ON COMPARABILITY OF IDEALS OF COMMUTATIVE RINGS

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INTRODUCTION

Throughout this paper $R$ denotes a commutative ring with identity, $Z(R)$ denotes the set of zerodivisors of $R$, and $N$ denotes the set of nonunit elements of $R$.

Let $A$ be a quasi-local domain with maximal ideal $M$ and quotient field $K$. David Anderson [1] studied several comparability conditions between $M$ and certain fractional ideals of $A$. Our main purpose is to generalize the study of comparability of ideals to the context of arbitrary rings with $Z(R)$ possibly nonzero.

It is fair to state that the whole work in this paper is motivated by [1] and [2].
SECTION 1

In this section we consider the following three comparability conditions that are analogs of those from [1, P. 453]:

(I) For every \(a, b \in R\), \(aR \subseteq bR\) or \(bR \subseteq aR\).

(II) For every \(a, b \in R\), \(aR \subseteq bN\) or \(bN \subseteq aR\).

(III) For every \(a, b \in R\), \(aN \subseteq bN\) or \(bN \subseteq aN\).

Clearly (I) \(\Rightarrow\) (II) \(\Rightarrow\) (III). Note that even for integral domains none of the implications is reversible, for examples see [1]. It is well-known that (I) is an equivalent condition for \(R\) to be a chained ring (a valuation ring).

The following theorem is an important tool in our work. The first part is taken from [2, Theorem 1] and the second part is [3, Theorem 2]. Recall from [3], a ring \(R\) is called a pseudo-valuation ring (PVR) in case each prime ideal \(P\) of \(R\) is strongly prime, in the sense that \(aP\) and \(bR\) are comparable for each \(a, b \in R\).

Theorem 0. (1) If for each \(a, b \in R\) either \(b|a\)' or \(a|b\)', then the prime ideals of \(R\) are linearly ordered and therefore \(R\) is quasilocal.

(2) \(R\) is a PVR if and only if it is quasi-local with maximal ideal \(M\) strongly prime.
Proof. We just provide the proof of (1) since it is so short. Suppose that there are two prime ideals \( P, Q \) of \( R \) that are not comparable. Let \( b \in P \setminus Q \) and \( a \in Q \setminus P \). Then neither \( b/a^2 \) nor \( a/b^2 \) can contradict this.

In the following theorem we show that (11) is an equivalent condition for \( R \) to be a pseudo valuation ring (PVR).

**Theorem 1.** A ring \( R \) satisfies (11) if and only if \( R \) is a PVR.

**Proof.** Suppose that \( R \) satisfies (11). Let \( a, b \in R \). Then \( b|a^2 \) or \( a|b^2 \). Hence, the prime ideals of \( R \) are linearly ordered by Theorem 0 (1). In particular, \( R \) is quasi-local with the maximal ideal \( N \). Thus, \( R \) is a PVR by Theorem 0 (2).

For the converse, suppose that \( R \) is a PVR. By [3, Lemma 1], \( R \) is quasi-local with the maximal ideal \( N \). Hence, \( aN \) and \( bR \) are comparable for every \( a, b \in R \).

In the next theorem, we show that if \( R \) satisfies (111) and \( N \) contains a non-zerodivisor, then \( R \) is quasi-local with the maximal ideal \( N \) and \( N:N = \{ x \in T : xN < N \} \) is a chained ring (valuation ring), where \( T = R_0 \) is the total quotient ring of \( R \) and \( S \).
is the set of non-zerodivisors of \( R \). Observe that if \( R \) satisfies (II) and \( N \) contains a non-zerodivisor of \( R \), then \( N:N \) is a chained ring with maximal ideal \( N \) by [3, Theorem 8]. This shows that the distinction between (II) and (III) is whether or not \( N \) is a maximal ideal of \( N:N \) (see [1, Example 3.2]).

**Theorem 2.** Suppose that \( R \) satisfies (III). Then

1. \( R \) is quasi-local with maximal ideal \( N \).
2. If \( N \) contains a non-zerodivisor element, then \( R \) is quasi-local with maximal ideal \( N \) and \( N:N \) is a chained ring.

**Proof.** (1). Let \( a,b \in R \). Then \( b|a^2 \) or \( a|b^2 \).

Once again, by Theorem 0 (1) \( R \) is quasi-local with maximal ideal \( N \). (2). Suppose that \( N \) contains a non-zerodivisor element. Now, let \( s \) be a non-zerodivisor element in \( N \) and \( x,y \in N:N \). Then \( x = a/d \) and \( y = b/d \) for some \( a,b \in R \) and a non-zerodivisor \( d \) of \( R \). Since \( aN \) and \( bN \) are comparable, we may assume that \( aN \subset bN \). Thus, \( as = bk \) for some \( k \in N \), and therefore in \( N:N \) we have \( (a/d)s = (b/d)k \). We consider two cases: case 1. Suppose that \( k \in Z(R) \). Then \( kN \subset sN \), for otherwise \( k|s^2 \) which is impossible since \( s \) is non-zerodivisor. Thus \( k/s \in N:N \) and \( y|x \) in \( N:N \). Case 2. Suppose that \( k \notin Z(R) \). Then \( k/s \in N:N \) or \( s/k \in N:N \) and hence \( y|x \) in \( N:N \) or \( x|y \) in \( N:N \). Thus, \( N:N \) is a chained ring.
Example 10 (a) in [1] shows that the non-zero divisor hypothesis is needed in Theorem 2 (2).

Our next result is motivated by [1, Proposition 1.3].

**Theorem 3.** Assume that for each $a, b \in R$, there is a maximal ideal $M$ of $R$ containing $Z(R)$ so that $aM$ and $bM$ are comparable. Then the prime ideals of $R$ are linearly ordered. In particular, $R$ is quasi-local.

**Proof.** Assume that $R$ has two distinct maximal ideals $M$ and $L$. Choose $a \in M \setminus L$ and $b \in L \setminus M$. By hypothesis there is a maximal ideal $P$ of $R$ containing $Z(R)$ so that $aP \subseteq bP$ or $bP \subseteq aP$. If $aP \subseteq bP$, then $a^2 \subseteq bP \subseteq L$. Thus, $P \subseteq L$ and hence $L = P$ since $P$ is maximal. Hence, $ab = bk$ for some $k \in L$ since $aP \subseteq bP$ and $P = L$ and $b \in L$. Hence, $b(a-k) = 0$ and therefore $a-k \in Z(R) \subseteq L$. Thus, $a \in L$, a contradiction. If $bP \subseteq aP$, then we leave this case for the reader to find a contradiction. Thus, $R$ is quasi-local with the maximal ideal $N$. Now, since for each $a, b \in R$ $aN$ and $bN$ are comparable, either $b \mid a^2$ or $a \mid b^2$. Thus, the prime ideals of $R$ are linearly ordered by Theorem 0 (11).

The ring $R = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ shows that the hypothesis $Z(R) \subseteq M$ is needed in Theorem 3.
In light of the above results, we state the following corollary (see [1, Corollary 3.4]):

**Corollary 4.** The following statements are equivalent:

1. $R$ satisfies (III).
2. For some maximal ideal $M$ of $R$ containing $Z(R)$, $aM$ and $bM$ are comparable for each $a, b \in R$.
3. For each $a, b \in R$, there is a maximal ideal $M$ of $R$ containing $Z(R)$ so that $aM$ and $bM$ are comparable.

We ask the reader to compare our next result with [1, Proposition 3.5]:

**Theorem 5.** Assume that for each $a, b \in R$ there is a maximal ideal $M$ of $R$ containing $Z(R)$ so that $aR$ and $bM$ are comparable. Then $R$ is a PVR.

**Proof.** Assume that $R$ has two distinct maximal ideals $M$ and $L$. Choose $a \in M \setminus L$ and $b \in L \setminus M$. By hypothesis there is a maximal ideal $P$ of $R$ containing $Z(R)$ so that $aP \subseteq bR$ or $bR \subseteq aP$. If $aP \subseteq bR$, then $aP \subseteq bR \subseteq L$. Thus, $P \subseteq L$ since $a \in L$. Hence, $P = L$ since $P$ is maximal. Since $P = L$ and $bL$ and $a'b = b'k$ for some $k \in R$, $b(a' - bk) = 0$. Thus, $a' - bk \in Z(R) \subseteq L$. Since $bk \in L$, $a \in L$, a contradiction. If $bR \supseteq aP$, then $bR \supseteq aP \subseteq M$. 
Thus, $b \in M$, a contradiction. Hence, $R$ is quasi-local. By Theorem 0 (2) $R$ is a PVR.

Example 6. Let $F$ and $K$ be any fields. The ring $R = F \times K$ shows that the hypothesis $Z(R) \subseteq M$ is needed in Theorem 5.

Now we state the following corollary:

Corollary 7. The following statements are equivalent:

1. $R$ is a PVR (and thus quasi-local).
2. For each $a, b \in R$ and maximal ideal $M$ of $R$, $aM$ and $bR$ are comparable.
3. For some maximal ideal $M$ of $R$ containing $Z(R)$, $aM$ and $bR$ are comparable for each $a, b \in R$.
4. For each $a, b \in R$, there is a maximal ideal $M$ of $R$ containing $Z(R)$ so that $aM$ and $bR$ are comparable.

SECTION 2

In this section we consider the following comparability condition:

(i) For each $a, b \in R$, $aM \subseteq bR$ or $bR \subseteq aR$.

Observe that (i) is the analog of (ii) from [1, P. 454]. Also, observe that if (i) holds, then
either \( b|a' \) or \( a|b' \), so the prime ideals of \( R \) are linearly ordered.

We have the following (see [1, Proposition 3.7]):

**Theorem 8.** Assume that for each \( a,b \in R \), there is a maximal ideal \( M \) of \( R \) containing \( Z(R) \) so that \( aM \subset bR \) or \( bM \subset aR \). Then the prime ideals of \( R \) are linearly ordered. In particular, \( R \) is quasi-local.

**Proof.** The proof is essentially the same as in Theorem 5. So we leave the proof to the reader. Hence \( R \) is quasi-local with the maximal ideal \( N \). Since for each \( a,b \in R \) either \( a|b' \) or \( b|a' \), by Theorem 0 (1) the prime ideals of \( R \) are linearly ordered.

Again, example 6 above shows that the \( Z(R) \subset M \) hypothesis is needed in Theorem 8.

In view of Theorem 8, we have the following:

**Corollary 9.** The following statements are equivalent:

1. \( R \) satisfies (i).
2. The prime ideals of \( R \) are linearly ordered and satisfies (i).
3. \( R \) is quasi-local and satisfies (i).
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(4) For some maximal ideal $M$ of $R$ containing
$Z(R)$, $aM \subseteq bR$ or $bM \subseteq aR$ for every $a, b \in R$.

(5) For each $a, b \in R$, there is a maximal ideal $M$ of $R$ containing $Z(R)$ so that $aM \subseteq bR$ or $bM \subseteq aR$.

Our last result is a generalization of
[1, Proposition 3.10].

Theorem 10. For a ring $R$, conditions (III) and (i) are equivalent.

Proof. We need only show (i) $\Rightarrow$ (III). Let $a, b \in R$ so that $aN \subseteq bR$ and $aN \notin bN$. Then for some $s \in N$, $as = 1$. Hence, $bN \subseteq aN$. Similarly, if $bN \subseteq aR$ and $bN \notin aN$, then $aN \subseteq bN$.

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REFERENCES


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