RINGS WITH PRIME NILRADICAL

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ABSTRACT: A commutative ring R is said to be a ϕ-ring if its nilradical $Nil(R)$ is both prime and divided, the latter meaning $Nil(R)$ is comparable with each principal ideal of R. Special types include ϕ-Noetherian (also known as nonnil-Noetherian), ϕ-Mori, ϕ-chained and ϕ-Prüfer. A ring R is ϕ-Noetherian if $Nil(R)$ is a divided prime and each ideal that properly contains $Nil(R)$ is finitely generated. If R is a ϕ-Noetherian ring and $x_1, x_2, \ldots, x_n$ are indeterminates, then an ideal I of $R[x_1, x_2, \ldots, x_n]$ which contains a nonnil element of R is finitely generated. Also, for a ring R where $Nil(R)$ is a nonzero prime ideal with $Nil(R)^2 = (0)$, there is a ring A whose nilradical $Nil(A)$ is a divided prime such that R embeds naturally in A with $R/Nil(R)$ isomorphic to $A/Nil(A)$, $R_{Nil(R)}$ isomorphic to $A_{Nil(A)}$, and the corresponding total quotient rings, $T(R)$ and $T(A)$, are such that $T(R) \subset T(A)$ and $A \cap T(R) = R + Nil(T(R))$.

1 Introduction

We assume throughout that all rings are commutative with $1 \neq 0$. For such a ring R, we let $Z(R)$ denote the set of zero divisors of R and $Nil(R)$ denote the nilradical. We say that $Nil(R)$ is *divided* if it compares with each principal ideal of R (see [9] and [2]). If $Nil(R)$ is both divided and a prime ideal, we say that R is a *ϕ-ring*. For convenience we let $\mathcal{H}$ denote the class of all ϕ-rings. In [2], [3], [4], [5], and [6] the first-named author investigated this class of rings. In [2] and [4] he introduced the concepts of ϕ-pseudo-valuation rings and ϕ-chained rings. Also, D.F. Anderson and the first-named author made further investigation on the class $\mathcal{H}$ in [1] and introduced the concepts of ϕ-Prüfer rings and ϕ-Bézout rings. See Section 4 for definitions of these specific types of ϕ-rings. The "ϕ" in the name refers to the canonical map $\phi : T(R) \to R_{Nil(R)}$ from $T(R)$, the total quotient ring of R, to R localized at $Nil(R)$ which maps a fraction $a/b \in T(R)$ to its image in $R_{Nil(R)}$.

An ideal I of a ring R is said to be *a nonnil ideal* if it is not contained in $Nil(R)$. Recall from [5] that a ring R is called a nonnil-Noetherian ring if every nonnil ideal of R is finitely generated. To establish more consistency in nomenclature, we will refer to such rings as ϕ-Noetherian rings. In the first section of this paper, we will show that many of the properties of Noetherian domains are valid for the nonnil ideals of a ϕ-Noetherian ring. In Section 3, we study polynomial rings with one or several variables over ϕ-Noetherian rings. For example, we show that if R is a ϕ-Noetherian ring and $x_1, x_2, \ldots, x_n$ are indeterminates, then an ideal I of $R[x_1, x_2, \ldots, x_n]$ which contains a nonnil element of R is finitely generated. On the other hand, if $Nil(R)$ is not finitely generated, then $x_1R[x_1, x_2, \ldots, x_n] + Nil(R[x_1, x_2, \ldots, x_n])$ is not finitely generated.

Most, but not all, of our non-domain examples of ϕ-Noetherian rings are provided by the idealization construction $R(+)B$ arising from a ring R and an R-module B as in Huckaba's book [13, Chapter VI]. We recall this construction.

For a ring R, let B be an R-module. Then the idealization of B over R is the ring $R(+)B$ obtained from taking the product $R \times B$ and defining addition and multiplication of elements (r, b) and (s, c) in $R \times B$ by $(r, b) + (s, c) = (r + s, b + c)$ and $(r, b)(s, c) = (rs, sb + rc)$. Under these definitions $R(+)B$ becomes a commutative ring with identity. If R is reduced, $Nil(R(+)B) = (0)(+)B$. If R is an integral domain, $Nil(R(+)B)$ is prime.

2 Properties of ϕ-Noetherian Rings

We start with the following characterization of ϕ-Noetherian rings. It is a combination of several results from [5].

Theorem 2.1. ([5, Corollary 2.3 and Theorems 2.2, 2.4 and 2.6]) Let $R \in \mathcal{H}$. Then the following are equivalent.

1. R is a ϕ-Noetherian ring.
2. $R/Nil(R)$ is a Noetherian domain.
3. $\phi(R)/Nil(\phi(R))$ is a Noetherian domain.
4. $\phi(R)$ is a ϕ-Noetherian ring.
5. Each nonnil prime ideal of R is finitely generated.

In the following result, we show that a ϕ-Noetherian ring is related to a pullback of a Noetherian domain.

Theorem 2.2. Let $R \in \mathcal{H}$. Then R is a ϕ-Noetherian ring if and only if $\phi(R)$ is ring-isomorphic to a ring A obtained from the following pullback diagram:

$$\begin{array}{ccc} A & \longrightarrow & S \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/M \end{array}$$

where T is a zero-dimensional quasilocal ring containing A with maximal ideal M, $S = A/M$ is a Noetherian subring of T/M, the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

Proof. Suppose $\phi(R)$ is ring-isomorphic to a ring A obtained from the given diagram. By Theorem 2.1, it suffices to show that $\phi(R)$ is ϕ-Noetherian. Since T is a zero-dimensional quasilocal ring and M is a prime ideal of both T and A, $A \in \mathcal{H}$ with $Nil(A) = Z(A) = M$. Since $S = A/M$ is a Noetherian domain, A is a ϕ-Noetherian ring by Theorem 2.1. Thus R is a ϕ-Noetherian ring.

Conversely, suppose that R is a ϕ-Noetherian ring. Then, letting $T = R_{Nil(R)}$, $M = Nil(R_{Nil(R)})$, and $A = \phi(R)$ yields the desired pullback diagram. □

A rather nice property of ϕ-Noetherian rings is that homomorphic images are either Noetherian or ϕ-Noetherian. Generally speaking this is an exclusive "or", the exception being when the image is either an integral domain or a local Artinian ring.

Lemma 2.3. Let $R \in \mathcal{H}$. Then $Nil(R)$ is finitely generated if and only if R is either an integral domain or a local Artinian ring with nonzero maximal ideal, $Nil(R)$. In particular, if R is Noetherian and not an integral domain, then it is a local Artinian ring with maximal ideal $Nil(R) \neq (0)$.

Proof. Obviously, $Nil(R)$ is finitely generated if R is either an integral domain or a local Artinian ring. Thus it suffices to prove that if $Nil(R)$ is finitely generated and not the maximal ideal of R, then R is an integral domain. Let M be a maximal ideal of R. Since $Nil(R)$ is a divided prime ideal of R, $Nil(R)R_M = Nil(R_M)$ is a divided prime of R_M. Hence R_M is also in $\mathcal{H}$ and $Nil(R_M)$ is finitely generated. Since $Nil(R_M)$ is a divided prime and properly contained in MR_M, $MNil(R_M) = Nil(R_M)$. Thus $Nil(R_M) = (0)$ by Nakayama's Lemma. As this happens for each maximal ideal of R, $Nil(R)$ must be (0) and we have that R is an integral domain. □

Proposition 2.4. Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring and let $I \neq R$ be an ideal of R. If $I \subset Nil(R)$, then R/I is a ϕ-Noetherian ring. If $I \not\subset Nil(R)$, then $Nil(R) \subset I$ and R/I is a Noetherian ring. Moreover, if $Nil(R) \subset I$, then R/I is both Noetherian and ϕ-Noetherian if and only if I is either a prime ideal or a primary ideal whose radical is a maximal ideal.

Proof. Suppose that $I \subset Nil(R)$. Then $Nil(R/I) = Nil(R)/I$ is a divided prime ideal of R/I. Hence, $R/I \subset \mathcal{H}$. Now, let Q be a nonnil prime ideal of R/I. Then $Q = P/I$ for some prime ideal $P \not\subset Nil(R)$ of R. Since P is finitely generated, Q is finitely generated. Thus R/I is a ϕ-Noetherian ring by Theorem 2.1.

Now, suppose that $I \not\subset Nil(R)$. Then $Nil(R) \subset I$ since $Nil(R)$ is divided. Let Q be a prime ideal of R/I. Then $Q = P/I$ for some nonnil prime ideal P of R such that $I \subset P$. Hence Q is finitely generated since P is finitely generated. Thus R/I is a Noetherian ring since each prime ideal is finitely generated.

The third statement follows from Lemma 2.3. □

Corollary 2.5. Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring. Then a homomorphic image of R is either a ϕ-Noetherian ring or a Noetherian ring.

An almost Dedekind domain that is not Dedekind is a ring that is locally Noetherian but not Noetherian. If a domain is locally Noetherian and each nonzero element is contained in at most finitely many maximal ideals, then the domain is Noetherian (see, for example, [14, Exercise # 10, page 73]). A similar statement holds for rings in $\mathcal{H}$.

Proposition 2.6. Let $R \in \mathcal{H}$ and suppose that R_M is ϕ-Noetherian for every maximal ideal M of R, and that each nonnil nonunit of R lies in only a finite number of maximal ideals of R, then R is a ϕ-Noetherian ring.

Proof. Set $D = R/Nil(R)$ (observe that D is an integral domain). If $Nil(R)$ is a maximal ideal of R, then D is Noetherian (being a field), and hence R is ϕ-Noetherian by Theorem 2.1. Thus assume that $Nil(R)$ is not a maximal ideal of R. Let J be a maximal ideal of D. Then $J = M/Nil(R)$ for some maximal ideal M of R. Since $R_M \in \mathcal{H}$ is ϕ-Noetherian by hypothesis and D_J is ring-isomorphic to $R_M/Nil(R)R_M$, we conclude that D_J is Noetherian. Since each nonnil nonunit of R lies in only a finite number of maximal ideals of R by hypothesis, we conclude that every nonzero nonunit of D lies in only a finite number of maximal ideals of D. Thus, D is Noetherian by [14, Exercise # 10, page 73]. Hence R is ϕ-Noetherian by Theorem 2.1. □

Note that as long as $Nil(R)$ is prime and locally divided (i.e., $Nil(R)R_M$ is a divided prime of R_M), $Nil(R)$ is divided. The key to the proof is that for each nonnil element $r \in R$, the conductor of r into $Nil(R)$ is $Nil(R)$ since $Nil(R)$ is prime. Thus no nonnil element can be transformed into a nilpotent element under localization. Hence for each maximal ideal M, rR_M will contain $Nil(R)R_M = Nil(R_M)$. For a given nilpotent $n \in Nil(R)$ and maximal ideal M, there are elements $s, t \in R$ with t not in M such that $sr = tn$. It follows that the ideal $(r :_R n) = R$; i.e., $n \in rR$. Thus the hypothesis in Proposition 2.6 could be superficially weakened to having $Nil(R)$ prime and each R_M a ϕ-Noetherian ring (thus a ϕ-ring) with each nonnil element in only finitely many maximal ideals. However, the assumption that $Nil(R)$ is prime cannot be eliminated. For example, let V be a two dimensional valuation domain with principal maximal ideal M and height one prime P. Choose any proper P-primary ideal J and let $R = V/J \oplus V/J$. Then the nilradical of R is $P/J \oplus P/J$ which is neither prime nor divided – simply consider the idempotents $(1,0)$ and $(0,1)$ and nilpotent multiples of each. On the other hand, $M/J \oplus V/J$ and $V/J \oplus M/J$ are the only maximal ideals of R. Localizing at either yields the ring V/J which is a ϕ-Noetherian ring. Thus R is not ϕ-Noetherian, but it is locally ϕ-Noetherian with only finitely many maximal ideals. Note that if J does not contain P^2, then V/J is definitely not formed by the idealization of a module over an integral domain. On the other hand, if we take $V = K + \mathrm{Y}K[[\mathrm{Y}]] + \mathrm{Z}K((\mathrm{Y}))[[\mathrm{Z}]]$ and $J = \mathrm{Z}^2 K((\mathrm{Y}))[[\mathrm{Z}]]$, then V/J is isomorphic to the idealization of $K((\mathrm{Y}))$ over $K[[\mathrm{Y}]]$.

Our next result shows that a ϕ-Noetherian ring will satisfy the conclusion of the Principal Ideal Theorem (and the Generalized Principal Ideal Theorem).

Theorem 2.7. Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring and let P be a prime ideal. If P is minimal over an ideal generated by n or fewer elements, then the height of P is less than or equal to n. In particular, each prime minimal over a nonnil element of R has height one.

Proof. The ring $D = R/Nil(R)$ is a Noetherian domain by Theorem 2.1. Assume P is minimal over the ideal $I = (a_1, a_2, \ldots, a_n)$. If $I \subset Nil(R)$, there is nothing to prove since we would have $P = Nil(R)$, the prime of height 0. Thus we may assume I is not nilpotent. Since $Nil(R)$ is divided, I properly contains $Nil(R)$. Thus $I/Nil(R)$ can be generated by n (or fewer) elements. Since D is Noetherian, the height of $P/Nil(R)$ is less than or equal to n by the Generalized Principal Ideal Theorem ([14, Theorem 152]). Thus P has height less than or equal to n. □

Other statements about primes of Noetherian rings that can be easily adapted to statements about primes of ϕ-Noetherian rings include the following. We leave the proofs to the reader.

Proposition 2.8. [14, Theorem 145] Let $R \in \mathcal{H}$ satisfy the ascending chain condition on radical ideals. If R has an infinite number of prime ideals of height 1, then their intersection is $Nil(R)$.

Proposition 2.9. [14, Theorem 153] Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring and P be a nonnil prime ideal of R of height n. Then there exist nonnil elements $a_1, \ldots, a_n$ in R such that P is minimal over the ideal $(a_1, \ldots, a_n)$ of R, and for any i $(1 \leq i \leq n)$, every (nonnil) prime ideal of R minimal over $(a_1, \ldots, a_i)$ has height i.

Proposition 2.10. [14, Theorem 154] Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring and let I be an ideal of R generated by n elements with $I \neq R$. If P is a prime ideal containing I with P/I of height k, then the height of P is less than or equal to $n+k$.

3 Polynomial Rings over ϕ-Noetherian Rings

By the Hilbert Basis Theorem, a polynomial ring in finitely many indeterminates over a Noetherian ring is Noetherian. The analogous statement cannot be made for a (nontrivial) ϕ-Noetherian R since a nonzero nilpotent element of R cannot be a multiple of an indeterminate. Also, if $Nil(R)$ is not finitely generated, then the ideal $xR[x] + Nil(R)R[x]$ is not finitely generated. On the other hand, a local Artinian ring that is not a field is a ϕ-Noetherian ring. So it is possible for a polynomial ring over a ϕ-Noetherian ring to be Noetherian. We start this section with dimension related statements.

Our first consideration involves the Jaffard property, we will show that if R is an n-dimensional ϕ-Noetherian ring, then $R[x_1, x_2, \ldots, x_m]$ has dimension $n+m$ for each $m > 0$.

Recall that if R is a Noetherian ring, P is a height n prime of R and Q is a prime of $R[x]$ that contracts to P but properly contains $PR[x]$, then $PR[x]$ has height n and Q has height $n+1$ [14, Theorem 149].

Proposition 3.1. Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring and let P be a height n prime of R. If Q is a prime of $R[x]$ that contracts to P but properly contains $PR[x]$, then $PR[x]$ has height n and Q has height $n+1$.

Proof. Since $Nil(R)$ is the minimal prime of R, $Nil(R[x]) = Nil(R)R[x]$ is the minimal prime of $R[x]$. Thus we may shift the setting to $D = R/Nil(R)$ and $D[x]$ (identified with $R[x]/Nil(R[x])$). By Theorem 2.1, D is a Noetherian domain. Moreover, $P/Nil(R)$ is a height n prime of D and $Q/Nil(R[x])$ is a prime of $D[x]$ that contracts to $P/Nil(R)$ and properly contains $(P/Nil(R))D[x]$. Thus the height of $(P/Nil(R))D[x]$ and $PR[x]$ is n and the height of $Q/Nil(R[x])$ and Q is $n+1$. $\square$

Similar height restrictions exist for the primes of $R[x_1, x_2, \ldots, x_m]$.

Proposition 3.2. Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring and let P be a height n prime of R. If Q is a prime of $R[x_1, \ldots, x_m]$ that contracts to P but properly contains $PR[x_1, \ldots, x_m]$, then $PR[x_1, \ldots, x_m]$ has height n and Q has height at most $n+m$. Moreover the prime $PR[x_1, \ldots, x_m] + (x_1, \ldots, x_m)R[x_1, \ldots, x_m]$ has height $n+m$.

Proof. Let $D = R/Nil(R)$. Then D is Noetherian as is each of the rings $D[x_1, \ldots, x_k] \equiv R[x_1, \ldots, x_k]/Nil(R)[x_1, \ldots, x_k]$. Repeated applications of [14, Theorem 149] shows that the image of Q in $D[x_1, \ldots, x_m]$ has height at most $n+m$. Hence the same restriction holds for the height of Q. $\square$

Corollary 3.3. If R is a finite dimensional ϕ-Noetherian ring of dimension n, then $dim(R[\mathrm{X}_1, \ldots, \mathrm{X}_m]) = n + m$ for each integer $m > 0$.

As noted above, if $Nil(R)$ is not finitely generated, then $\mathrm{X}R[\mathrm{X}]+Nil(R)[\mathrm{X}]$ is not finitely generated. In Examples 3.6 and 3.7, we will show how to construct a ϕ-Noetherian ring R that is not Noetherian, but where $R[\mathrm{X}]$ has finitely generated primes that contract to $Nil(R)$. In our next result we show that each ideal of $R[\mathrm{X}]$ that contracts to a nonnil ideal of R is finitely generated.

Proposition 3.4. Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring. If I is an ideal of $R[\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n]$ for which $I \cap R$ is not contained in $Nil(R)$, then I is a finitely generated ideal of $R[\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n]$.

Proof. The key to the proof is that if $I \cap R$ is not contained in $Nil(R)$, then any single nonnil element in this intersection is enough to generate the nilradical of $R[\mathrm{X}_1, \ldots, \mathrm{X}_n]$. Since $R/Nil(R)$ is Noetherian, $(I/Nil(R))[\mathrm{X}_1, \ldots, \mathrm{X}_n]$ is finitely generated. Let $\{f_1, f_2, \ldots, f_m\} \subset I$ generate the image of I modulo $Nil(R)[\mathrm{X}_1, \ldots, \mathrm{X}_n]$. To get a finite set of generators for I, simply add any single nonnil element r of $I \cap R$ to the set $\{f_1, f_2, \ldots, f_m\}$. Since $rNil(R) = Nil(R)$, the set $\{r, f_1, \ldots, f_m\}$ is a finite set of generators for I. $\square$

Since three distinct comparable primes of $R[\mathrm{X}]$ cannot contract to the same prime of R, a consequence of Proposition 3.4 is that the search for primes of $R[\mathrm{X}]$ that are not finitely generated can be restricted to those of height one. A similar statement can be made for primes of $R[\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n]$.

Corollary 3.5. Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring and let P be a prime of $R[\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n]$. If P has height greater than n, then P is finitely generated.

The ring in our next example shows that the converse of Proposition 3.4 does not hold even for prime ideals.

Example 3.6. Let $R = D(+)L$ be the idealization of $L = K((\mathrm{Y}))/D$ over $D = K[[\mathrm{Y}]]$. Then R is a quasilocal ϕ-Noetherian ring with nilradical $Nil(R)$ isomorphic to L. Consider the polynomial $g(\mathrm{X}) = 1 - \mathrm{YX}$. Since the coefficients of g generate D as an ideal and g is irreducible, $P = gD[\mathrm{X}]$ is a height one principal prime of $D[\mathrm{X}]$ with $P \cap D = (0)$. Each nonzero element of L can be written in the form d/y^n where n is a positive integer, y denotes the image of Y in L and $d = d_0 + d_1\mathrm{Y} + \cdots + d_{n-1}\mathrm{Y}^{n-1}$ with $d_0 \neq 0$. Given such an element, let $f(\mathrm{X}) = 1 + y\mathrm{X} + \cdots + y^{n-1}\mathrm{X}^{n-1} \in L[\mathrm{X}]$. Then $g(\mathrm{X})(df(\mathrm{X})/y^n) = d/y^n$ since $dy^n/y^n = 0$ in L. It follows that $g(\mathrm{X})R[\mathrm{X}]$ is a height one principal prime of $R[\mathrm{X}]$ that contracts to $Nil(R)$.

Since the domain D in the above example is a DVR, each prime of $D[\mathrm{X}]$ that contracts to (0) is principal and generated by an irreducible polynomial whose coefficients generate D as an ideal. Let $h(X)$ be such a polynomial. If $h(X)$ has the form $1 - YXk(X)$ for some polynomial $k(X) \in D[X]$, then the corresponding prime $Q = hD[X]$ will be such that $QR[X] = hR[X]$ is a height one principal prime of R that contracts to $Nil(R)$. The basic scheme of the proof is the same as for $g(X) = 1 - YX$. Given $d = d_0 + d_1 Y + \cdots + d_{n-1} Y^{n-1}$ with $d_0 \neq 0$ and n positive, simply replace $f(X)$ by the polynomial $k(X) = 1 + yXk(X) + (yXk(X))^2 + \cdots + (yXk(X))^{n-1}$. Then, as above, $h(X)(dk(X)/y^n) = d/y^n$. It follows that $(h(X), 0)$ generates a height one principal prime of $R[X]$. On the other hand, the irreducible polynomial $j(X) = Y - X$ also generates a height one prime of $D[X]$, but it does not generate a height one principal prime of $R[X]$ since its leading coefficient is a unit. A slightly more complicated argument shows that if $r(X)$ is an irreducible polynomial whose coefficients generate D as an ideal, then $rR[X]$ is prime if and only if $r(X)$ is of the form $u - YXt(X)$ for some unit u of D and some nonzero polynomial $t(X) \in D[X]$. Let $d(X) \in L[X]$ be such that $r(X)d(X)$ is a nonzero element of $L[X]$. Since D is a valuation domain, there is a largest positive integer m such that some coefficient of $d(X)$ is of the form v/y^m where v is (the image of) a unit of D. Let q be the largest integer such that X^q has a coefficient of this form. Also let p be the largest integer such that r_p, the coefficient on X^p, is a unit of D. In the product $r(X)d(X)$ the coefficient on X^{p+q} has the form $r_0 d_{p+q} + \cdots r_{p-1} d_{q+1} + r_p d_q + r_{p+1} d_{q-1} + \cdots r_{p+q} d_0$. Since the exponents on Y in $r_0, \ldots, r_{p-1}$ are nonnegative and those on y in $d_{p+q}, \ldots, d_{q+1}$ are larger than $-m$, the resulting power on y in the corresponding products are all strictly larger than $-m$. For $i > p$, the coefficient on Y in r_i is positive and that on y in the corresponding d_j is greater than or equal to $-m$. Thus the coefficient on X^{p+q} is a unit multiple of y^{-m}. It follows that a necessary condition for $rR[X]$ to contract to $Nil(R)$ is that $r(X)$ have the form $u - YXt(X)$ for some unit u of D and some nonzero polynomial $t(X) \in D[X]$.

There are non-Noetherian ϕ-Noetherian rings with height one principal primes that contract to the nilradical but are not generated by a polynomial whose constant term is a unit nor do the coefficients generate the entire ring as an ideal.

Example 3.7. Let $D = K[Y, Z]$ and let $L = K(Y, Z)/D_P$ where P is the principal prime YD. Then the ring $R(+)L$ is a ϕ-Noetherian ring. Since D is a UFD, each height one prime of $D[X]$ that contracts to (0) is principal and generated by an irreducible polynomial whose coefficients generate an ideal of D whose inverse is D. Let $Q = g(X)D[X]$ be such a prime and suppose that $g(X) = a - YXt(X)$ where $a \in D \backslash YD$ (and $t(X) \in D[X]$ is nonzero). Since D_P is a DVR with maximal ideal YD_P, each nonzero element of L can be written as a quotient d/y^n for some $d \in D_P \backslash PD_P$ and some positive integer n. As in the previous example, let n be a positive integer and let $f(X) = a^{n-1} + a^{n-2} yXt(X) + \cdots y^{n-1} X^{n-1} t(X)^{n-1}$ and $d/y^n \in L$ with $d \in D_P \backslash PD_P$. Since a is a unit of D_P, $a^{-n} d/y^n$ is an element of L. Hence we have $(a - YX)(f(X)a^{-n} d/y^n) = d/y^n \in g(X)R[X]$. It follows that $g(X)R[X]$

is a principal height one prime of $R[\text{x}]$ that contracts to $Nil(R)$.

The argument given above characterizing the prinicipal height one primes of the polynomial ring in Example 3.6 can be adapted to this ring. Let $r(\text{x})D[\text{x}]$ be a height one prime of $D[\text{x}]$ that contracts to (0) where $r(\text{x})$ is an irreducible polynomial whose coefficients generate an ideal of D with inverse equal to D. Let $d(\text{x}) \in L[\text{x}]$ be such that $r(\text{x})d(\text{x})$ is a nonzero element of $L[\text{x}]$. As above let p be the largest integer where the coefficient on x^p in $r(\text{x})$ is not in $\text{y}D$ and let m and q be such m is the largest integer such that some coefficient of $d(\text{x})$ is of the form d_k/y^m for some unit d_k of D and let q be the largest value of k where this maximum occurs. Similar analysis of the coefficient on x^{p+q} shows that this coefficient is of the form e/y^m for some unit e of D_P. It follows that the height one principal primes of $R[\text{x}]$ that contract to $Nil(R)$ are of the form $r(\text{x})R[\text{x}]$ where $r(\text{x}) = r_0 - \text{yx}k(\text{x})$ is an irreducible polynomial with $r_0 \in D \backslash \text{y}D$ and $k(\text{x})$ nonzero. Moreover, each such polynomial generates such a prime.

We end this section with the following problem.

Problem. Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring. Characterize the finitely generated height one primes of $R[\text{x}]$.

4 Embedding R into a ϕ-ring

We begin this section with a question.

Question. Let R be a ring whose nilradical $Nil(R)$ is prime. Is it possible to embed R into a ϕ-ring A with nilradical $Nil(A)$ in such a way that all of the following hold?

(I) $A_{Nil(A)}$ is isomorphic to $R_{Nil(R)}$,

(II) $A/Nil(A)$ is isomorphic to $R/Nil(R)$, and R embeds in A in such a way that

(III) $Z(R) \subset Z(A)$,

(IV) $T(R) \subset T(A)$,

(V) $A \cap T(R) = R + Nil(T(R))$, and

(VI) $A = R + Nil(A)$.

For all but (I), it is easy to answer "Yes" if R is formed by taking the idealization of a module over an integral domain. For a D-module B, we let $Z_D(B) = \{r \in D \mid rb = 0 \text{ for some nonzero } b \in B\}$. If D is an integral domain, then for $R = D(+)B$, $Z(R) = \{(r, c) \mid r \in Z_D(B), c \in B\}$ [13, Theorem 25.3].

Lemma 4.1. Let D be an integral domain and let B be a D-module. Then there is a D-module C containing B which is a divisible D-module such that $Z_D(C) = Z_D(B)$.

Proof. Let $S = D \backslash Z_D(B)$ and let $U(D_S)$ denote the units of D_S. Then B_S is both a D-module and a D_S-module and B naturally embeds as a D-submodule of B_S. Also, each nonunit of D_S is a zero divisor on B_S, so $Z_D(B_S) = Z_D(B)$ and $Z_{D_S}(B_S) = Z_D(B)_S = D_S \backslash U(D_S)$. By [10, Proposition VII.1.4], there is a divisible D_S-module C containing B_S. Since each element of S is a unit of D_S, $Z_D(C) = Z_D(B)$. We also have that C is a divisible D-module. □

An immediate consequence is the following theorem.

Theorem 4.2. Let $R = D(+)B$ where D is an integral domain and B is a D-module. Then there is a divisible D-module C such that $A = D(+)C$ is a ϕ-ring for which

(i) $A/Nil(A)$ is isomorphic to $R/Nil(R)$,

(ii) $Z(R) \subset Z(A)$,

(iii) $T(R) \subset T(A)$,

(iv) $A \cap T(R) = R + Nil(T(R))$, and

(v) $A = R + Nil(A)$.

Proof. As in Lemma 4.1, let $S = D \backslash Z_D(B)$ and let C be a divisible D_S-module that contains B_S. Set $A = D(+)C$. Then $Nil(R) = (0)(+)B$ and $Nil(A) = (0)(+)C$. Thus $R/Nil(R) = D = A/Nil(A)$ and $A = R + Nil(A)$. As in the proof of Lemma 4.1, we have $Z_D(B) = Z_D(C)$ and $Z_{D_S}(B_S) = Z_{D_S}(C) = D_S \backslash U(D_S)$ where $U(D_S)$ is the set of units of D_S. With regard to zero divisors, we have $Z(R) = Z_D(B) \times B$ and $Z(A) = Z_D(C) \times C$. From this it is easy to see that $T(R)$ can be identified with $D_S(+)B_S$ and $T(A)$ with $D_S(+)C$. Thus $A \cap T(R) = R(+)B_S$.

To see that $Nil(A)$ is divided, consider $(0, c) \in Nil(A)$ and $(r, b) \in A \backslash Nil(A)$. Since C is a divisible D-module, there is an element $f \in C$ such that $rf = c$. Clearly, $(r, b)(0, f) = (0, c)$ and therefore $Nil(A)$ is divided. □

What is missing in the conclusions in Theorem 4.2 is the statement about the localizations $R_{Nil(R)}$ and $A_{Nil(A)}$. In some sense, these were not simply ignored. The module C could be too large (since $C \oplus V$ is divisible for each vector space V over the quotient field of D). Rather than showing that it is possible to cut C down to a divisible module C' for which $A' = D(+)C'$ would satisfy all six of the desired properties, we will establish the existence of a ϕ-ring satisfying all six under the somewhat more general assumption that the square of the nilradical is the zero ideal (no matter whether the original ring R is formed by idealization or not).

In Chapter VII of [10], L. Fuchs and L. Salce present a useful technique for constructing a divisible module of projective dimension one over a given integral domain. They then use this module to generate a divisible module containing a specific D-module B. We make use of the general notion of their construction to establish the following theorem.

Theorem 4.3. Let R be a ring with nilradical $Nil(R)$. If $Nil(R)$ is prime and $Nil(R)^2 = (0)$, then there is a ring A with nilradical $Nil(A)$ such that (i) $Nil(A)$ is a divided prime of A, (ii) $A_{Nil(A)}$ is isomorphic to $R_{Nil(R)}$, (iii) $A/Nil(A)$ is isomorphic to $R/Nil(R)$, and R embeds in A in such a way that (iv) $Z(R) \subset Z(A)$, (v) $T(R) \subset T(A)$ and (vi) $A \cap T(R) = R + Nil(T(R))$, and (vii) $A = R + Nil(A)$.

Proof. Assume $Nil(R)$ is a prime ideal of R with $Nil(R)^2 = (0)$. Then $Nil(T(R))$ is a common prime ideal of $T(R)$ and $R + Nil(T(R))$ whose square is zero. Note that there are natural isomorphisms between $R/Nil(R)$ and $[R + Nil(T(R))]/Nil(T(R))$ and between the localizations $R_{Nil(R)}$ and $[R + Nil(T(R))]_{Nil(T(R))}$. Thus we may assume $Nil(R) = Nil(T(R))$.

We start by constructing a ring A' such that (i) $Nil(A')$ is a divided prime of A', (ii) $A'_{Nil(A')}$ is isomorphic to $T(R)_{Nil(R)}$, (iii) $A'/Nil(A')$ is isomorphic to $T(R)/Nil(R)$, and $T(R)$ embeds in A' (and thus in $T(A')$) in such a way that (iv) A' can be identified with $T(R) + Nil(A')$ and (v) the image of $T(R)$ in $A'_{Nil(A')}$ is the same as the image of A'. Given such a ring A', the ring A resulting from taking the pullback of $R/Nil(R)$ along $Nil(A')$ will satisfy conditions (i)–(vii). Pictorially, A is the subring obtained from the following pullback diagram with i the inclusion map and σ the restriction of the isomorphism from $T(R)/Nil(R)$ to $A'/Nil(A')$.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & R/Nil(R) \arrow[d, "\sigma"] \\ A' \arrow[r] & A'/Nil(A') \end{tikzcd}$$

Let $S = R\backslash Z(R)$, $N = Nil(R)\backslash\{0\}$ and $C = Z(R)\backslash N$. If C contains only 0 (equivalently, $Z(R) = Nil(R)$), there is nothing to prove as $Nil(R)$ will be a divided prime (since we have assumed $Nil(R) = Nil(T(R))$). So we may assume $Z(R)$ properly contains $Nil(R)$.

Let $\mathcal{F}$ denote the subset of $N \times C^{\mathbb{N}}$ consisting of those elements of the form $\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots)$ such that there is a positive integer n_α where $\alpha_j \neq 0$ for each $j \leq n_\alpha$ and $\alpha_k = 0$ for each $k > n_\alpha$. Let $E = T(R)[\mathcal{X}]$ where $\mathcal{X} = \{\mathrm{x}_\alpha\}$ is a set of indeterminates indexed over $\mathcal{F}$. For each $\alpha \in F$ with $n_\alpha > 1$, let $\alpha - k = (\alpha_0, \alpha_1, \dots, \alpha_{n_\alpha - k}, 0, \dots)$ for $1 \leq k < n_\alpha$. Let J be the ideal of E generated by (i) the products of the form $\mathrm{x}_\alpha \mathrm{x}_\beta$, (ii) the products of the form $m\mathsf{x}_\alpha$ for each $m \in Nil(R)$, (iii) the elements of the form $\alpha_{n_\alpha}\mathsf{x}_\alpha - \mathsf{x}_{\alpha-1}$ for those α with $n_\alpha > 1$, and (iv) the elements of the form $\alpha_1\mathsf{x}_\alpha - \alpha_0$ for those α with $n_\alpha = 1$. By (i), the square of each x_α is in J. Also, J is contained in the ideal $Q = Nil(R) + \mathcal{X}T(R)[\mathcal{X}]$, the set of polynomials with constant term a nilpotent of $T(R)$ (and R). Since $Nil(R)$ is a prime ideal of $T(R)$, Q is a prime ideal of E. As a power of each element of Q is contained in J, Q/J is the nilradical of $A' = E/J$. Moreover, by the construction of J, J contains Q^2. Thus $(Q/J)^2 = (0)$ and we have that $Nil(A')^2 = (0)$. We next show that $J \cap T(R) = (0)$ and no single x_α is in J. Note that the former implies that $T(R)$ embeds naturally into A'. Moreover, since $Nil(A') = Q/J$ with $Q = Nil(R) + \mathcal{X}T(R)[\mathcal{X}]$, we may view A' as $T(R) + Nil(A')$ and we have that there is a natural isomorphism between $A'/Nil(A')$ and $T(R)/Nil(R)$.

Since J is contained in $Nil(R) + \mathcal{X}T(R)[\mathcal{X}]$, $J \cap T(R)$ is contained in $Nil(R)$. By way of contradiction assume that some nonzero nilpotent of $T(R)$ is contained in J. Based on the generating set described above, an arbitrary element of J can be written as a finite sum of the form $\sum p_\alpha(\alpha_1\mathsf{x}_\alpha - \alpha_0)\prod \mathsf{x}_\beta + \sum q_\gamma(\gamma_{n_\gamma}\mathsf{x}_\gamma - \mathsf{x}_{\gamma-1})\prod \mathsf{x}_\mu + \sum r_\sigma \prod \mathsf{x}_\tau + \sum y_\lambda \prod \mathsf{x}_\rho$ where $n_\alpha = 1$ for each α, the products $\prod \mathsf{x}_\beta$, $\prod \mathsf{x}_\mu$, $\prod \mathsf{x}_\tau$ and $\prod \mathsf{x}_\rho$ are finite with $\prod \mathsf{x}_\beta$ and $\prod \mathsf{x}_\mu$ arbitrary (including "empty products"), at least one term in each $\prod \mathsf{x}_\gamma$, at least two in each $\prod \mathsf{x}_\rho$ and $p_\alpha, q_\gamma, r_\sigma, y_\lambda \in T(R)$ with each r_σ nilpotent. To have such a sum reduce to a nonzero nilpotent of $T(R)$, it must be that some $\prod \mathsf{x}_\beta$ are empty. To then "cancel out" the terms $p_\alpha\alpha_1\mathsf{x}_\alpha$, not all of the remaining sums can contain only products of two or more variables. The part of the expression that contributes nonzero constant terms and constant multiples of single variables has the form $\sum u_\alpha(\alpha_1\mathsf{x}_\alpha - \alpha_0) + \sum v_\alpha(\alpha_1\mathsf{x}_\alpha - \alpha_0)\mathsf{x}_\beta + \sum z_\gamma(\gamma_{n_\gamma}\mathsf{x}_\gamma - x_{\gamma-1}) + \sum w_\sigma\mathsf{x}_\tau$. All others involve only products of two or more indeterminates. Note that each w_σ is nilpotent. If each u_α is nilpotent, then there is no nonzero constant term since $Nil(R)^2 = (0)$. So we may assume some u_α is not nilpotent. For each such nonnil u_α, $u_\alpha\alpha_1$ is not in $Nil(R)$ since $Nil(R)$ is prime and α_1 is not nilpotent. In the sums $\sum v_\alpha(\alpha_1\mathsf{x}_\alpha - \alpha_0)\mathsf{x}_\beta$ and $\sum w_\sigma\mathsf{x}_\tau$ all α_0 and w_σ are nilpotent, so none of these can cancel with the nonnil $u_\alpha\alpha_1$'s. The same is true for each z_γ that is nilpotent. Thus there are nonnil z_γ's. Since $n_\gamma > 1$ for each γ, there must be γ's with maximal values of n_γ and z_γ not nilpotent. Such terms cannot be cancelled in the sum. Thus no nonzero element of J is contained in $Nil(R)$. Note that if x_ψ is in J for some ψ, then so is $\mathsf{x}_{\psi-1} = \psi_{n_\psi}\mathsf{x}_\psi - (\psi_{n_\psi}\mathsf{x}_\psi - \mathsf{x}_{\psi-1})$ (or ψ_0 if $n_\psi = 1$). By continuing, we would eventually have ψ_0 in J, a contradiction.

For the remainder of the proof we let x_α denote the image of x_α in A'.

Let t be a nonnil element of A'. Then there is a nonnil element $r \in T(R)\backslash Nil(R)$ and a nilpotent element $g \in Nil(A')$ such that $t = r - g$. Since $g^2 = 0$, $t(r + g) = r^2$. Thus to show that each nonnil element of A' divides each nilpotent element, it suffices to show that each nonnil element of

$T(R)$ divides each nilpotent in A'. There is nothing to prove for the units of $T(R)$ as each remains a unit under the embedding into A'. So we may further reduce the problem to showing that each nonnil element $r \in C$ divides each nilpotent of A'. For a nonzero nilpotent $b \in Nil(R)$, let $\alpha = (b, r, 0, \dots)$. Then $rx_\alpha = b$ since $r\mathrm{x}_\alpha - b$ is one of the generators of J. Similarly, for x_α with $n_\alpha > 1$, let $\beta = (\alpha_0, \alpha_1, \dots, \alpha_n, r, 0, \dots)$. Then $rx_\beta = x_\alpha$ since $r\mathrm{x}_\beta - \mathrm{x}_\alpha$ is in J. Since $Nil(R)$ and the set $\{x_\alpha\}$ generate $Nil(A')$, $Nil(A')$ is a divided prime.

It is only slightly more complicated to take care of statement (ii) dealing with the localizations $T(R)_{Nil(R)}$ and $A'_{Nil(A')}$.

The easiest thing to prove is that $T(R)_{Nil(R)}$ embeds naturally in $A'_{Nil(A')}$. For this consider two equivalent fractions t/s and v/q of $A'_{Nil(A')}$ where $s, t, q, v \in T(R)$. To be equivalent, there must be a nonnil element $z \in A'$ such that $z(sv - qt) = 0$. Since $A' = T(R) + Nil(A')$, there are elements $u \in T(R)$ and $f \in Nil(A')$ such that $z = u + f$. The product $z(u - f) = u^2 \in T(R) \backslash Nil(R)$ since $f^2 = 0$. It follows that $u^2(sv - qt) = 0$ and the fractions t/s and v/q are also equivalent in $T(R)_{Nil(R)}$.

To complete the proof that this embedding is surjective it suffices to show that each member of the set $\{x_\alpha\}$ is equivalent to a quotient of the form n/q for some nilpotent $n \in T(R)$ and nonnil $q \in T(R)$. This is quite simple. Let $x_\sigma \in \{x_\alpha\}$ and consider the product $q = \prod \sigma_i$ with i ranging from 1 to n_σ (the last nonzero σ_j). This is a product of nonnil elements of $T(R)$ so q is nonnil. By the construction of J, $\sigma_n x_\sigma = x_{\sigma-1}$, $\sigma_j x_{\sigma-k} = x_{\sigma-(k+1)}$ for $k = n_\sigma - j$ and $1 < j < n_\sigma$, and $\sigma_1 x_{\sigma-k} = \sigma_0$ for $k = n_\sigma - 1$. Thus $qx_\sigma = \sigma_0$. In the localization $T(R)_{Nil(R)}$, we have $x_\sigma = \sigma_0/q$. Therefore we may view $T(R)_{Nil(R)}$ and $A'_{Nil(A')}$ as the same ring. □

As mentioned in the introduction, several specific types of ϕ-rings have been studied. A general definition for a ϕ-BLANK ring is that a ϕ-ring R is a ϕ-BLANK ring if $R/Nil(R)$ is a BLANK domain. For each of the types that have been studied so far, the specific definition involves properties of the nonnil ideals of R, sometimes with regard to the image of such ideals in $\phi(R)$. In each case, a consequence of the definition is that R is a ϕ-BLANK ring if and only if $R/Nil(R)$ is a BLANK domain. For example, given a ϕ-ring R, it is (i) a ϕ-chained ring if the nonnil ideals are linearly ordered – equivalent to $R/Nil(R)$ being chained (i.e., a valuation domain); (ii) a ϕ-pseudo-valuation ring if each nonnil prime is strongly prime with respect to $T(R)$ (meaning if $rt \in P$ for some $r, t \in T(R)$, then either $r \in P$ or $t \in P$) – equivalent to $R/Nil(R)$ being a pseudo-valuation domain; (iii) a ϕ-Prüfer ring if for each finitely generated nonnil ideal I, $\phi(I)$ is an invertible ideal of $\phi(R)$ – equivalent to $R/Nil(R)$ being a Prüfer domain; (iv) a ϕ-Bézout ring if each finitely generated nonnil ideal is principal – equivalent to $R/Nil(R)$ being a Bézout domain; and (v) a ϕ-Mori ring if it satisfies the ascending chain condition on those nonnil ideals I for which $\phi(I)$ is divisorial as an ideal of $\phi(R)$ – equivalent to $R/Nil(R)$ being a Mori domain (a domain with a.c.c. on divisorial ideals). [For references see [4], [2], [3], [1], and [8].]

For each of these special types of ϕ-rings, we have the following corollary to Theorem 4.3.

Corollary 4.4. Let R be a ring with nonzero nilradical $Nil(R)$. For BLANK any one of "chained", "pseudo-valuation", "Prüfer", "Bézout", "Mori", and "Noetherian", if $Nil(R)$ is prime with $R/Nil(R)$ a BLANK domain and $Nil(R)^2 = (0)$, then there is a ϕ-BLANK ring A for which statements (I)–(VI) hold for the pair R and A.

Note that with regard to idealization, if D is an integral domain and B is a D-module, then the process used in the proof of Theorem 4.3 can be used to build a divisible D-module C such that the pair $R = D(+)B$ and $A = D(+)C$ satisfy statements (I)–(VI). It follows that if D is Noetherian, the resulting ring A is a ϕ-Noetherian ring.

We have been unable to extend Theorem 4.3 to rings with prime nilradicals whose squares are not zero. For some such rings it is not hard to show that it is impossible to find a ϕ-ring A that satisfies conditions (I) through (VI). In particular, one necessary condition for the existence of A is that each nonzero nilpotent with a nonnil annihilator must annihilate $Nil(R)$. To see this assume $rn = 0$ for some nonnil element $r \in R$ and nonzero nilpotent n. If $Nil(A)$ is divided, then for each $b \in Nil(R)$, there is a nilpotent $m \in Nil(A)$ such that $rm = b$. Since $nr = 0$, we must have $nb = 0$ as well. Hence $nNil(R) = (0)$. A specific example of a ring with prime nilradical which cannot be embedded in a ϕ-ring is the ring $R = K[\text{Y}, \text{Z}]/(\text{Y}^3, \text{YZ})$. The nilradical of R is the prime ideal $yK[y, z]$ (where the lower case letters represent the images of the indeterminates in R) and y has a nonnil annihilator, namely the element z. As $y^2 \neq 0$, it is impossible to embed R in a ring whose nilradical is divided.

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