ON CHAINED OVERRINGS OF PSEUDO-VALUATION RINGS

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ABSTRACT. A prime ideal $P$ of a commutative ring $R$ with identity is called strongly prime if $aP$ and $bR$ are comparable for every $a, b$ in $R$. If every prime ideal of $R$ is strongly prime, then $R$ is called a pseudo-valuation ring. It is well-known that a (valuation) chained overring of a Prüfer domain $R$ is of the form $R_p$ for some prime ideal $P$ of $R$. In this paper, we show that this statement is valid for a certain class of chained overrings of a pseudo-valuation ring.

1. INTRODUCTION

Throughout this paper, all rings are commutative with identity and if $R$ is a ring, then $Z(R)$ denotes the set of zero-divisors of $R$ and $T$ denotes the total quotient ring of $R$. We say a ring $A$ is an overring of a ring $R$ if $A$ is between $R$ and $T$. Recall that a ring $R$ is called a chained ring if the principal ideals of $R$ are linearly ordered, that is, if for every $a, b \in R$ either $a | b$ or $b | a$. It is well-known that a chained overring of a Prüfer domain $R$
is of the form $R_\pi$ (see [9, Theorem 65]) for some prime ideal $P$ of $R$. In this paper, we show that this statement is still valid for a certain class of chained overrings of a pseudo-valuation ring. Recall from [5] that a prime ideal $P$ of a ring $R$ is called a strongly prime ideal if $aP$ and $bR$ are comparable for all $a, b \in R$. If $R$ is an integral domain, this is equivalent to the original definition of strongly prime introduced by Hedstrom and Houston in [8]. If every prime ideal of a ring $R$ is strongly prime, we say that $R$ is a pseudo-valuation ring, abbreviated a PVR. It is easy to see that a PVR is quasilocal, see [5, Lemma 1].

2. RESULTS

We start with the following lemma.

**Lemma 1.** Let $R$ be a PVR and let $a, b \in R$. If $a \in Z(R)$ and $b$ is a nonzerodivisor of $R$, then $b|a$. In particular, if $c/d \in T \setminus R$ for some $c, d \in R$, then $c$ is a nonzerodivisor of $R$ and therefore $d/c \in T$.

**Proof.** Deny. Let $M$ be the maximal ideal of $R$. Since $M$ is strongly prime and $b$ does not divide $a$, we must have $bM \subset aR$. Hence, $b^2 = ac$ for some $c$ in $R$, which is impossible since $b^2$ is a nonzerodivisor of $R$ and $a \in Z(R)$. Thus, our denial is invalid and $b|a$.

The following lemma is trivial, but it is needed in the proof of our main result.
Lemma 2. Let $R$ be a PVR and let $A$ be an overring of $R$. Then $Z(R) = Z(A)$.

Proof. This is clear by Lemma 1.

Theorem 3. Let $R$ be a PVR with maximal ideal $M$, and let $V$ be a chained overring of $R$ with the maximal ideal $N$. If $P = N \cap R$ is different from $M$, then $V = R_p$.

Proof. By Lemma 2, $Z(R) \subseteq P$. Hence, if $s \in R \setminus P$, then $s$ is a nonzerodivisor of $R$ and $s^{-1} = 1/s \in T$. Now, for any $s \in R \setminus P$, we must have $s^{-1} \in V$, for otherwise $s \in N$ and so $s \in P$. Thus, $R_p \subseteq V$. Now, we show that $V \subseteq R_p$. Since $P$ is a nonmaximal prime ideal of $R$, we note that $R_p$ is a chained ring by [5, Theorem 12]. Suppose that there is a $v \in V$ and $v$ is not in $R_p$. Write $v = a/s$ for some $a, s \in R$. Since $v$ is not in $R_p$, $v \in T \setminus R$. Hence, $a$ is a nonzerodivisor of $R$ by Lemma 1 and $v^{-1} \in T$.

Since $R_p$ is a chained ring and $v$ is not in $R_p$, we must have $v^{-1} = s/a \in R_p$. Thus, we may assume $a \in P$. Since $v^{-1} \in R_p$ and $v$ is not in $R_p$, we must have $s \in P$, for otherwise, $v^{-1} = s/a$ would be a unit in $R_p$ and $v \in R_p$, which we assumed is not the case. Since $s \in P$, we must have $s \in N$ and $sv \in N$. But $a = sv \in P$, a contradiction. Thus, $V \subseteq R_p$. Hence, $V = R_p$.

It was shown in [5, Lemma 20] that if $R$ is a PVR with maximal ideal $M$ and $B$ is an overring of $R$ containing an element of the form $1/s$ for some nonzerodivisor $s$ of $M$, then $B$ is a chained ring. In view of Theorem 3, now we can show that
such an overring of \( \mathcal{R} \) is of the form \( \mathcal{R}_p \) for some prime ideal \( P \) of \( \mathcal{R} \).

**Corollary 4.** Let \( \mathcal{R} \) be a PVR with maximal ideal \( M \), and \( B \) be an overring of \( \mathcal{R} \) containing an element of the form \( 1/s \) for some nonzerodivisor \( s \) of \( M \). Then \( B \) is a chained ring of the form \( \mathcal{R}_p \) for some prime ideal \( P \) of \( \mathcal{R} \).

**Proof.** By [5, Lemma 20] \( B \) is a chained ring. Let \( N \) be the maximal ideal of \( B \). Since \( B \) contains an element of the form \( 1/s \) for some nonzerodivisor \( s \) of \( M \), \( s \) is not in \( N \). Hence, \( N \cap \mathcal{R} \) is different from the maximal ideal of \( \mathcal{R} \). Thus, \( B = \mathcal{R}_p \) where \( P = N \cap \mathcal{R} \) by Theorem 3. \( \blacksquare \)

It was shown in [2, Proposition 4.3] that if \( \mathcal{P} \) is a nonmaximal strongly prime ideal of an integral domain \( \mathcal{R} \), then \( \mathcal{P} : \mathcal{P} \) is valuation domain. Since \( \mathcal{P} \) is divided (comparable to every principal ideal of \( \mathcal{R} \)) by [5, Lemma 1(a)] and nonmaximal, \( \mathcal{P} : \mathcal{P} = \{ x \in \mathcal{T} : x\mathcal{P} \subset \mathcal{P} \} \) contains an element of the form \( 1/s \) for some nonunit \( s \in M \setminus \mathcal{P} \). Hence, by Corollary 4, \( \mathcal{P} : \mathcal{P} = \mathcal{R}_p \). Thus, we have:

**Corollary 5.** Let \( \mathcal{P} \) be a nonmaximal strongly prime ideal of an integral domain \( \mathcal{R} \). Then \( \mathcal{P} : \mathcal{P} = \mathcal{R}_p \) is a valuation domain. \( \blacksquare \)

Recall that an ideal of \( \mathcal{R} \) is called regular if it contains a nonzerodivisor of \( \mathcal{R} \). If every regular ideal of \( \mathcal{R} \) is generated by its
set of nonzerodivisors, then \( R \) is called a Marot ring. We have the following result.

**Proposition 6.** Let \( R \) be a PVR. Then:

1. \( R \) is a Marot ring.
2. \( Z(R) \) is a prime ideal of \( R \) and \( T = R_{Z(R)} \).
3. If \( R \neq T \), then \( T \) is a chained ring.

**Proof.** (1). This is clear by Lemma 1. (2). Since the prime ideals of \( R \) are linearly ordered by [5, Lemma 1(a)] and \( Z(R) \) is a union of prime ideals of \( R \), \( Z(R) \) is a prime ideal of \( R \) and hence \( T = R_{Z(R)} \). If \( R \neq T \), then \( Z(R) \) is a nonmaximal ideal of \( R \). Hence, \( T = R_{Z(R)} \) is a chained ring by [5, Theorem 12].

We say an overring \( B \) of \( R \) is a valuation overring of \( R \) if there is an ideal \( J \) of \( B \) such that for each \( t \in T \setminus B \) there is an element \( r \in J \) such that \( rt \in B \setminus J \). See [9] for more information.

**Proposition 7.** Let \( R \) be a PVR which is not its own total quotient ring, and let \( B \) be an overring of \( R \). Then the following are equivalent:

1. \( B \) is a chained overring of \( R \).
2. \( B \) is a valuation overring of \( R \).

**Proof.** There is nothing to prove if \( R = T \), so we may assume that \( R \neq T \). (1) \( \implies \) (2). This is clear by [9, Theorem 5.1]. (2) \( \implies \) (1). Since \( T \) is a chained ring by Proposition 6(3) and \( Z(R) = Z(T) \subset B \) by Lemma 2, \( B \) is a chained overring of \( R \) by [9, Theorem 23.2].
Now, we state the main result in this paper.

**Theorem 8.** Let $R$ be a PVR with maximal ideal $M$. Then the following are equivalent:

1. Every overring of $R$ is a PVR.
2. Every chained overring of $R$ other than $M : M$ is of the form $R_p$ for some nonmaximal prime ideal $P$ of $R$.
3. $M : M$ is the integral closure of $R$ in $T$.

Proof. There is nothing to prove if $R = T$, so we may assume $R \neq T$. Since $M : M = \{ x \in T : xM \subseteq M \}$ is a chained ring with maximal ideal $M$ by [5, Theorem 8], it is the only valuation overring of $R$ that has maximal ideal $M$ (see [9, Theorem 5.1]). Hence $M : M$ is the only chained overring of $R$ that has maximal ideal $M$ by Proposition 7. $(1) \iff (3)$. This is clear by [5, Theorem 21].

$(1) \implies (2)$. Since every subring of $M : M$ containing $R$ is a PVR with maximal ideal $M$ by [7, Corollary 18] and $M : M$ is the only chained overring of $R$ that can have $M$ as a maximal ideal, each chained overring of $R$ other than $M : M$ contains an element of the form $1/s$ where $s$ is a nonzerodivisor of $M$ and thus each is of the form $R_p$ for some prime ideal $P$ of $R$ by Corollary 4.

$(2) \implies (3)$. First, $R$ is a Marot ring by Proposition 6. Thus, by [8, Theorem 9.3], the integral closure of $R$ in $T$ is the intersection of the valuation overrings of $R$. By Proposition 7, each valuation overring of $R$ is chained, so except possibly for $M : M$, each is of the form $R_p$ for some prime ideal $P$ of $R$. All such rings contain $M : M$. Therefore, the integral closure of $R$ in $T$ is $M : M$. \[\square\]
An immediate consequence of the above theorem is the following corollary.

**Corollary 9.** Let $R$ be a PVR with maximal ideal $M$ and integral closure $R'$ such that $R' \neq M : M$. Then there exists a chained overring $W$ of $R$ such that $R' \subset W \subset M : M$, and $W$ is not of the form $R_p$ for some prime ideal $P$ of $R$.

**Example 10.** David F. Anderson provided us with a concrete example of a PVR $R$ that has a valuation overring which is not of the form $R_p$ for some prime ideal $P$ of $R$. Let $\mathbb{R}$ be the set of real numbers and $\mathbb{C}$ be the set of complex numbers. Set $V = \mathbb{C}[t] + XC(t)[[X]]$ is a valuation (chained) domain with maximal ideal $M = XC(t)[[X]]$, and $R = \mathbb{R} + XC(t)[[X]]$ is a PVR with maximal ideal $M$. Then $W = \mathbb{C}[t][0] + XC(t)[[X]]$ is a valuation (chained) overring of $R$ which is not of the form of $R_p$ for some prime ideal $P$ of $R$. Observe that $R' = \mathbb{C} + XC(t)[[X]] \subset W \subset M : M = V$.

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REFERENCES


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