

MTH 213, Exam I

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Score = $\frac{50}{50}$

QUESTION 1. Consider the following permutation function

$$f : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 7 & 6 & 1 & 8 & 3 & 4 \end{pmatrix}$$

(a) **(4 points)** Find the least positive integer m such that $f^m = f \circ f \circ \cdots \circ f = I$, the identity function

$\cancel{(1\ 2\ 5\ 1)(3\ 7)(4\ 6\ 8)}$ $(1\ 2\ 5)(3\ 7)(4\ 6\ 8)$

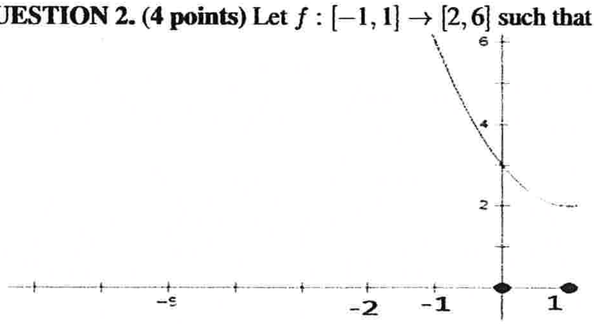
~~4-cycle 2-cycle 3-cycle~~ 3-cycle 2-cycle 3-cycle

$\cancel{m = LCM[4,2,3] = 12}$ $m = LCM[3,2] = 6$ ✓

(b) **(2 points)** Find f^3

$f^2 = f \circ f : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 1 & 3 & 8 & 2 & 4 & 7 & 6 \end{pmatrix}$, $f^3 = f \circ f^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 7 & 4 & 5 & 6 & 3 & 8 \end{pmatrix}$ ✓

QUESTION 2. (4 points) Let $f : [-1, 1] \to [2, 6]$ such that $y = (x-1)^2 + 2$, see picture.



By staring at the picture, is f is a bijective (i.e., one-to-one and onto) function?If no explain, if yes, then find the domain and the codomain of f^{-1} and find the formula of f^{-1}.

f is one to one and onto (bijective) as every horizontal line $y = b$ s.t. $b \in Range(f)$ intersects the curve at exactly one point.

$f^{-1} : \underbrace{[2,6]}_{\text{domain}} \to \underbrace{[-1,1]}_{\text{codomain}}$, ✓ $\quad x = (y-1)^2 + 2$

$(y-1)^2 = x - 2$

$y - 1 = -\sqrt{x-2}$

$y = -\sqrt{x-2} + 1$ (we choose the − square root as the codomain has negative numbers in it).

QUESTION 3. (a) **(4 points)** Find all solutions of the equation $9x = 6$ over the planet Z_{12}

$\left.\begin{array}{l} gcd(9,12) = 3 \\ 3 \mid 6 \text{ in } Z \end{array}\right\} \Rightarrow$ there are 3 distinct solutions in Z_{12}

$\cancel{x = 18}$ $\quad x = 2$

$f = \frac{n}{d} = \frac{12}{3} = 4$

solution set, ~~$x \in \{18, 22.$~~ $x \in \{2, 6, 10\}$

(b)**(6 points)** Let x be your score on an exam out of 63. Given $x\ (mod\ 7) = 3$ and $x\ (mod\ 9) = 7$. Use the CRT and find x.

$x\ (mod\ 7) = 3 = r_1$

$x\ (mod\ 9) = 7 = r_2$

$m_1 = 7$, $m_2 = 9$, $gcd(7,9) = 1 \Rightarrow$ we can use CRT (as the question says!)

$m = m_1 m_2 = 7 \times 9 = 63$

$b_1 = \frac{m}{m_1} = \frac{\cancel{7} \times 9}{\cancel{7}} = 9$

$b_2 = \frac{m}{m_2} = \frac{7 \times \cancel{9}}{\cancel{9}} = 7$

Solve

$9d_1 = 1$ over Z_7

$2d_1 = 1$

$d_1 = 4$

Solve

$7d_2 = 1$ over Z_9

$d_2 = 4$

$x = [9 \times 4 \times 3 + 7 \times 4 \times 7](mod\ 63) = 304\ mod\ 63$

$x = 52$

side note: Need to do better than that to get an A!

more like 63 out of 63!

Call it

QUESTION 4. (a) **(5 points)** Use math induction and prove that $3 | (n-5)(n-4)(n-3), \forall\, n \geq 6$ (*)

① Prove it for $n=6$, $n=6 \Rightarrow (n-5)(n-4)(n-3) = (6-5)(6-4)(6-3) = \cancel{24}\ 6$

$3 | 6$ ✓

② Assume (*) is true for some $n \geq 6$

$3 | (n-5)(n-4)(n-3)$ for some $n \geq 6$

③ Prove it for $n+1$, $(n+1-5)(n+1-4)(n+1-3) = (n-4)(n-3)(n-2)$

$= (n-2-3+3)(n-4)(n-3) = \underbrace{3(n-4)(n-3)}_{\text{divisible by } 3} + \underbrace{(n-5)(n-4)(n-3)}_{\text{divisible by 3 by ②}}$

$\Rightarrow 3 | (n+1-5)(n+1-4)(n+1-3)$, $\therefore 3 | (n-5)(n-4)(n-3)$ by math induction $\forall n \geq 6$.

(b) **(5 points)** Use math induction and prove that $\sum_{i=1}^{n}(2i+1) = \frac{n(4+2n)}{2}, \forall\, n \geq 1$ ← Call it (*)
[hint: $\sum_{i=1}^{n+1}(2i+1) = 2(n+1)+1+\sum_{i=1}^{n}(2i+1) = 2n+3+\sum_{i=1}^{n}(2i+1)$]

① Prove it for $n=1$, $n=1 \Rightarrow \sum_{i=1}^{1}(2i+1) = 3$

$\frac{n(4+2n)}{2} = \frac{1(4+2)}{2} = 3$ } equal

② Assume (*) is true for some $n \geq 1$

$\sum_{i=1}^{n}(2i+1) = \frac{n(4+2n)}{2}$ for some $n \geq 1$

③ Prove it for $n+1$, show that $\sum_{i=1}^{n+1}(2i+1) = \frac{(n+1)(4+2(n+1))}{2}$

$= \frac{(n+1)(6+2n)}{2}$

$\sum_{i=1}^{n+1}(2i+1) = 2(n+1)+1+\sum_{i=1}^{n}(2i+1)$

$= 2n+3+\underbrace{\frac{n(4+2n)}{2}}_{\text{by ②}} = \frac{4n+6+n(4+2n)}{2} = \frac{4n+6+4n+2n^2}{2}$

$= \frac{2n^2+8n+6}{2} = \frac{(n+1)(6+2n)}{2}$ ✓

$\Rightarrow \sum_{i=1}^{n+1}(2i+1) = \frac{(n+1)(4+2(n+1))}{2}$

$\therefore \sum_{i=1}^{n}(2i+1) = \frac{n(4+2n)}{2}$, $\forall n \geq 1$ by math induction.

(c) **(5 points)** Use the 4th method and prove that $\sqrt{61}$ is an irrational number

Deny. $\Rightarrow$ $\sqrt{61}$ is irrational $\Rightarrow$ $\sqrt{61} = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $b \neq 0$ where $\gcd(a,b)=1$

$61 = \frac{a^2}{b^2}$, where $\gcd(a^2, b^2) = 1$

both a and b are odd

$\Rightarrow a = 2n+1$, $n \in \mathbb{Z}$

$b = 2m+1$, $m \in \mathbb{Z}$

$61 = \frac{(2n+1)^2}{(2m+1)^2} = \frac{4n^2+4n+1}{4m^2+4m+1}$, $61.4m^2+61.4m+61 = 4n^2+4n+1$

$61.4m^2+61.4m+60 = 4n^2+4n$

divide both sides by 4 $\Rightarrow$ $61m^2+61m+15 = n^2+n$

$\underbrace{\underbrace{61m(m+1)}_{\text{always even}}+14+1}_{\text{always odd}} = \underbrace{n(n+1)}_{\text{always even}}$

(even + 1 = odd)

odd = even

$\Rightarrow$ contradiction, so our denial is invalid and $\sqrt{61}$ must be irrational.

QUESTION 5. (4 points) Find $11^{122} \pmod{21}$

$21 = 3*7$, $\phi(21) = (3-3^0)*(7-7^0) = 12$

$\gcd(11,21) = 1 \Rightarrow 11^{\phi(21)} \pmod{21} = 1$

$11^{12} \pmod{21} = 1$, multiply both sides by 11^{12} 9 times

$11^{12}.(11^{12})^9 \pmod{21} = 1$, $(11^{12})^{10} \bmod(21) = 1$, $11^{120} \bmod(21) = 1$

$11^{120}.11^2 \bmod(21) = 16$, $11^{122} \bmod(21) = 16$ $(= 11^2 \bmod(21))$

QUESTION 6. (4 points) Write down T or F

(i) If $\exists! x \in Z$ such that $x^2 - 4 = 0$, then $\exists y \in \mathbb{R}$ such that $y^2 + 1 = 0$ T

$S_1 = F$ $S_2 = F$

(ii) $\exists\, x \in \mathbb{Q}$ such that $x^2 = x$ if and only if $\exists! y \in Z$ such that $y^3 = 4y$. $\Leftarrow$ $y^3 = 4y$, F

$S_1 = T$ $S_2 = F$

(iii) $\exists y \in \mathbb{Q}$ such that $\forall x \in Z$, we have $x + 3y = 1$ F

(iv) $\forall x \in Z, \exists y \in \mathbb{Q}$ such that $x^2 + 5y = 3$ T

$y = \frac{3-x^2}{5}$

$x = -2$
$x = 2$

$x^2 - x = 0$
$x(x-1) = 0$
$x = 0$, $x = 1$

$y^3 - 4y = 0$
$y(y^2-4) = 0$
$y = 0$, $y = -2$, $y = 2$

$x + 3y = 1$
$y = \frac{1-x}{3}$

QUESTION 7. (7 points) Let $A = \{3, \{4\}, 4, 5\}$ and $B = \{4, 7, \{7\}\}$. Then

(a) Find $B - A$

$B - A = \{7, \{7\}\}$

(b) Find $|\ \mathbb{P}(A \times B)\ |$

$|A \times B| = |A||B| = 4 \times 3 = 12$

$|\mathbb{P}(A \times B)| = 2^{|A \times B|} = 2^{12} = 4096$

(c) Write down T or F

(i) $\{(4, \{7\}), (\{4\}, 4)\} \in \mathbb{P}(A \times B)$.

T

(ii) $\{7\} \in B$

T

(iii) $\{4, \{4\}, 5\} \subset A$

T

(iv) $\{\{7\}\} \in \mathbb{P}(B)$

T

(v) $\{\{7, 4\}, \{4\}\} \subset \mathbb{P}(B)$

T

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