geq 2$ such that $gr(\Gamma_m(R)) = 4$ and $gr(\Gamma_n(R)) = \infty$.

□

Remark 3.9. Let $R = D \times B$, where D is an integral domain with $|D| \geq 3$ and $B = \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ as in the proof of Theorem 3.8 above. Then $\Gamma(R) = \overline{K}_{m,3}$ with $m = |D| - 1 \geq 2$, where $\overline{K}_{m,3}$ is the graph obtained by joining the complete bipartite graph $G_1 = K_{m,3}(= A \cup C$ with $|A| = m$ and $|C| = 3)$ to the star graph $G_2 = K_{1,m}$ by identifying the center of G_2 to a point of C ([12, Theorem 2.3]). Let $|\{(d^n, 0) \mid d \in D^*\}| = \alpha$. As in the proof of Theorem 3.8, we have $\Gamma_n(R) = K_{1,\alpha}$ for n an even positive integer and $\Gamma_n(R) = K_{2,\alpha}$ for $n \geq 3$ an odd integer. Note that α depends on D. If D is infinite, then clearly α is an infinite cardinal. Let D be a finite integral domain; so D is a field with D^* cyclic. Thus $\alpha = 1$ if $n = k(|D| - 1)$ for any positive integer k, and $\alpha \geq 2$ otherwise. Hence $\Gamma_n(R)$ can have girth 4 or ∞ when D is finite, depending on n.

In view of Theorem 3.8, we have the following result.

Corollary 3.10. *Let* R *be a nonreduced commutative ring such that* $\Gamma_n(R)$ *is not connected for some integer* $n \geq 2$. *Then* $gr(\Gamma(R)) \in \{3, \infty\}$.

The converses of Theorem 3.8 and Corollary 3.10 need not be true. We have the following examples.

Example 3.11. (a) Let $R = \mathbb{Z}_9 \times \mathbb{Z}_9$; so R is not reduced. Then $(3, 3) - (0, 3) - (3, 0) - (3, 3)$ is a cycle of length 3 in $\Gamma(R)$; so $gr(\Gamma(R)) = 3$. It is clear that $Z_n(R)^* \subseteq \{(x, 0), (0, y) \mid x, y \in U(\mathbb{Z}_9)\}$ for every integer $n \geq 2$; so $\Gamma_n(R)$ is a complete bipartite graph. Thus $\Gamma_n(R)$ is connected with $gr(\Gamma_n(R)) \in \{4, \infty\}$ for every integer $n \geq 2$. Hence the converses of Theorem 3.8 and Corollary 3.10 do not hold.

(b) Let $R = \mathbb{Z}_4 \times \mathbb{Z}_4$; so R is not reduced. Then $(2, 2) - (0, 2) - (2, 0) - (2, 2)$ is a cycle of length 3 in $\Gamma(R)$; so $gr(\Gamma(R)) = 3$. For every even positive integer n, we have $Z_n(R)^* = \{(1,0),(01)\}$; so $\Gamma_n(R) = K_2 = K_{1,1}$ is connected with $gr(\Gamma_n(R)) = \infty$. For every odd integer $n \geq 3$, we have $Z_n(R)^* = \{(1,0),(3,0),(0,1),(0,3)\}$; so $\Gamma_n(R) = K_{2,2}$ is connected with $gr(\Gamma_n(R)) = 4$. Thus the converses of Theorem 3.8 and Corollary 3.10 do not hold.

(c) Let $R = \mathbb{Z}_2[X,Y,Z]/(X^2, XZ, XY) = \mathbb{Z}_2[x,y,z]$ (cf. Example 2.1(a)); so R is not reduced. Then $\Gamma(R) = K_{1,\infty}$ with center x; so $gr(\Gamma(R)) = \infty$. For every integer $n \geq 2$, $\Gamma_n(R)$ is not connected by Theorem 3.1 since $nil_n(y) \subseteq Nil_n(R)$. Note that $\Gamma_n(R) = \overline{K_{\aleph_0}}$ for every integer $n \geq 2$.

(d) Let $R = \mathbb{Z}_2[X,Y,Z]/(X^2, XZ, YZ) = \mathbb{Z}_2[x,y,z]$; so R is not reduced. Then $Z(R) = \{ax + yf(y) + zg(z) + xyh(y) \mid a \in \mathbb{Z}_2, f(T), g(T), h(T) \in \mathbb{Z}_2[T]\}$, and $gr(\Gamma(R)) = 3$ since $z - x - xy - z$ is a cycle in $\Gamma(R)$ of length 3. For every integer $n \geq 2$, $Z_n(R) \subseteq \{yf(y) + zg(z) + xyh(y) \mid f(T), g(T), h(T) \in \mathbb{Z}_2[T]\}$; so $\Gamma_n(R) = K_{\aleph_0,\aleph_0}$. Thus $\Gamma_n(R)$ is connected with $gr(\Gamma_n(R)) = 4$ and $y^n - z^n - y^{2n} - z^{2n} - y^n$ is a 4-cycle in $\Gamma_n(R)$ for every integer $n \geq 2$. Hence the "reduced" hypothesis is needed in Theorem 2.8.

We may also have $\Gamma_m(S)$ connected and $\Gamma_n(S)$ not connected for some integers $m, n \geq 2$. In this case, $diam(\Gamma_m(S)) \in \{0,1,2,3\}$, but $diam(\Gamma_n(S)) = \infty$ by definition. See Example 2.1(d) for a "non-ring" example.

Example 3.12. Let $R = \mathbb{Z}_2[X,Y]/(X^3, XY) = \mathbb{Z}_2[x,y] = \{a + bx + cx^2 + yf(y) \mid a,b,c \in \mathbb{Z}_2, f \in \mathbb{Z}_2[T]\}$ and $S = Z(R) = \{bx + cx^2 + yf(y) \mid b,c \in \mathbb{Z}_2, f \in \mathbb{Z}_2[T]\}$. Note that $gr(\Gamma(R)) = 3$ since $x - y - x^2 - x$ is a 3-cycle. We have $S_2 = Z_2(R) = \{bx^2 + y^2f(y^2) \mid b \in \mathbb{Z}_2, f \in \mathbb{Z}_2[T]\}$; so $\Gamma(S_2) = \Gamma_2(R) = \Gamma_2(S) = K_{1,\aleph_0}$ is a star graph with center x^2, and thus $gr(\Gamma_2(R)) = \infty$. Moreover, $S_n = \{y^nf(y)^n \mid f \in \mathbb{Z}_2[T]\}$; so $Z(S_n) = \{0\}$, while $Z_n(S) = \{y^nf(y)^n \mid f \in \mathbb{Z}_2[T]\} = S_n$, for every integer $n \geq 3$. Thus the $\Gamma_n(S)$'s are all distinct. Also, $\{0\} = Z(S_n) = Z(Z_n(S)) \subsetneq Z_n(S)$ for every integer $n \geq 3$; so $\Gamma(S_n) \neq \Gamma_n(S)$ and $\Gamma_n(S)$ is not connected for every integer $n \geq 3$ by Theorem 2.2. Moreover, $\Gamma_n(R) = \Gamma_n(S) = \overline{K_{\aleph_0}}$ is not connected (in fact, totally disconnected) and $\Gamma(S_n) = \emptyset$ for every integer $n \geq 3$.

We can replace X^3 by X^m for any integer $m \geq 4$ in the definition of the ring R to get that $\Gamma_n(R)$ is connected for $1 \leq n \leq m-1$ and $\Gamma_n(R)$ is not connected for every integer $n \geq m$. Details are left to the reader.

4. $\Gamma_n(R)$ WHEN R IS π-REGULAR

In this section, we study $\Gamma_n(R)$ when R is a π-regular (i.e., zero-dimensional) or von Neumann regular (i.e., reduced and zero-dimensional) commutative ring. We show that the $\Gamma_n(R)$'s are all connected, and in certain nice cases, the $\Gamma_n(R)$'s eventually repeat in blocks. We also consider commutative rings such that some power of every element (or zero-divisor) is idempotent.

Recall that a (not necessarily commutative) ring R is *strongly π-regular* if for every $x \in R$, there is a positive integer n and $y \in R$ such that $x^{n+1}y = x^n$ and $xy = yx$; and R is *π-regular* if for every $x \in R$, there is a positive integer n and $y \in R$ such that $x^{2n}y = x^n$. If R is a commutative ring, then R is strongly π-regular if and only if R is π-regular, if and only if R is zero-dimensional ([22, Theorem 3.1]).

The following theorem gives another case when $\Gamma_n(R)$ is connected for every positive integer n, when R is zero-dimensional (e.g., finite). Example 2.1(a) shows that the $\Gamma_n(R)$'s need not be connected when R is not zero-dimensional.

Theorem 4.1. *Let R be a π-regular (i.e., zero-dimensional) commutative ring. Then $\Gamma_n(R)$ is connected for every positive integer n. In particular, $\Gamma_n(R)$ is connected for every positive integer n when R is a finite commutative ring.*

Proof. We may assume that $n \geq 2$ and $Z_n(R)^* \neq \emptyset$. We show that for every $x \in Z_n(R)^*$, there is a $y \in Z_n(R)^*$ such that $xy = 0$. Let $x \in Z_n(R)^*$. Then $x = eu + w$ for an $e \in Id(R)$, $u \in U(R)$, and $w \in Nil(R)$ by [13, Corollary 1]. Since $x \in Z_n(R)^*$, $e \neq 1$. First, assume that $w = 0$. Then $e \neq 0, 1$ since $x \in Z_n(R)^*$. Thus $y = 1 - e \in Z(R)^*$ is idempotent; so $y = y^n \in Z_n(R)^*$ and $xy = eu(1-e) = 0$. Next, assume that $e = 0$. Then $0 \neq x = w \in Nil(R)$. Let $m \geq 2$ be the least positive integer such that $x^m = w^m = 0$. Since $Z_n(R)$ is a semigroup with 0 and $x \in Z_n(R)^*$, we have $y = x^{m-1} = w^{m-1} \in Z_n(R)^*$ and $xy = 0$ (cf. Lemma 3.2). Now, assume that $e \neq 0$ (note that $e \neq 1$) and $w \neq 0$. Since $w \in Nil(R)$, let k be the least positive integer such that $[(1-e)w]^k = (1-e)w^k = 0$. Note that $(1-e)x = (1-e)(eu+w) = (1-e)w$. So, if $k = 1$, then $y = 1 - e \in Z_n(R)^*$ and $xy = (1-e)x = (1-e)w = 0$. Hence we may assume that $k \geq 2$. Then $y = (1-e)w^{k-1} = [(1-e)w]^{k-1} = [(1-e)x]^{k-1} \in Z_n(R)^*$ since $Z_n(R)$ is a semigroup and $1 - e, x \in Z_n(R)^*$, and $xy = x[(1-e)x^{k-1}] = [(1-e)x]^k = [(1-e)w]^k = 0$. Thus for every $x \in Z_n(R)^*$, there is a $y \in Z_n(R)^*$ such that $xy = 0$; so $\Gamma_n(R)$ is connected by Theorem 2.2.

The "in particular" statement is clear. □

We next give a particular case when the $\Gamma_m(R)$'s eventually repeat in blocks of length n, when $Z_n(R)^* = Id(R) \setminus \{0, 1\}$ for some positive integer n. However, this may happen even when $Z_n(R)^* \neq Id(R) \setminus \{0, 1\}$ (see Example 4.13(b)).

Theorem 4.2. *Let R be a commutative ring and n a positive integer. Then the following statements are equivalent.*

(1) *$x^n \in Id(R)$ for every $x \in Z(R)$.*
(2) *$Z_n(R)^* = Id(R) \setminus \{0, 1\}$.*

Moreover, if either of the above holds, then $Z_{kn+j}(R)^* = Z_{n+j}(R)^*$ for every positive integer k and integer j with $0 \leq j < n$, and thus $\Gamma_{kn+j}(R) = \Gamma_{n+j}(R)$ for every positive integer k and integer j with $0 \leq j < n$, i.e., $\Gamma_r(R) = \Gamma_s(R)$ for integers $r, s \geq n$ if $r \equiv s (mod\ n)$.

Proof. The equivalence of statements (1) and (2) is clear since $Id(R) \setminus \{0, 1\} \subseteq Z_m(R)^*$ for every positive integer m.

For the "moreover" statement, let $x \in Z(R)$. Then $x^{kn+j} = (x^n)^k x^j = x^n x^j = x^{n+j}$ since $x^n \in Id(R)$. Thus $Z_{kn+j}(R)^* = Z_{n+j}(R)^*$ for every positive integer k and integer j with $0 \leq j < n$. □

In view of the above theorem, it is important to know when there is a positive integer n such that x^n is idempotent for every $x \in Z(R)$. As in [15], R is a *Euler ring* if for every $x \in R$, there is a positive integer n such that x^n is idempotent; and R is an *exact-Euler ring* if there is a positive integer n such that x^n is idempotent for every $x \in R$. We define a commutative ring R to be a *Z-Euler ring* if for every $x \in Z(R)$, there is a positive integer n such that x^n is idempotent; and R is a *Z-exact-Euler ring* if there is a positive integer n such that x^n is idempotent for every $x \in Z(R)$. An exact-Euler (resp., Z-exact-Euler) ring is certainly a Euler (resp., Z-Euler) ring, but the converse need not hold, see Example 4.8(c) and Example 4.13(c)(resp., Example 3.4).

For a commutative ring R, let $\gamma(R)$ (resp., $\gamma_Z(R)$) be the least positive integer n such that x^n is idempotent for every $x \in R$ (resp., $x \in Z(R)$); if no such n exists, set $\gamma(R) = \infty$ (resp., $\gamma_Z(R) = \infty$). Clearly, $\gamma_Z(R) \leq \gamma(R)$. Example 4.8 shows that the inequaliy may be strict.

We have the following characterization of exact-Euler commutative rings. Note that a finite commutative ring is always an exact-Euler ring.

Theorem 4.3. ([15, Theorem 4.1 and Proposition 4.2]) *Let R be commutative ring. Then the following statements are equivalent.*

(1) *R is an exact-Euler ring.*
(2) *R is π-regular (i.e., zero-dimensional), and there are positive integers m and n such that $x^m = 0$ for every $x \in Nil(R)$ and $u^n = 1$ for every $u \in U(R)$. Moreover, in this case, x^{mn} is idempotent for every $x \in R$.*

In particular, a finite commutative ring is an exact-Euler ring.

Corollary 4.4. *Let R be a finite commutative ring. Then there is a positive integer n such that $\Gamma_{kn+j}(R) = \Gamma_{n+j}(R)$ for every positive integer k and integer j with $0 \leq j < n$, i.e., $\Gamma_r(R) = \Gamma_s(R)$ for integers $r, s \geq n$ if $r \equiv s (mod\ n)$. Moreover, either $\Gamma_k(R) = \emptyset$ for every integer $k \geq n$ or $|\Gamma_k(R)| \geq 2$ for every positive integer k.*

Proof. Since R is finite, there is a positive integer n such that $x^n \in Id(R)$ for every $x \in R$ by Theorem 4.3. If $Z(R) = Nil(R)$, then $Z_k(R) = \{0\}$ for $k \geq n$. If $Z(R) \neq Nil(R)$, then $Z_n(R)^* = Id(R) \setminus \{0, 1\} \neq \emptyset$. If $Z_k(R) = \{0\}$, then $\Gamma_k(R) = \emptyset$. Otherwise, $|Z_k(R)^*| \geq |Id(R) \setminus \{0, 1\}| \geq 2$. The proof now follows from Theorem 4.2. □

Remark 4.5. (a) Let R be a π-regular (i.e., zero-dimensional) commutative ring. If there are positive integers m and n such that $x^m = 0$ for every $x \in Nil(R)$ and $u^n = 1$ for every $u \in U(R)$, then $\gamma(R) \leq mn$ by Theorem 4.3. However, we may have $\gamma(R) < mn$. For example, let $R = \mathbb{Z}_3 \times \mathbb{Z}_4$. Then $m = n = 2$ in Theorem 4.3, but x^2 is idempotent for every $x \in R$; so $\gamma(R) = 2 < 4 = 2 \cdot 2$ (cf. Example 4.14(b)). As another example, let $T = \mathbb{Z}_8$. Then $m = 3, n = 2$ in Theorem 4.3, but $x^4 \in Id(T)$ for every $x \in T$ and $3^3 = 3 \notin Id(T)$; so $\gamma(T) = 4 < 6 = 3 \cdot 2$ (cf. Example 4.14(a)).

(b) Let R be a local ring with maximal ideal M. If R is Euler (resp., exact-Euler), then $M = Nil(R)$ (resp., the index of nilpotency $n_M < \infty$). If R is finite with n the least positive integer such that $u^n = 1$ for every $u \in U(R)$ and $m = n_M$, then $\gamma_Z(R) = m$ and $\gamma(R) = min\{kn \mid kn \geq m, k \text{ a positive integer}\}$ since $u^j = 1$ for every $u \in U(R)$ if and only if $n|j$ by a standard "division algorithm" argument.

In some cases, to show that R is an exact-Euler ring, we only need to check the elements of $Z(R)$ (i.e., show that R is a Z-exact-Euler ring). To prove this, we will need the following lemma.

Lemma 4.6. *Let R be a commutative ring, $e \in R$ a nontrivial idempotent, and n a positive integer. If $f = (ex)^n$ is idempotent for $x \in R \setminus Z(R)$, then $f = e$. Moreover, if in addition, $(1 - e)x^n = 1 - e$, then $x^n = 1$.*

Proof. Assume that $f = (ex)^n = ex^n$ is idempotent. Then $(1-e)f = (1-e)ex^n = 0$ and $(1 - f)ex^n = (1 - f)f = 0$. Thus $f = ef$, and $(1 - f)e = 0$ since $x \in R \setminus Z(R)$. Hence $ef = e$; so $f = ef = e$.

Fot the "moreover" statement, assume that $(1-e)x^n = 1-e$. Then $ex^n = f = e$ and $(1-e)x^n = 1-e$; so $x^n = ex^n + (1-e)x^n = e + (1-e) = 1$. $\square$

Theorem 4.7. *Let R be a commutative ring with $Z(R) \neq Nil(R)$ and n a positive integer. Then the following statements are equivalent.*

(1) *x^n is idempotent for every $x \in R$, i.e., R is an exact-Euler ring..*

(2) *x^n is idempotent for every $x \in Z(R)$, i.e., R is a Z-exact-Euler ring.*

In particular, $\gamma(R) = \gamma_Z(R)$ when $Z(R) \neq Nil(R)$.

Proof. (1) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (1) Since $Z(R) \neq Nil(R)$ and x^n is idempotent for every $x \in Z(R)$, there is an idempotent $e \in Z(R)^*$. Now, let $y \in R \setminus Z(R)$. Then $ey, (1-e)y \in Z(R)$; so $(ey)^n = ey^n$ and $[(1-e)y]^n = (1-e)y^n$ are idempotent by hypothesis. Thus $y^n = 1$ by Lemma 4.6; so y^n is idempotent. Hence x^n is idempotent for every $x \in R$.

The "in particular" statement is clear. $\square$

The following three examples show that the hypothesis "$Z(R) \neq Nil(R)$" is crucial in Theorem 4.7. Note that if $Z(R) = Nil(R)$, then (2) of Theorem 4.7 holds (i.e., R is a Z-exact-Euler ring) if and only if $n_x \leq n$ for every $x \in Nil(R)$. Recall that the *idealization* $R(+)M$ of an R-module M is the commutative ring $R \times M$ with $(a,m)+(b,n) = (a+b, m+n)$, $(a,m)(b,n) = (ab, am+bn)$, and identity $(1,0)$. Note that $(\{0\}(+)M)^2 = \{(0,0)\}$.

Example 4.8. (a) Let R be an integral domain. Then $Z(R) = Nil(R) = \{0\}$; so R is clearly Z-Euler and exact-Z-Euler with $\gamma_Z(R) = 1$. However, it is easy to show that R is Euler (resp., exact Euler) if and only if R is a field which is an algebraic extension of a finite field (resp., a finite field). For $R = \mathbb{F}_{p^n}$, we have $\gamma(R) = p^n - 1$ (since R^* is cyclic) and $\gamma_Z(R) = 1$.

(b) Let $R = \mathbb{Z}(+)\mathbb{Z}$. Then $Z(R) = Nil(R) = \{0\}(+)\mathbb{Z}$ and $x^2 = 0$ for every $x \in Z(R)$; so x^2 is idempotent for every $x \in Z(R)$. However, $(2,0)^2 = (4,0)$; so x^2 is not idempotent for some $x \in R$. Thus the "(2) $\Rightarrow$ (1)" implication of Theorem 4.7 fails. In fact, $(2,0)^n = (2^n, 0)$ is not idempotent for any positive integer n; so R is not even a Euler ring. Note that $\gamma_Z(R) = 2$, $\gamma(R) = \infty$, and R is neither local nor zero-dimensional.

(c) For a zero-dimensional local example, let $R = K[X]/(X^2) = K[x] = \{a + bx \mid a, b \in K\}$, where K is a field. Then $Z(R) = Nil(R) = (x)$, $U(R) = \{a + bx \mid a \in K^*, b \in K\}$, and $y^2 = 0$ is idempotent for every $y \in (x)$; so $\gamma_Z(R) = 2$. If K is finite, then $y^n = 1$ is idempotent for every $y \in K^*$ and n a positive integral multiple of $|K| - 1$ since the multiplicative group K^* is cyclic. Thus $(a + bx)^n = a^n + na^{n-1}bx = 1$ when $a \neq 0$, $char(K)|n$, and $(|K| - 1)|n$. However, if K is infinite, then there is no positive integer n such that y^n is idempotent for every $y \in K$; so $\gamma(R) = \infty$ when K is infinite. Hence, as in (a) above, R is a Euler (resp., exact-Euler) ring if and only if K is an algebraic extension of a finite field (resp., a finite field). For $K = \mathbb{F}_{p^n}$, we have $\gamma(R) = lcm(p^n - 1, p) = p(p^n - 1)$ and $\gamma_Z(R) = 2$.

We next show that the $Z_n(R)$'s, and thus the $\Gamma_n(R)$'s, are eventually repeating in blocks for certain nice zero-dimensional commutative rings R. The "$Z(R) = Nil(R)$" case was handled in Theorem 3.3.

Theorem 4.9. *Let R be a commutative ring with $Z(R) \neq Nil(R)$. Then the following statements are equivalent.*

(1) *R is an exact-Euler ring.*
(2) *R is π-regular (i.e., zero-dimensional), and x^{mn} is idempotent for every $x \in R$, where m and n are positive integers such that $x^m = 0$ for every $x \in Nil(R)$ and $u^n = 1$ for every $u \in U(R)$.*
(3) *$Z_{kmn}(R)^* = Z_{mn}(R)^* = Id(R) \setminus \{0,1\} \neq \emptyset$ for every positive integer k, where m and n are positive integers such that $x^m = 0$ for every $x \in Nil(R)$ and $u^n = 1$ for every $u \in U(R)$.*

Moreover, if the above hold, then $Z_{kmn+j}(R)^* = Z_{mn+j}(R)^*$, and thus $\Gamma_{kmn+j}(R) = \Gamma_{mn+j}(R)$ and $|\Gamma_k(R)| \geq 2$, for every positve integer k and integer j with $0 \leq j < mn$, i.e., $\Gamma_r(R) = \Gamma_s(R)$ for integers $r, s \geq mn$ if $r \equiv s (mod\ mn)$.

Proof. (1) $\Rightarrow$ (2) This is clear by Theorem 4.3.

(2) $\Rightarrow$ (3) This follows directly from Theorem 4.2 and Theorem 4.3.

(3) $\Rightarrow$ (1) Since $x^{mn} \in Id(R)$ for every $x \in Z(R)$ and $Z(R) \neq Nil(R)$, we have $x^{mn} \in Id(R)$ for every $x \in R$ by Theorem 4.7. Thus R is an exact-Euler ring.

The first part of the "moreover" statement, also follows from Theorem 4.2. In addition, $|\Gamma_k(R)| \geq 2$ for every positive integer k since $\emptyset \neq Id(R) \setminus \{0,1\} \subseteq Z_k(R)^*$ for every positive integer k. ☐

A commutative ring R is *von Neumann regular* if for every $x \in R$, there is a $y \in R$ such that $x^2y = x$. Recall that a commutative ring R is von Neumann regular if and only if R is reduced and zero-dimensional ([22, Theorem 3.1]), if and only if for every $x \in R$, there is an $e \in Id(R)$ and $u \in U(R)$ such that $x = eu$ ([22, Corollary 3.3]). Thus a commutative von Neumann regular ring is just a reduced π-regular ring. For a recent article on von Neumann regular rings, see [4]. The zero-divisor graph $\Gamma(R)$ for a commutative von Neumann regular ring R has been studied in [24] and [10].

If R is a commutative von Neumann regular ring, but not a field, then $Z(R) \neq Nil(R)$, and thus $\gamma(R) = \gamma_Z(R)$ by Theorem 4.7. The next result shows that, in this case, $\gamma(R)$ is the least positive integer m such that $u^m = 1$ for every $u \in U(R)$. Moreover, if $u^n = 1$ for every $u \in U(R)$, then $\gamma(R)|n$.

Theorem 4.10. *Let R be a commutative von Neumann regular ring that is not a field and n a positive integer. Then the following statements are equivalent.*

(1) *$x^n \in Id(R)$ for every $x \in R$, i.e., R is an exact-Euler ring.*
(2) *$x^n \in Id(R)$ for every $x \in Z(R)$, i.e., R is a Z-exact-Euler ring.*
(3) *$u^n = 1$ for every $u \in U(R)$.*
(4) *$\gamma(R)|n$.*

Moreover, $\gamma(R) = \gamma_Z(R)$ is the least positive integer m such that $u^m = 1$ for every $u \in U(R)$. If no such m exists, then $\gamma(R) = \gamma_Z(R) = \infty$.

Proof. (1) $\Leftrightarrow$ (2) This is clear by Theorem 4.7.

(1) $\Rightarrow$ (3) This is clear since $Id(R) \cap U(R) = \{1\}$.

(3) $\Rightarrow$ (1) Let $x \in R$. Then $x = eu$ for some $e \in Id(R)$ and $u \in U(R)$ since R is von Neumann regular. Thus $x^n = (eu)^n = e^n u^n = e \in Id(R)$ since $u^n = 1$ by hypothesis.

(3) $\Rightarrow$ (4) Let $\gamma(R) = m$; so m is the least positive integer such that $u^m = 1$ for every $u \in U(R)$ by (1) $\Leftrightarrow$ (3) above. A standard "division algorithm" argument then shows that $m|n$.

(4) $\Rightarrow$ (1) This is clear by definition.

The "moreover" statement is clear . □

The next theorem shows that the $Z_k(R)^*$'s, and thus the $\Gamma_k(R)$'s, repeat in blocks of length n when R is a commutative von Neumann regular ring in which the elements of $U(R)$ have bounded order n (this is the "$m = 1$" case for Theorem 4.9). Example 4.13(b) shows that the $\Gamma_k(R)$'s can all be equal, all distinct, or repeat in blocks when R is a commutative von Neumann regular ring with $\gamma(R) = \infty$.

Theorem 4.11. *Let R be a commutative von Neumann regular ring that is not a field such that there is a positive integer n such that $u^n = 1$ for every $u \in U(R)$. Then $Z_{kn}(R)^* = Z_n(R)^* = Id(R) \setminus \{0, 1\} \neq \emptyset$ and $Z_{kn+j}(R)^* = Z_j(R)^*$ for every positive integer k and integer j with $1 \leq j \leq n$. Thus $\Gamma_{kn+j}(R) = \Gamma_j(R)$ for every positive integer k and integer j with $1 \leq j \leq n$, i.e., $\Gamma_r(R) = \Gamma_s(R)$ for positive integers r, s if $r \equiv s(mod\ n)$. In particular, $\Gamma_{kn+1}(R) = \Gamma(R)$ and $|\Gamma_k(R)| \geq 2$ for every positive integer k.*

Proof. Let $x \in Z(R)$. Then $x = eu$ for some $e \in Id(R) \setminus \{1\}$ and $u \in U(R)$ since R is von Neumann regular. Since $u^n = 1$ for every $u \in U(R)$, we have $x^n = (eu)^n = e^n u^n = e \in Z_n(R)$. Thus $Z_{kn}(R)^* = Z_n(R)^* = Id(R) \setminus \{0, 1\}$ for every positive integer k. Let k be a positive integer and j an integer with $1 \leq j \leq n$. Then $x^{kn+j} = x^{kn}x^j = (x^n)^k x^j = e^k(eu)^j = e(eu^j) = eu^j = (eu)^j = x^j$; so $Z_{kn+j}(R)^* = Z_j(R)^*$, and hence $\Gamma_{kn+j}(R) = \Gamma_j(R)$.

The "in particular" statement is clear since $Id(R) \setminus \{0, 1\} \subseteq Z_k(R)^*$ for every positive integer k and $|Id(R) \setminus \{0, 1\}| \geq 2$ since R is reduced and not a field. □

Corollary 4.12. (cf. Example 4.14(c)) *Let R be a reduced finite commutative ring that is not a field. Then there is a positive integer n such that $\Gamma_{kn+j}(R) = \Gamma_j(R)$ for every positive integer k and integer j with $1 \leq j \leq n$, i.e., $\Gamma_r(R) = \Gamma_s(R)$ for positive integers r, s if $r \equiv s(mod\ n)$. Moreover, $|\Gamma_k(R)| \geq 2$ for every positive integer k.*

Proof. Since R is a reduced finite commutative ring, R is von Neumann regular and there is a positive integer n such that $u^n = 1$ for every $u \in U(R)$. The result now follows by Theorem 4.11. □

We next give several examples to illustate Theorem 4.11. We use the easily proved fact that $\gamma(R_1 \times R_2) = \gamma_Z(R_1 \times R_2) = lcm(\gamma(R_1), \gamma(R_2))$ for any two integral domains R_1 and R_2. Moreover, $\gamma(R_1 \times R_2) = \gamma_Z(R_1 \times R_2)$ for any two commutative rings R_1 and R_2 by Theorem 4.7 since $Z(R_1 \times R_2) \neq Nil(R_1 \times R_2)$. However, $\gamma(\mathbb{Z}_8) = 4$ and $\gamma(\mathbb{Z}_9) = 6$, but $\gamma(\mathbb{Z}_8 \times \mathbb{Z}_9) = 6 < 12 = lcm(4, 6)$ (cf. Example 4.14(b)).

Example 4.13. (a) (cf. Example 2.1(c)) Let R be a Boolean ring that is not a field. Then $Nil(R) = \{0\}$ and $U(R) = \{1\}$; so we may choose $n = 1$ in Theorem 4.11 (or $m = n = 1$ in Theorem 4.9). Thus $Z_k(R)^* = Z(R)^* = Id(R) \setminus \{0, 1\} \neq \emptyset$, and hence $\Gamma_k(R) = \Gamma(R) \neq \emptyset$, for every positive integer k.

(b) Let $R = \prod_{\alpha \in \Lambda} K_\alpha$, where every K_α is a field and $|\Lambda| \geq 2$. Then R is a commutative von Neumann regular ring that is not a field, $U(R) = \{(x_\alpha) \in R \mid x_\alpha \neq 0 \text{ for every } \alpha \in \Lambda\}$, $Z(R) = R \setminus U(R) = \{(x_\alpha) \in R \mid x_\alpha = 0 \text{ for some } \alpha \in \Lambda\}$, and $Id(R) = \{(x_\alpha) \in R \mid x_\alpha = 0 \text{ or } 1 \text{ for every } \alpha \in \Lambda\}$. Note that the elements of $U(R)$ have bounded order if and only if every K_α is finite and $\{|K_\alpha|\}_{\alpha \in \Lambda}$ is finite. We consider several cases when $K_\alpha = K$ for every $\alpha \in \Lambda$.

(1) Let $K = \mathbb{C}$. In this case, $Z_n(R)^* = Z(R)^*$ for every positive integer n; so $\Gamma_n(R) = \Gamma(R)$ for every positive integer n, and $\gamma(R) = \gamma_Z(R) = \infty$.
(2) Let $K = \mathbb{R}$. In this case, $Z_n(R)^* = Z(R)^*$ for every odd positive integer n, and $Z_n(R) = \{(x_\alpha) \in Z(R) \mid x_\alpha \geq 0\}$ for every even positive integer n. So $\Gamma_n(R) = \Gamma(R)$ for every odd positive integer n, $\Gamma_n(R) = \Gamma_2(R)$ for every even positive integer n, and $\Gamma_2(R) \subsetneq \Gamma(R)$. Also, $\gamma(R) = \gamma_Z(R) = \infty$.
(3) Let $K = \mathbb{Q}$. In this case, the $Z_n(R)^*$'s, and thus the $\Gamma_n(R)$'s, are all distinct and nonempty since $(2^m, 0, \ldots) \in Z_m(R)^* \setminus Z_n(R)^*$ when $m < n$. However, $\Gamma_n(R) \subseteq \Gamma_m(R)$ when $m|n$, and $\gamma(R) = \gamma_Z(R) = \infty$.
(4) Let $K = \mathbb{F}_{p^m}$. In this case, $n = p^m - 1$ in Theorem 4.11 since $U(K) = K^*$ is cyclic, and thus $\gamma(R) = \gamma_Z(R) = p^m - 1$ by Theorem 4.10. Hence $Z_{kn+j}(R)^* = Z_j(R)^*$, and thus $\Gamma_{kn+j}(R) = \Gamma_j(R)$ for every positive integer k and integer j with $1 \leq j \leq n$, i.e., $\Gamma_r(R) = \Gamma_s(R)$ for positive integers r, s if $r \equiv s (\text{mod n})$.

(c) Let $R = \prod_{i=1}^{\infty} \mathbb{Z}_2 + \oplus_{i=1}^{\infty} \mathbb{F}_{2^i} \subsetneq T = \prod_{i=1}^{\infty} \mathbb{F}_{2^i}$. Then R and T are both commutative von Neumann regular rings, and every $u \in U(R)$ has finite order, but the orders are not bounded. Thus R is a Euler ring, but not an exact-Euler ring; so $\gamma(R) = \gamma_Z(R) = \infty$. The $Z_n(R)^*$'s are all distinct, and thus the $\Gamma_n(R)$'s are all distinct. Also, T is not a Euler ring, $\gamma(T) = \gamma_Z(T) = \infty$, and the $Z_n(T)^*$'s and $\Gamma_n(T)$'s are all distinct.

In the next example, we compute $\gamma(R)$ and $\gamma_Z(R)$ when R is either $\mathbb{Z}_n$ or a finite commutative von Neumann regular ring.

Example 4.14. (a) We first consider $R = \mathbb{Z}_{p^m}$ for a prime p and integer $m \geq 1$. If p is odd, then $U(\mathbb{Z}_{p^m})$ is cyclic of order $p^{m-1}(p-1)$ and its maximal ideal $p\mathbb{Z}_{p^m}$ has index of nilpotence $n_{p\mathbb{Z}_{p^m}} = m$. Thus $\gamma_Z(\mathbb{Z}_{p^m}) = m$, and $\gamma(\mathbb{Z}_{p^m}) = p^{m-1}(p-1)$ since $p^{m-1}(p-1) \geq m$ for every $m \geq 1$. If $p = 2$, then $U(\mathbb{Z}_{2^m})$ is cyclic of order 1 and 2 for $m = 1, 2$, respectively, and isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}}$ for $m \geq 3$; so $u^{2^{m-2}} = 1$ for every $u \in U(\mathbb{Z}_{2^m})$ when $m \geq 3$. Since $2\mathbb{Z}_{2^m}$ has index of nilpotency $n_{2\mathbb{Z}_{2^m}} = m$, we have $\gamma_Z(\mathbb{Z}_{2^m}) = m$, $\gamma(\mathbb{Z}_{2^m}) = 2^{m-1}$ when $m = 1, 2, 3$ (cf. Remark 4.5(b) for $m = 3$), and $\gamma_Z(\mathbb{Z}_{2^m}) = m$,$\gamma(\mathbb{Z}_{2^m}) = 2^{m-2}$ when $m \geq 4$ since $2^{m-2} \geq m$ for every $m \geq 4$.

(b) Let $R = \mathbb{Z}_{p_1^{n_1}} \times \cdots \times \mathbb{Z}_{p_k^{n_k}}$, where $k \geq 2$, the p_i are primes with $p_1 \leq \cdots \leq p_k$, and the n_i are positive integers. When the primes p_i are all distinct, we have $R = \mathbb{Z}_n$ for $n = p_1^{n_1} \cdots p_k^{n_k}$. Since $Z(R) \neq Nil(R)$, we have $\gamma(R) = \gamma_Z(R) = m$ by Theorem 4.9. We consider three case to compute m.

(1) Let $p_1 = \cdots = p_k = 2$ and $n_1 \leq \cdots \leq n_k$. Then $\gamma(R) = \gamma_Z(R) = \gamma(\mathbb{Z}_{2^{n_k}}) = 2^{n_k - 1}$ when $n_k = 1, 2, 3$, and $2^{n_k - 2}$ when $n_k \geq 4$, by part (a) above.
(2) Let $p_1 = \cdots = p_i = 2$ for i with $1 \leq i < k$, $n_1 \leq \cdots \leq n_i$, and $p_{i+1} > 2$. Then $\gamma(R) = \gamma_Z(R) = lcm(p_{i+1}^{n_{i+1}-1}(p_{i+1} - 1), \ldots, p_k^{n_k - 1}(p_k - 1))$ for $n_i \leq 2$ since $p_{i+1} - 1 \geq 2$ is even, and $\gamma(R) = \gamma_Z(R) = lcm(2^{n_i - 2}, p_{i+1}^{n_{i+1}-1}(p_{n_{i+1}} - 1), \ldots, p_k^{n_k - 1}(p_k - 1))$ for $n_i \geq 4$. For $n_i = 3$, $\gamma(R) = \gamma_Z(R) = 4$ if $p_{i+1} = \cdots = p_k = 3$ and $n_{i+1} = \cdots = n_k = 1$, and $\gamma(R) = \gamma_Z(R) = lcm(p_{i+1}^{n_{i+1}-1}(p_{i+1} - 1), \ldots, p_k^{n_k - 1}(p_k - 1))$ otherwise (i.e., some $n_j \geq 2$ or $p_j \geq 5$ for $i + 1 \leq j \leq k$) since then either $p_{j+1}^{n_{j+1}-1} \geq 3$ or $p_{j+1} - 1 \geq 4$ is even (cf. Remark 4.5(b)).
(3) Let $p_1 > 2$. Then $\gamma(R) = \gamma_Z(R) = lcm(p_1^{n_1 - 1}(p_1 - 1), \ldots, p_k^{n_k - 1}(p_k - 1))$.

In a similar manner, one can compute $\gamma(R)$ and $\gamma_Z(R)$ when R is any Artinian commutative ring.

(c) Let R be a finite commutative von Neumann regular ring that is not a field. Then $R = \mathbb{F}_{p_1^{n_1}} \times \cdots \times \mathbb{F}_{p_k^{n_k}}$, where the p_i are primes, n_i positive integers, and $k \geq 2$. Since every $U(\mathbb{F}_{p_i^{n_i}})$ is cyclic of order $p_i^{n_i} - 1$, we have $m = lcm(p_1^{n_1} - 1, \ldots, p_k^{n_k} - 1)$ is the least positive integer such that $u^m = 1$ for every $u \in U(R)$. Thus $\gamma(R) = \gamma_Z(R) = lcm(p_1^{n_1} - 1, \ldots, p_k^{n_k} - 1)$ by Theorem 4.10. For $R = \mathbb{F}_{p_1^{n_1}}$, we have $\gamma(R) = p_1^{n_1} - 1$ and $\gamma_Z(R) = 1$.

Recall that a commutative ring R is a *p.p. ring* if every principal ideal of R is projective, equivalently, if every element of R is the product of an idempotent and a regular element of R ([20] and [25, Proposition 15]). Thus a commutative p.p. ring that is not an integral domain has nontrivial idempotents. For example, a commutative von Neumann regular ring is a p.p. ring, and $\mathbb{Z} \times \mathbb{Z}$ is a p.p. ring that is not von Neumann regular. Also, note that a finite commutative ring is a p.p. ring if and only if it is von Neumann regular, if and only if it is a finite product of finite fields.

The nextl result gives a characterization of certain p.p. rings.

Theorem 4.15. *Let R be a reduced commutative ring that is not an integral domain and n a positive integer. Then the following statements are equivalent.*

(1) *R is a p.p. ring and x^n is idempotent for every $x \in Z(R)$.*
(2) *R is a p.p. ring and x^n is idempotent for every $x \in R$.*
(3) *R is a von Neumann regular ring and x^n is idempotent for every $x \in R$.*
(4) *R is a von Neumann regular ring and x^n is idempotent for every $x \in Z(R)$.*
(5) *R is a von Neumann regular ring and $u^n = 1$ for every $u \in U(R)$.*
(6) *$Z_n(R)^* = Id(R) \setminus \{0, 1\} \neq \emptyset$.*

Moreover, if any of the above hold, then $\Gamma_{kn+j}(R) = \Gamma_j(R) \neq \emptyset$ for every positive integer k and integer j with $1 \leq j \leq n$, i.e., $\Gamma_r(R) = \Gamma_s(R)$ for positive integers r, s if $r \equiv s (mod\ n)$.

Proof. (1) $\Rightarrow$ (2) $Z(R) \neq Nil(R)$ since R is reduced and not an integral domain; so $x^n \in Id(R)$ for every $x \in R$ by Theorem 4.7.

(2) $\Rightarrow$ (3) Since $x^n \in Id(R)$ for every $x \in R$, every regular element of R is a unit. Let $y \in R$. Then $y = eu$ for some $e \in Id(R)$ and $u \in U(R)$ since R is a p.p. ring and every regular element of R is a unit; so R is von Neumann regular.

(3) $\Rightarrow$ (4) This is clear.

(4) $\Rightarrow$ (5) This follows from Theorem 4.10.

(5) $\Rightarrow$ (6) This follows from Theorem 4.11.

(6) $\Rightarrow$ (1) We have $Z(R) \neq Nil(R)$ as in (1) $\Rightarrow$ (2) above. Thus $x^n \in Id(R)$ for every $x \in R$ by Theorem 4.7. Hence R is an exact-Euler ring, and thus R is π-regular by Theorem 4.3. Since R is reduced and π-regular, R is also von Neumann regular. Hence R is a p.p. ring and $x^n \in Id(R)$ for every $x \in Z(R)$.

The "morever" statement follows from Theorem 4.11. □

We end this section with a short discussion summarizing when $\Gamma_n(R)$ is connected (cf. Theorem 2.2). We say that a commutative ring R (or commutative semigroup S with 0) satisfies property $(*_n)$ for a positive integer n if either $Z_n(R) = \{0\}$ or $x \in Z(R) \Rightarrow x^n \in Z(Z_n(R))$, i.e., either $Z_n(R) = \{0\}$ or $Z(Z_n(R)) = Z_n(R)$;

and that R satisfies property $(*)$ if it satisfies $(*_n)$ for every positive integer n. Every commutative ring clearly satisfies $(*_1)$.

Theorem 4.16. *Let R be a commutative ring and n a positive integer.*

(a) *R satisfies $(*_n)$ if and only if $\Gamma_n(R)$ is connected.*

(b) *R satisfies $(*_n)$ if and only if $\Gamma_n(R) = \Gamma(Z_n(R))$.*

(c) *$T(R)$ satisfies $(*_n)$ if and only if R satisfies $(*_n)$.*

(d) *Let $\{R_\alpha\}_{\alpha\in\Lambda}$ be a family of commutative rings. Then $R = \prod_{\alpha\in\Lambda} R_\alpha$ satisfies $(*_n)$ if and only if R_α satisfies $(*_n)$ for every $\alpha \in \Lambda$.*

(e) *If R is reduced, zero-dimensional, or $Z(R) = Nil(R)$, then R satisfies $(*)$.*

(f) *If R is Artinian, then R satisfies $(*)$.*

(g) *If R is local with maximal ideal $Nil(R)$, then R satisfies $(*)$.*

Proof. (a) This follows from Theorem 2.2.

(b) This also follows from Theorem 2.2.

(c) This follows from Theorem 2.11 and (a).

(d) Let $R = \prod_{\alpha\in\Lambda} R_\alpha$. The result is clear if $|\Lambda| = 1$; so assume that $|\Lambda| \geq 2$. In this case, $Z_n(R) \neq \{0\}$ since R has nontrivial idempotents. First, suppose that R satisfies $(*_n)$, and let $\alpha \in \Lambda$. For $0 \neq x_\alpha \in Z(R_\alpha)$, let $0 \neq x = (1, \ldots, 1, x_\alpha, 1, \ldots) \in Z(R)$. Then $0 \neq x^n \in Z(Z_n(R))$ by hypothesis; so there is a $0 \neq y^n = (0, \ldots, 0, y_\alpha^n, 0, \ldots) \in Z_n(R)$ with $x^n y^n = 0$. Thus $0 \neq y_\alpha^n \in Z_n(R_\alpha)$ and $x_\alpha^n y_\alpha^n = 0$. Hence $x_\alpha^n \in Z(Z_n(R_\alpha))$; so R_α satisfies $(*_n)$.

Conversely, suppose that R_α satisfies $(*_n)$ for every $\alpha \in \Lambda$. Note that $Z_n(R) \neq \{0\}$. Let $0 \neq x = (x_\alpha) \in Z(R)$. First, suppose that $x_\beta = 0$ for some $\beta \in \Lambda$. Let $y = (0, \ldots, 0, 1_\beta, 0, \ldots) \in Z(R)$. Then $0 \neq y = y^n \in Z_n(R)$ and $x^n y^n = 0$. Thus $x^n \in Z(Z_n(R))$. So we may assume that $x_\alpha \neq 0$ for every $\alpha \in \Lambda$. Hence $0 \neq x_\beta \in Z(R_\beta)$ for some $\beta \in \Lambda$. By a similar argument, we may assume that $x_\beta^n \neq 0$. Since R_β satisfies $(*_n)$ by hypothesis and $x_\beta^n \neq 0$, there is a $y_\beta \in Z(R_\beta)$ with $x_\beta^n y_\beta^n = 0$ and $y_\beta^n \neq 0$. Let $y = (0, \ldots, 0, y_\beta, 0, \ldots) \in Z(R)$. Then $0 \neq y^n \in Z_n(R)$ and $x^n y^n = 0$. Thus $x^n \in Z(Z_n(R))$; so R satisfies $(*_n)$.

(e) The reduced (resp., zero-dimensional, $Z(R) = Nil(R)$) case follows from Theorem 2.4 (resp., Theorem 4.1, Theorem 3.3) and (a).

(f) This is a special case of (e) since an Artinian commutative ring is zero-dimensional.

(g) This is a special case of (e) since, in this case, $Z(R) = Nil(R)$. □

Example 2.1(a) shows that, unlike the Artinian case, a Noetherian ring R need not satisfy $(*)$. Example 3.12 shows that for every integer $n \geq 2$, there is a commutative ring R_n that satisfies $(*_m)$ if and only if $m < n$.

5. Additional n-divisor graphs

In this final section, we consider the n-zero-divisor graph analog for several other related zero-divisor graphs, namely, the extended zero-divisor graph, annihilator graph, and congruence-based zero-divisor graphs. Let S be a commutative semigroup S with 0.

The *extended zero-divisor graph* of S is the (simple) graph $\overline{\Gamma}(S)$ with vertices $Z(S)^*$, and distinct vertices x and y are adjacent if and only if $x^m y^n = 0$ for positive integers m and n with $x^m \neq 0$ and $y^n \neq 0$; and the *annihilator graph* of S is the (simple) graph $AG(S)$ with vertices $Z(S)^*$, and distinct vertices x and y are adjacent if and only if $ann_S(x) \cup ann_S(y) \neq ann_S(xy)$ (i.e., $ann_S(x) \cup ann_S(y) \subsetneq$

$ann_S(xy)$). All three graphs $\Gamma(S), \overline{\Gamma}(S)$, and $AG(S)$ have the same set of vertices $Z(S)^*$. The graphs $\overline{\Gamma}(S)$ and $AG(S)$ were first defined when S is a commutative ring in [16] and [14], repectively, and then extended to commutative semigroups with 0 in [11] and [1], respectively. For a unified treatment of these three graphs, see [11].

We always have $\Gamma(S) \subseteq \overline{\Gamma}(S)$ and $\overline{\Gamma}(S) = \overline{\Gamma}(Z(S))$. If $S \neq Z(S)$ (e.g., S has an identity element), then we also have $\overline{\Gamma}(S) \subseteq AG(S)$ (cf. [1, Theorem 3.1] and [11]). So we often assume that $S = R$. In this case, all four possible inclusions (i.e., each $\subseteq$ is either $\subsetneq$ or $=$) for $\Gamma(R) \subseteq \overline{\Gamma}(R) \subseteq AG(R)$ are possible ([11, Example 2.3]). However, we need not have $AG(S) = AG(Z(S))$.

The following example shows that we may have $\Gamma(S) = \overline{\Gamma}(S) \not\subseteq AG(S)$ when $S \neq Z(S)$ and $AG(T) \neq AG(Z(T))$ even if T has an identity element.

Example 5.1. Let X be a set with $|X| = \alpha \geq 1$. Define $S = X \cup \{0\}$ to be a commutative semigoup with 0 by defining $xy = 0$ for every $x, y \in S$; so $S = Z(S)$. Then $\Gamma(S) = \overline{\Gamma}(S) = K_\alpha$ and $AG(S) = \overline{K_\alpha}$ since $ann_S(x) = S$ for every $x \in S$. Thus $\Gamma(S) = \overline{\Gamma}(S) = K_\alpha \not\subseteq \overline{K_\alpha} = AG(S)$. Now define $T = S \cup \{1\}$ to be the commutative semigroup with $\{0\}$ obtained by adjoining an identity element 1 to S. Then $Z(T) = S$ and $AG(T) = K_\alpha$ since $ann_T(x) = S$ for every $0 \neq x \in S$ and $ann_T(0) = T$. Hence $AG(T) = K_\alpha \neq \overline{K_\alpha} = AG(Z(T))$.

In a similar manner as to $\Gamma_n(S)$, we define $\overline{\Gamma}_n(S)$ and $AG_n(S)$ to be the induced subgraphs of $\overline{\Gamma}(S)$ and $AG(S)$, respectively, with vertices $Z_n(S)^*$. Note that $\Gamma_n(S) \subseteq \overline{\Gamma}_n(S)$, and thus $\overline{\Gamma}_n(S)$ is connected when $\Gamma_n(S)$ is connected, for every integer $n \geq 2$. If $S \neq Z(S)$ (e.g., S has an identity element), then also $\overline{\Gamma}_n(S) \subseteq AG_n(S)$, and hence $AG_n(S)$ is connected when $\overline{\Gamma}_n(S)$ is connected, for every integer $n \geq 2$. Moreover, if $\Gamma_n(S)$ is connected, then $\overline{\Gamma}_n(S) = \overline{\Gamma}(S_n) = \overline{\Gamma}(Z_n(S))$ when $|Z_n(S)^*| \geq 2$.

Clearly $\overline{\Gamma}(S) = \Gamma(S)$ when S is reduced, and $\overline{\Gamma}_n(S) = \Gamma_n(S)$ when $Z_n(S)$ is reduced. We next consider some cases when $\overline{\Gamma}_n(S) = \Gamma_n(S)$.

Theorem 5.2. *Let S be a commutative semigroup with 0.*

(a) *If S is reduced, then $\overline{\Gamma}_n(S) = \Gamma_n(S)$ for every positive integer n.*

(b) *Let $N = sup\{ n_x \mid x \in Nil(S)\}$. If $N < \infty$, then $\overline{\Gamma}_n(S) = \Gamma_n(S)$ for every integer $n \geq N$. In particular, if S is finite, then $\overline{\Gamma}_n(S) = \Gamma_n(S)$ for all large n.*

(c) *If $\overline{\Gamma}_n(S) = \Gamma_n(S)$, then $\overline{\Gamma}_{kn}(S) = \Gamma_{kn}(S)$ for every positive integer k. In particular, if $\overline{\Gamma}(S) = \Gamma(S)$, then $\overline{\Gamma}_n(S) = \Gamma_n(S)$ for every positive integer n.*

Proof. (a) Suppose that $(x^n)^i(y^n)^j = 0$ for $x, y \in Z(S)^*$ and positive integers n, i, j with $(x^n)^i, (y^n)^j \neq 0$. Then $xy \in Nil(S) = \{0\}$; so $x^ny^n = 0$. Thus $\overline{\Gamma}_n(S) = \Gamma_n(S)$.

(b) Note that $Z_n(S)$ is reduced for $n \geq N$. The proof is then similar to that in part (a) above.

(c) Suppose that $\overline{\Gamma}_n(S) = \Gamma_n(S)$ and $(x^{kn})^i(y^{kn})^j = 0$ for positive integers n, k, i, j with $(x^{kn})^i, (y^{kn})^j \neq 0$. Then $(x^n)^{ki}(y^n)^{kj} = 0$ with $(x^n)^{ki}, (y^n)^{kj} \neq 0$; so $x^ny^n = 0$. Thus $x^{kn}y^{kn} = 0$, and hence $\overline{\Gamma}_{kn}(S) = \Gamma_{kn}(S)$. □

The following is an example where $\Gamma_n(R) \subsetneq \overline{\Gamma}_n(R) \subsetneq AG_n(R)$ for every positive integer n.

Example 5.3. (a) Let $R = \mathbb{Z}_2[\{X_n, Y_n\}_{n=1}^\infty]/(\{X_n^{3n}, Y_n^{3n}, X_n^{2n}Y_n^{2n}\}_{n=1}^\infty) = \mathbb{Z}_2[\{x_n, y_n\}_{n=1}^\infty]$. Then R is a zero-dimensional commutative local ring with maximal ideal $Z(R) =$

$Nil(R) = (\{x_n, y_n\}_{n=1}^{\infty})$. Thus $\Gamma_n(R)$, and hence $\overline{\Gamma}_n(R)$ and $AG_n(R)$, are connected for every positive integer n by Theorem 3.3 or Theorem 4.1. Clearly $Z_m(R) \neq Z_n(R)$ for positive integers $m < n$ since $x_m^m \in Z_m(R) \setminus Z_n(R)$. Note that $\Gamma_n(R) \subsetneq \overline{\Gamma}_n(R)$ since $(x^n)^2(y^n)^2 = 0$ with $(x^n)^2, (y^n)^2 \neq 0$, but $x^n y^n \neq 0$.

(b) Let $R = A \times B$, where $A = \mathbb{Z}_2[\{x_n, y_n\}_{n=1}^{\infty}]$ as in part (a) above and $B = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then R is a zero-dimensional commutative ring and $\Gamma_n(R)$, and thus $\overline{\Gamma}_n(R)$ and $AG_n(R)$, are connected for every positive integer n by Theorem 4.1. It is easily checked that $Z_m(R) \neq Z_n(R)$ for all positive integers $m < n$ and $\Gamma_n(R) \subsetneq \overline{\Gamma}_n(R) \subsetneq AG_n(R)$ for every positive integer n.

(c) We may have $\Gamma(R) \subsetneq \overline{\Gamma}(R) \subsetneq AG(R)$ for a commutative ring R and $\Gamma_n(R) = \overline{\Gamma}_n(R) = AG_n(R)$ for some positive integer n. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_8$. Then it is easily checked that $\Gamma(R) \subsetneq \overline{\Gamma}(R) \subsetneq AG(R)$ and $\Gamma_4(R) = \overline{\Gamma}_4(R) = AG_4(R) = K_2 = K_{1,1}$.

Let R be a commutative ring with $1 \neq 0$ and $\sim$ a multiplicative congruence relation on R, i.e., $\sim$ is an equivalence relation and $x \sim y \Rightarrow xz \sim yz$ for every $x, y, z \in R$. Let $R/{\sim} = \{[x] \mid x \in R\}$ be the set of congruence classes of $\sim$. Then $S = R/{\sim}$ is a commutative monoid under the multiplication $[x][y] = [xy]$ with zero element $[0]$ and identity element $[1]$. As in [8], let $\Gamma_{\sim}(R) = \Gamma(R/{\sim})$ be the *$\sim$-zero-divisor graph* of R. We then define $\overline{\Gamma}_{\sim}(R) = \overline{\Gamma}(R/{\sim})$ and $AG_{\sim}(R) = AG(R/{\sim})$ as in [11]. All three graphs have the same set of vertices $Z(R/{\sim})^*$, and $\Gamma_{\sim}(R) \subseteq \overline{\Gamma}_{\sim}(R) \subseteq AG_{\sim}(R)$ ([11, Theorem 3.1(a)]). Note that $I = [0]$ is a semigroup ideal of R, and $[x]$ and $[y]$ are adjacent in $\Gamma_{\sim}(R)$ (resp., $\overline{\Gamma}_{\sim}(R), AG_{\sim}(R)$) if and only if $xy \in I$ (resp., $x^m y^n \in I$ for positive integers m and n with $x^m, y^n \notin I$, $(I : x) \cup (I : y) \neq (I : xy)$).

For a positive integer n, we define $\Gamma_{n\sim}(R) = \Gamma_n(R/{\sim})$, $\overline{\Gamma}_{n\sim}(R) = \overline{\Gamma}_n(R/{\sim})$, and $AG_{n\sim}(R) = AG_n(R/{\sim})$ with vertices $Z_n(R/{\sim})^*$. Thus $\Gamma_{n\sim}(R) \subseteq \overline{\Gamma}_{n\sim}(R) \subseteq AG_{n\sim}(R)$ for every positive integer n.

When $\sim$ is defined by $x \sim y \Leftrightarrow ann_R(x) = ann_R(y)$, then $\Gamma_{\sim}(R) = \Gamma_E(R)$ is the compressed zero-divisor graph (see [6] and [7]) and $[x][y] = [0] \Leftrightarrow xy = 0$. Moreover, $R_E = R/{\sim}$ is a Boolean monoid when R is reduced; so $\Gamma_{n\sim}(R) = \Gamma_{\sim}(R) = \overline{\Gamma}_{\sim}(R) = \overline{\Gamma}_{n\sim}(R)$ for every positive integer n when R is reduced.

Let I be an ideal of R. When $\sim$ is defined by $x \sim y \Leftrightarrow x = y$ or $x, y \in I$, then $R/{\sim}$ is the Rees semigroup of R with respect to I and $Z(R/{\sim}) = Z_I(R) = \{x \in R \setminus I \mid xy \in I$ for some $y \in R \setminus I\}$. Then $\Gamma_{\sim}(R)$, $\overline{\Gamma}_{\sim}(R)$, and $AG_{\sim}(R)$ are the usual ideal-based graphs $\Gamma_I(R)$, $\overline{\Gamma}_I(R)$, and $AG_I(R)$, respectively, and x and y are adjacent in $\Gamma_I(R)$ (resp., $\overline{\Gamma}_I(R), AG_I(R)$) if and only if $xy \in I$ (resp., $x^m y^n \in I$ for positive integers m and n with $x^m, y^n \notin I$, $(I : x) \cup (I : y) \neq (I : xy)$).

We leave a more detailed study of these graphs to a later time and place.

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