1 INTRODUCTION

Throughout this paper, all rings are commutative with identity and if $R$ is a ring, then $Z(R)$ denotes the set of zerodivisors of $R$ and $\text{Nil}(R)$ denotes the set of nilpotent elements of $R$. Our main purpose is to provide another generalization of pseudo-valuation domains (as introduced in [10]) to the context of arbitrary rings (with $Z(R)$ possibly nonzero). Recall from [10] that an integral domain $R$ with quotient field $K$ is called a pseudo-valuation domain (PVD) in case each prime ideal $P$ of $R$ is strongly prime (or a strong prime), in the sense that $xy \in P$, $x \in K$, $y \in K$ implies that either $x \in P$ or $y \in P$. Anderson, Dobbs, and the author in [7] generalized the study of pseudo-valuation domains to the context of arbitrary rings. Recall from [7] that a prime ideal $P$ of a ring $R$ is said to be strongly prime (or a strong prime) if $aP$ and $bR$ are comparable for all $a, b \in R$. If $R$ is an integral domain this is equivalent to the original definition of strongly prime as introduced by Hedstrom and Houston in [10] (cf. [1, Proposition 3.1], [2 Proposition 4.2], and [5, Proposition 3]). If each prime ideal of $R$ is strongly prime, then $R$ is called a pseudo-valuation ring (PVR).
First, recall from [6] and [8] that a prime ideal of $R$ is called divided if it is comparable to every principal ideal of $R$; equivalently, if it is comparable to every ideal of $R$. If every prime ideal of $R$ is divided, then $R$ is called a divided ring.

In the following proposition, we show that if a ring $R$ admits a strongly prime ideal, then $\text{Nil}(R)$ is a strongly prime ideal and thus $\text{Nil}(R)$ is a divided prime. This result justifies our focus in studying pseudo-valuation rings to be restricted to rings $R$ where $\text{Nil}(R)$ is a divided prime.

**PROPOSITION 0** Let $P$ be a strongly prime ideal of a ring $R$. Then the prime ideals of $R$ contained in $P$ are strongly prime and are linearly ordered. In particular, $\text{Nil}(R)$ is strongly prime and therefore it is a divided prime.

**Proof:** Let $Q$ be a prime ideal of $R$ contained in $P$. By applying the same argument as in the proof of [7, Theorem 2], we conclude that $Q$ is strongly prime. By [7, Lemma 1], $P$ is comparable to every prime ideal of $R$ and the prime ideals of $R$ contained in $P$ are linearly ordered. Hence, $\text{Nil}(R)$ is prime and therefore it is strongly prime and divided.

Now we state our definition of $\phi$-pseudo-valuation rings.

**DEFINITION** Let $R$ be a ring such that $\text{Nil}(R)$ is a divided prime, let $S$ be the set of nonzerodivisors of $R$, let $T = R$, be the total quotient ring of $R$, and let $K = \mathbb{R}_{\text{Nil}(R)}$. Define $\phi : T \to K$ by $\phi(a/b) = a/b$ for every $a \in R$ and $b \in S$. Then $\phi$ is a ring homomorphism from $T$ into $K$, and $\phi$ restricted to $R$ is also a ring homomorphism from $R$ into $K$ given by $\phi(x) = x/1$ for every $x \in R$. Also, observe that $\phi(R)$ is a subring of $K$ with identity. A prime ideal $Q$ of $\phi(R)$ is called $K$-strongly prime if $xy \in Q$, $x \in K$, $y \in K$ implies that either $x \in Q$ or $y \in Q$. If each prime ideal of $\phi(R)$ is $K$-strongly prime, then $\phi(R)$ is called a $K$-pseudo-valuation ring ($K$-PVR). A prime ideal $P$ of $R$ is called $\phi$-strongly prime, if $\phi(P)$ is a $K$-strongly prime ideal of $\phi(R)$. If each prime ideal of $R$ is $\phi$-strongly prime then $R$ is called a $\phi$-pseudo-valuation ring ($\phi$-PVR). Observe that $Q$ is a prime ideal of $\phi(R)$ if and only if $Q = \phi(P)$ for some prime ideal $P$ of $R$, and $R$ is a $\phi$-PVR if and only if $\phi(R)$ is a $K$-PVR.
Throughout this section, $R$ denotes a commutative ring with identity such that $\text{Nil}(R)$ is a divided prime. Given a ring $R$, let $K = R_{\text{Nil}(R)}$ and $T = R_S$, where $S$ is the set of nonzerodivisors of $R$.

Observe that an integral domain $R$ is a PVD if and only if it is a $\phi$-PVR. In fact, in Corollary 7, we show that a PVR (in the sense of [7]) is always a $\phi$-PVR. Also, observe that a quasi-local zero-dimensional ring is a $\phi$-PVR. The following is an example of a zero-dimensional $\phi$-PVR that is not a PVR.

**EXAMPLE 1 ([7, Remark 15])** Let $K$ be a field, $X,Y,$ and $Z$ be indeterminates, and $R = K[X,Y,Z] / (X^2, Y^2, Z^2) = K[x,y,z]$. Then $R$ is quasi-local zero-dimensional with maximal ideal $\text{Nil}(R) = (X,Y,Z) / (X^2, Y^2, Z^2) = (x,y,z)$; hence $R$ is a $\phi$-PVR. However, $R$ is not a PVR since $xz \notin yR$ and $y \notin x\text{Nil}(R)$.

**PROPOSITION 2** For a ring $R$, we have the following:
1. $\text{Ker}(\phi)$ is contained in $\text{Nil}(R)$.
2. $\phi(R)$ is an integral domain if and only if for every nonzero $w \in \text{Nil}(R)$ there exists a $z \in Z(R) \setminus \text{Nil}(R)$ such that $zw = 0$ in $R$.

**Proof:**
(1). Let $x \in \text{Ker}(\phi)$. Then $x = a/b$ for some $a \in R$ and $b \in S$ such that $\phi(a/b) = a/b = 0/1$ in $K$. Hence, $za = 0$ in $R$ for some $z \in Z(R) \setminus \text{Nil}(R)$. Thus, $a \in \text{Nil}(R)$ since $\text{Nil}(R)$ is prime. Hence, $x = a/b = w \in \text{Nil}(R)$ since $b \in S$ and $\text{Nil}(R)$ is divided. (2). Suppose that $\phi(R)$ is an integral domain. Since $R/\text{Ker}(\phi) = \phi(R)$ and $\text{Ker}(\phi) \subset \text{Nil}(R)$, we have $\text{Ker}(\phi) = \text{Nil}(R)$, and the claim is now clear. Conversely, since for every nonzero $w \in \text{Nil}(R)$ there is a $z \in Z(R) \setminus \text{Nil}(R)$ such that $zw = 0$ in $R$, we have $\text{Ker}(\phi) = \text{Nil}(R)$. Since $\text{Nil}(R)$ is prime and $R/\text{Nil} = \phi(R)$, $\phi(R)$ is an integral domain.

**PROPOSITION 3** For a ring $R$, we have the following:
1. $\text{Nil}(T) = \text{Nil}(R)$ and $\text{Nil}(K) = \text{Nil}(\phi(R)) = \phi(\text{Nil}(R))$.
2. Let $x \in \text{Nil}(K)$ and write $x = a/b$ for some $a \in R$ and $b \in R \setminus \text{Nil}(R)$. Then $a \in \text{Nil}(R)$ and $x = a/b = w/1$ in $K$ for some $w \in \text{Nil}(R)$. 
(3). Let \( x \in K \) and write \( x = a/b \) for some \( a \in R \) and 
\( b \in R \setminus \text{Nil}(R) \). If \( a/b = i/1 \) in \( K \) for some \( i \in R \), then \( b | a \) in \( R \); in particular, \( a = (i+w)b \) in \( R \) for some \( w \in \text{Nil}(R) \), and therefore \( a \) is contained in every prime ideal of \( R \) which 
contains \( i \).

(4). Let \( x \in R \) and \( y \in R \setminus \text{Nil}(R) \). If \( x/1 = y/1 \) in \( K \), then \( x 
= uy \) in \( R \) for some unit \( u \) of \( R \); in particular, \( (x) = (y) \) in \( R \).

Proof: (1). Note that \( \text{Nil}(T) = \text{Nil}(R) \) since \( \text{Nil}(R) \) is a 
divided prime ideal of \( R \). For the second equality, we only 
need show that \( \text{Nil}(K) = \text{Nil}(\phi(R)) \). Let \( x \in \text{Nil}(K) \) and write 
\( x = a/b \) for some \( a \in R \) and \( b \in R \setminus \text{Nil}(R) \). Since \( \text{Nil}(R) \) is 
prime, it follows that \( a \in \text{Nil}(R) \). Since \( \text{Nil}(R) \) is a divided 
prime and \( a \in \text{Nil}(R) \) and \( b \in R \setminus \text{Nil}(R) \), \( x = a/b = w/1 \) for some 
\( w \in \text{Nil}(R) \). Thus, \( x \in \text{Nil}(\phi(R)) \). (2). Clear by the proof of 
(1). (3). Since \( a/b = i/1 \) in \( K \), \( z(a-bi) = 0 \) in \( R \) for some 
\( z \in R \setminus \text{Nil}(R) \). Thus, \( a-bi = c \in \text{Nil}(R) \) since \( \text{Nil}(R) \) is prime. 
Since \( b \in R \setminus \text{Nil}(R) \) and \( \text{Nil}(R) \) is a divided prime, \( c = wb \) for some 
\( w \in \text{Nil}(R) \). Hence, \( a-bi = c = wb \). Thus, \( a = (i+w)b \). 

(4). Since \( x/1 = y/1 \) in \( K \), \( z(x-y) = 0 \) in \( R \) for some 
\( z \in R \setminus \text{Nil}(R) \). Thus, \( x-y = w \in \text{Nil}(R) \). Once again, since \( y 
\in R \setminus \text{Nil}(R) \), \( w = dy \) for some \( d \in \text{Nil}(R) \). Hence, \( x-y = w = dy \). 
Thus, \( x = (1+d)y \). Since \( 1+d \) is a unit of \( R \), the claim is 
clear.

In light of the above proposition, observe that \( K \) is 
quasilocal, zero-dimensional, and a \( K \)-PVR with maximal ideal 
\( \text{Nil}(\phi(R)) \). In general, let \( A \) be a divided ring and \( I \) be an 
ideal of \( A \), and let \( R = A/I \). Then \( K \) is a \( K \)-PVR with maximal 
ideal \( \text{Nil}(\phi(\text{Rad}(I))/I) \), where \( \text{Rad}(I) \) is the radical ideal of 
\( I \) in \( A \).

The following result is an analogue of [10, Corollary 
1.3] and [7, Lemma 1], also see [4, Proposition 1].

PROPOSITION 4  Let \( P \) be a \( \phi \)-strongly prime ideal of \( R \). Then 
\( P \) (resp., \( \phi(P) \)) is a divided prime. In particular, if \( R \) is a 
\( \phi \)-PVR, then \( R \) (resp., \( \phi(R) \)) is a divided ring and hence is 
quasilocal.

Proof: Deny. Then for some ideal \( I \) of \( R \), there is an \( i \in 
I \setminus P \) and a \( p \in P \setminus I \). Since \( \text{Nil}(R) \subset P \), \( i \in R \setminus \text{Nil}(R) \). Hence,
\[(p/1)(i/1) = p/1 \in \phi(P).\] Since \(i/1 \in \phi(P)\) by Proposition 3(4), \(p/1 \in \phi(P)\). Hence, \(i \mid p\) in \(R\) by Proposition 3(3). Thus, \(p \in I\) which is a contradiction.

The following result is an analogue of [10, Theorem 1.4], [2, Proposition 4.8], [4, Proposition 2], and [7, Theorem 2].

**PROPOSITION 5** 1. Let \(P\) be a \(\phi\)-strongly prime ideal of \(R\) and suppose that \(Q\) is a prime ideal of \(R\) contained in \(P\). Then \(Q\) is \(\phi\)-strongly prime. In particular, \(R\) is a \(\phi\)-PVR if and only if some maximal ideal of \(R\) is \(\phi\)-strongly prime.

2. Let \(P\) be a \(K\)-strongly prime ideal of \(\phi(R)\). If \(Q\) is a prime ideal of \(\phi(R)\) contained in \(P\), then \(Q\) is \(K\)-strongly prime. In particular, \(\phi(R)\) is a \(K\)-PVR if and only if some maximal ideal of \(\phi(R)\) is \(K\)-strongly prime.

**Proof:** (1). Suppose that \(xy \in \phi(Q)\) for some \(x \in K\) and \(y \in K\). If \(xy \in \text{Nil}(\phi(R))\), then either \(x \in \text{Nil}(\phi(R)) \subset \phi(Q)\) or \(y \in \text{Nil}(\phi(R)) \subset \phi(Q)\) since \(K\) is a \(K\)-PVR with maximal ideal \(\text{Nil}(\phi(R))\). Hence, we may assume that \(xy \in \text{Nil}(\phi(R))\) and \(x \in K\setminus\phi(R)\). Since \(xy \in \phi(P)\) and \(x \in K\setminus\phi(R)\), we must have \(y \in \phi(P)\). Since \(x(y^2/xy) = y \in \phi(P)\) and \(x \in K\setminus\phi(R)\), we must have \(y^2/xy = p/1 \in \phi(P)\) for some \(p \in P\). Thus, \(y^2 = (xy)(p/1)\) in \(K\). Since \(xy \in \phi(Q)\), \(y^2 \in \phi(Q)\). Thus, \(y \in \phi(Q)\). (2). Since every prime ideal of \(\phi(R)\) is of the form \(\phi(G)\) for some prime ideal \(G\) of \(R\), the claim is clear.

The following lemma is an analogue of [10, Proposition 1.2]. Since the proof is exactly the same as in [10], we leave the proof to the reader.

**LEMMA 6** A prime ideal \(P\) of \(R\) is \(\phi\)-strongly prime if and only if \(x^{-1}\phi(P) \subset \phi(P)\) for every \(x \in K\setminus\phi(R)\).

**COROLLARY 7** (1). A prime ideal \(P\) of \(R\) is \(\phi\)-strongly prime if and only if for every \(a, b \in R \setminus \text{Nil}(R)\), either \(a \mid b\) in \(R\) or \(aP \subset bP\).

(2). A ring \(R\) is a \(\phi\)-PVR if and only if for every \(a, b \in R \setminus \text{Nil}(R)\), either \(a \mid b\) in \(R\) or \(b \mid ac\) in \(R\) for every nonunit \(c\) of \(R\).

(3). If \(R\) is a PVR, then \(R\) is a \(\phi\)-PVR.
Proof: (1). Suppose that $P$ is $\phi$-strongly prime and $a, b \in R \setminus \text{Nil}(R)$ such that $a \mid b$ in $R$. Then $b/a \in K \setminus \Phi(R)$ by Proposition 3(3). Let $p \in P$. Then $(a/b)(p/1) = q/l$ in $K$ for some $q \in P$ by Lemma 6. Thus, $ap = (q+w)b$ in $R$ for some $w \in \text{Nil}(R)$ by Proposition 3(3). Hence, $ap \in bP$ in $R$. Thus, $ap \in bP$ in $R$. Conversely, suppose that for every $a, b \in R \setminus \text{Nil}(R)$ either $a \mid b$ or $aP \subset bP$. Let $x \in K \setminus \Phi(R)$. Then $x = b/a$ for some $a, b \in R \setminus \text{Nil}(R)$ (observe that $b \not\in \text{Nil}(R)$ since $\text{Nil}(R)$ is divided). Hence, $a/b$ in $R$ by Proposition 3(3). Thus, $aP \subset bP$ in $R$. Hence, $(a/b) \Phi(P) \subset \Phi(P)$. Thus, $P$ is $\phi$-strongly prime by Lemma 6. (2). If $R$ is a $\phi$-PVR with maximal ideal $M$, then the claim is clear by (1). Conversely, since for every $a, b \in R$ either $a \mid b^n$ or $b \mid a^n$ for some $n, m \geq 1$, the prime ideals of $R$ are linearly ordered by [5, Theorem 1]. Hence $R$ is quasilocal with maximal ideal $M$. Once again, the claim is clear by (1). (3). This is clear by [7, Theorem 5].

REMARK 8 It was shown in [7, Theorem 5] that a ring $R$ is a PVR if and only for every $a, b \in R$, either $a \mid b$ or $b \mid ac$ for every nonunit $c$ of $R$. Thus, Corollary 7(2) gives a clear difference between a PVR and a $\phi$-PVR.

The first part of the following proposition follows easily since the prime ideals of a divided ring $R$ are linearly ordered and $Z(R)$ is a union of prime ideals of $R$.

**PROPOSITION 9** Let $R$ be a divided ring. Then

(1). $Z(R)$ is a prime ideal of $R$.

(2). If $x \in T \setminus R$, then $x^{-1} \in T$.

Proof: (2). Let $x = a/b \in T \setminus R$ for some $a \in R$ and $b \in S$. Then $a \in S$ since $R$ is divided. Hence, $x^{-1} = b/a \in T$.

Given an ideal $I$ of $R$, then $I:I = \{x \in T : xI \subset I\}$ and $\Phi(I) : \Phi(I) = \{x \in K : x\Phi(I) \subset \Phi(I)\}$

**PROPOSITION 10** Let $R$ be a quasilocal ring with maximal ideal $M$. Then

(1). $R$ is a $\phi$-PVR if and only if $M:M$ is a $\phi$-PVR with maximal ideal $M$.

(2). $\Phi(R)$ is a $K$-PVR if and only if $\Phi(M) : \Phi(M)$ is a $K$-PVR with maximal ideal $\Phi(M)$.
Proof: (1). Suppose that $R$ is a $\phi$-PVR. Let $x \in M:M \backslash R$. Then $\phi(x) \in K \backslash \phi(R)$ by Proposition 3(3). Since $x$ is a unit of $T$ by Proposition 9(2), $\phi(x^{-1}) \phi(M) = \phi(x^{-1}) \phi(M) \subset \phi(M)$ by Lemma 6. Thus, $x^{-1} \in M:M$. Thus, $x$ is a unit of $M:M$. Hence, $M$ is the maximal ideal of $M:M$. Thus, $M:M$ is a $\phi$-PVR since $\phi(M)$ is $K$-strongly prime. The converse is clear. (2). This follows by a similar argument to that in (1).

Recall that a ring $B$ is called an overring of $R$ (resp., $\phi(R)$) if $R \subset B \subset T$ (resp., $\phi(R) \subset B \subset K$).

**Proposition 11** Suppose that $R$ is a $\phi$-PVR with maximal ideal $M$.

(1). If $B$ is an overring of $\phi(R)$ which contains an element of the form $1/s$ for some nonunit $s \in R \backslash \text{Nil}(R)$, then $x^{-1} \in B$ for every $x \in K \backslash \phi(R)$. Furthermore, $B$ is a $K$-PVR.

(2). If $B$ is an overring of $R$ which contains an element of the form $1/s$ for some nonunit $s \in S$, then $x^{-1} \in B$ for every $x \in T \backslash R$. Furthermore $B$ is a $\phi$-PVR.

**Proof: (1).** Suppose that $B$ is an overring of $\phi(R)$ which contains an element of the form $1/s$ for some nonunit $s \in R \backslash \text{Nil}(R)$. Let $x \in K \backslash \phi(R)$. Then $x^{-1}(s/1) \in \phi(M) \subset \phi(R)$ by Lemma 6. Hence, $x^{-1} = (x^{-1}s)/s \in B$ since $s^{-1} \in B$. Now, let $N$ be a maximal ideal of $B$ and $xy \in N$ for some $x, y \in K$ with $x \in K \backslash \phi(R)$. Then $y = x^{-1}(xy) \in N$ since $x^{-1} \in B$. Thus, $N$ is $K$-strongly prime. Hence, $B$ is a $K$-PVR. (2). Suppose that $B$ is an overring of $R$ which contains an element of the form $1/s$ for some nonunit $s \in S$. Then $1/s \in \phi(B)$. Hence, $\phi(B)$ is a $K$-PVR by (1) and therefore $B$ is a $\phi$-PVR. Let $x = a/b \in T \backslash R$ for some $a \in R$ and $b \in S$. Then $x^{-1} = b/a \in T$ by Proposition 9(2). Since $b/a \in R$, $a|b \in R$ by Corollary 7(2). Hence, $sb = ga$ in $R$ for some $g \in R$. Thus, $x^{-1} = b/a = g/s \in B$ since $s^{-1} \in B$.

**Corollary 12** Let $R$ be a $\phi$-PVR with maximal ideal $M$. Then

(1). For every prime ideal $P$ of $R$, $P:P$ is a $\phi$-PVR.

(2). For every prime ideal $P$ of $\phi(R)$, $P:P$ is a $K$-PVR.

(3). For every prime ideal $P$ of $\phi(R)$, $\phi(P)$ is a $K$-PVR.

**Proof:** (1). If $P$ is maximal, then the claim follows by Proposition 10. Hence, assume that $P$ is nonmaximal. Since $P$ is divided, $P:P$ either contains an element of the form $1/s$
for some nonunit \( s \in S \), and in this case \( P/P \) is a \( \phi \)-PVR by Proposition 11; or \( P/P \) does not contain such an element, and in this case it is a \( \phi \)-PVR since it equals \( R \). (2). This follows by a similar argument to that in (1). (3). Once again, if \( P \) is maximal, then \( \phi(R) = \phi(R) \) is a \( K \)-PVR. If \( P \) is nonmaximal, then \( \phi(R) \), contains an element of the form \( 1/s \) for some nonunit \( s \in R \setminus \text{Nil}(R) \) and therefore it is a \( K \)-PVR by Proposition 11.

Recall that a ring \( B \) is called a chained ring if the principal ideals of \( B \) are linearly ordered.

**PROPOSITION 13** Let \( R \) be a \( \phi \)-PVR and let \( B \) be an overring of \( R \) (resp., \( \phi(R) \)) which contains an element of the form \( 1/s \) for some nonunit \( s \in S \) (resp., \( s \in R \setminus \text{Nil}(R) \)). Then \( B \) is a chained ring if and only if for every \( a, b \in \text{Nil}(R) \) (resp., \( \text{Nil}(\phi(R)) \)) either \( a \mid b \) in \( B \) or \( b \mid a \) in \( B \).

**Proof:** We only need prove the converse. Suppose that \( B \) is an overring of \( \phi(R) \). Let \( x, y \in B \) such that neither \( x \in \text{Nil}(\phi(R)) \) nor \( y \in \text{Nil}(\phi(R)) \) and \( x/y \) in \( B \). Then \( d = x^{-1}y \in K \setminus \phi(R) \). Hence, \( d^{-1} = xy^{-1} \in B \) by Proposition 11. Thus, \( x = (xy^{-1})y \) in \( B \). Next, suppose that \( B \) is an overring of \( R \). Let \( x, y \in B \) such that neither \( x \in \text{Nil}(R) \) nor \( y \in \text{Nil}(R) \) and \( y/x \) in \( B \). Since each \( d \in B \setminus R \) is a unit of \( B \) by Proposition 11, we may assume that \( x, y \in R \). Since \( y/x \) in \( B \), \( y/x \) in \( R \), and therefore \( x \mid ys \) in \( R \) by Corollary 7(2). Hence, \( ys = cx \) for some \( c \in R \). Hence, \( y = (c/s)x \). Thus, \( x \mid y \) in \( B \) since \( c/s \in B \).

Given a ring \( R \), then \( R' \) denotes the integral closure of \( R \) in \( T \), and \( \phi(R)' \) denotes the integral closure of \( \phi(R) \) in \( K \). The following result is an analogue of [7, Lemma 17 and Theorem 19].

**PROPOSITION 14** Let \( R \) be a \( \phi \)-PVR with maximal ideal \( M \). Then \( \phi(R)' \) is a \( \phi \)-PVR with maximal ideal \( \phi(M) \).

**Proof:** (1). Let \( x \in R' \setminus R \). Then \( x^{-1} \in R \). For, if \( x^{-1} \in R \), then \( x = 1/s \) for some nonunit \( s \in S \) which is impossible by [12,
Theorem 15. Since \( x^{-1} \in R \), \( \phi(x^{-1}) \in \phi(R) \) by Proposition 3(3), and hence \( \phi(x)\phi(M) \subset \phi(M) \) by Lemma 6. Thus, \( xM \subset M \). Hence, \( x \in M:M \) and \( M \) is a prime ideal of \( R' \) (observe that if \( zw \in M \) for some \( z, w \in T \), then either \( z \in M \) or \( w \in M \) since \( M \) is \( \phi \)-strongly prime). Since \( R \subset R' \) satisfies the INC condition by [12, Theorem 47], \( M \) is the maximal ideal of \( R \). Hence, \( R' \) is a \( \phi \)-PVR. (2). Apply a similar argument as in (1).

Our final result is an analogue of [11, Proposition 2.7], [9, Proposition 4.2], and [7, Theorem 21].

PROPOSITION 15 Let \( R \) be a \( \phi \)-PVR with maximal ideal \( M \). Then
(1). Every overring of \( R \) is a \( \phi \)-PVR if and only if \( R' = M:M \).
(2). Every overring of \( \phi(R) \) is a \( \phi \)-PVR if and only if
\( \phi(R)' = \phi(M):\phi(M) \).

Proof: (1). Let \( C \) be an overring of \( R \) that does not contain
an element of the form \( 1/s \) for some nonunit \( s \in S \). Then
observe that \( C \subset M:M \), and use a similar argument as in [7,
Theorem 21]. (2). Once again, let \( C \) be an overring of \( \phi(R) \)
that does not contain an element of the form \( 1/s \) for some
nonunit \( s \in R \setminus \text{Nil}(R) \). Then observe that \( C \subset \phi(M):\phi(M) \), and
use a similar argument as in the proof of [7, Theorem 21].

ACKNOWLEDGMENT

I am very grateful to the referee for his many corrections.

REFERENCES

[2] D.F. Anderson, When the dual of an ideal is a ring,
[3] D.F. Anderson and D.E. Dobbs, Pairs of rings with the
domains, in Commutative Ring Theory, Lecture Notes in Pure