ON ϕ-DEDEKIND RINGS AND ϕ-KRULL RINGS

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Abstract. The purpose of this paper is to introduce two new classes of rings that are closely related to the classes of Dedekind domains and Krull domains. Let $\mathcal{H} = \{R \mid R$ is a commutative ring with $1 \neq 0$ and $Nil(R)$ is a divided prime ideal of $R\}$. Let $R \in \mathcal{H}$, $T(R)$ be the total quotient ring of R, and set $\phi : T(R) \longrightarrow R_{Nil(R)}$ such that $\phi(a/b) = a/b$ for every $a \in R$ and $b \in R \setminus Z(R)$. Then ϕ is a ring homomorphism from $T(R)$ into $R_{Nil(R)}$, and ϕ restricted to R is also a ring homomorphism from R into $R_{Nil(R)}$ given by $\phi(x) = x/1$ for every $x \in R$. A nonnil ideal I of R is said to be ϕ-invertible if $\phi(I)$ is an invertible ideal of $\phi(R)$. If every nonnil ideal of R is ϕ-invertible, then we say that R is a *ϕ-Dedekind ring*. Also, we say that R is a *ϕ-Krull ring* if $\phi(R) = \cap V_i$, where each V_i is a discrete ϕ-chained overring of $\phi(R)$, and for every nonnilpotent element $x \in R$, $\phi(x)$ is a unit in all but finitely many V_i. We show that the theories of ϕ-Dedekind and ϕ-Krull rings resemble those of Dedekind and Krull domains.

1. Introduction

Let R be a commutative ring with $1 \neq 0$ and $Nil(R)$ its set of nilpotent elements. Recall from [11] and [9] that a prime ideal of R is called a *divided prime* if $P \subset (x)$ for every $x \in R \setminus P$; thus a divided prime ideal is comparable to every ideal of R. In [2], [3], [4], [5], [6], and [7], the second-named author investigated the class of rings $\mathcal{H} = \{R \mid R$ is a commutative ring and $Nil(R)$ is a divided prime ideal of $R\}$ (Observe that if R is an integral domain, then $R \in \mathcal{H}$.) Recently, the authors [1] generalized the concept of Prüfer and Bezout domains to the context of rings that are in the class $\mathcal{H}$. Also, Lucas and the second-named author [8] generalized the concept of Mori domain to the context of rings that are in the class $\mathcal{H}$. In this paper, we give a generalization of Dedekind domains and Krull domains to the context of rings that are in the class $\mathcal{H}$.

We assume throughout that all rings are commutative with $1 \neq 0$. Let R be a ring. Then $T(R)$ denotes the total quotient ring of R, and $Z(R)$ denotes the set of zerodivisors of R. We start by recalling some background material. A non-zerodivisor of a ring R is called a *regular element* and an ideal of R is said to be *regular* if it contains a regular element. An ideal I of a ring R is said to be a *nonnil ideal* if $I \not\subseteq Nil(R)$. If I is a nonnil ideal of a ring $R \in \mathcal{H}$, then $Nil(R) \subset I$. In particular, this holds if I is a regular ideal of a ring $R \in \mathcal{H}$.

Recall from [2] that for a ring $R \in \mathcal{H}$ with total quotient ring $T(R)$, the map $\phi : T(R) \longrightarrow R_{Nil(R)}$ such that $\phi(a/b) = a/b$ for $a \in R$ and $b \in R \setminus Z(R)$ is a ring homomorphism from $T(R)$ into $R_{Nil(R)}$, and ϕ restricted to R is also a ring homomorphism from R into $R_{Nil(R)}$ given by $\phi(x) = x/1$ for every $x \in R$. Observe that if $R \in \mathcal{H}$, then $\phi(R) \in \mathcal{H}$, $Ker(\phi) \subseteq Nil(R)$, $Nil(T(R)) = Nil(R)$, $Nil(R_{Nil(R)}) = \phi(Nil(R)) = Nil(\phi(R)) = Z(\phi(R))$, $T(\phi(R)) = R_{Nil(R)}$

is quasilocal with maximal ideal $Nil(\phi(R))$, and $R_{Nil(R)}/Nil(\phi(R)) = T(\phi(R))/Nil(\phi(R))$ is the quotient field of $\phi(R)/Nil(\phi(R))$.

Recall from [4] that a ring $R \in \mathcal{H}$ is called a *ϕ-chained ring* if $x^{-1} \in \phi(R)$ for every $x \in R_{Nil(R)} \setminus \phi(R)$; equivalently, if for every $a, b \in R \setminus Nil(R)$, either $a \mid b$ or $b \mid a$ in R (i.e., $R/Nil(R)$ is a valuation domain). Let V be an overring of $\phi(R)$ (i.e., $\phi(R) \subseteq V \subseteq T(\phi(R))$). Then observe that $Nil(V) = Nil(\phi(R))$ and $T(V) = T(\phi(R)) = R_{Nil(R)}$, and hence V is a ϕ-chained overring of $\phi(R)$ if and only if $x^{-1} \in V$ for every $x \in R_{Nil(R)} \setminus V$. Clearly a chained ring is also a ϕ-chained ring. It was shown in [4] that for each integer $n \geq 1$, there is a ϕ-chained ring with Krull dimension n which is not a chained ring. We say that a ring $R \in \mathcal{H}$ is a *discrete ϕ-chained ring* if R is a ϕ-chained ring with at most one nonnil prime ideal and every nonnil ideal of R is principal. Also, recall from [6] that a ring $R \in \mathcal{H}$ is called a *nonnil-Noetherian ring* if every nonnil ideal of R is finitely generated. It was shown in [6] that a ring $R \in \mathcal{H}$ is a nonnil-Noetherian ring iff $R/Nil(R)$ is a Noetherian domain. Recall that an ideal I of a ring R is called a *divisorial ideal* of R if $(I^{-1})^{-1} = I$, where $I^{-1} = \{x \in T(R) \mid xI \subseteq R\}$. If a ring R satisfies the ascending chain condition (a.c.c.) on divisorial regular ideals of R, then R is called a *Mori ring* in the sense of [16]. A ring $R \in \mathcal{H}$ is called a *ϕ-Mori ring* in the sense of [8] if $\phi(R)$ is a Mori ring. It was shown in [8] that a ring $R \in \mathcal{H}$ is a ϕ-Mori ring iff $R/Nil(R)$ is a Mori domain.

An integral domain R is called a *Dedekind domain* if every nonzero ideal of R is invertible, i.e., if I is a nonzero ideal of R, then $II^{-1} = R$. Also, recall from [12] that an integral domain R is called a *Krull domain* if $R = \cap V_i$, where each V_i is a discrete valuation overring of R, and every nonzero element of R is a unit in all but finitely many V_i. Many characterizations and properties of Dedekind and Krull domains are given in [12], [13], and [15]. Let $R \in \mathcal{H}$. We say that a nonnil ideal I of R is *ϕ-invertible* if $\phi(I)$ is an invertible ideal of $\phi(R)$. Recall from [1] that R is called a *ϕ-Prüfer ring* if every finitely generated nonnil ideal of R is ϕ-invertible. If every nonnil ideal of R is ϕ-invertible, then we say that R is a *ϕ-Dedekind ring*. Also, we say that R is a *ϕ-Krull ring* if $\phi(R) = \cap V_i$, where each V_i is a discrete ϕ-chained overring of $\phi(R)$, and for every nonnilpotent element $x \in R$, $\phi(x)$ is a unit in all but finitely many V_i. We say that a ring $R \in \mathcal{H}$ is *ϕ-(completely) integrally closed* if $\phi(R)$ is (completely) integrally closed in $T(\phi(R)) = R_{Nil(R)}$. Among many results in this paper, we show (Theorems 2.10 and 2.15) that a ring $R \in \mathcal{H}$ is a ϕ-Dedekind ring iff R is a ϕ-integrally closed nonnil-Noetherian ring of dimension ≤ 1, iff R is a nonnil-Noetherian ring and R_M is a discrete ϕ-chained ring for each maximal ideal M of R, iff every nonnil ideal of R is a product of (nonnil) prime ideals of R. Also, we show (Theorem 3.4) that a ring $R \in \mathcal{H}$ is a ϕ-Krull ring iff R is a ϕ-completely integrally closed ϕ-Mori ring. We also use idealization-constructions as in [14, Chapter VI, page 161] to construct examples of ϕ-Dedekind and ϕ-Krull rings which are not integral domains.

2. On ϕ-Dedekind Rings

We start this section with the following proposition.

Proposition 2.1. *Let $R \in \mathcal{H}$. Then R is a ϕ-Dedekind ring if and only if every nonnil ideal of $\phi(R)$ is invertible.*

Proof. Suppose that R is ϕ-Dedekind. Let J be a nonnil ideal of $\phi(R)$. Then it is clear that $J = \phi(I)$ for some nonnil ideal I of R. Hence $J = \phi(I)$ is an invertible ideal of $\phi(R)$. Conversely, suppose that every nonnil ideal of $\phi(R)$ is invertible. Then it is clear that every nonnil ideal of R is ϕ-invertible. Thus R is ϕ-Dedekind. ☐

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We define a ring R to be a *Dedekind ring* if every regular ideal I of R is invertible. Hence Proposition 2.1 can be restated as in the following corollary.

Corollary 2.2. *Let $R \in \mathcal{H}$. Then R is a ϕ-Dedekind ring if and only if $\phi(R)$ is a Dedekind ring.*

We recall the following two lemmas from [1].

Lemma 2.3. *(*[1, Lemma 2.3]*) Let $R \in \mathcal{H}$ with $Nil(R) = Z(R)$, and let I be an ideal of R. Then I is an invertible ideal of R if and only if $I/Nil(R)$ is an invertible ideal of $R/Nil(R)$.*

Lemma 2.4. *(*[1, Lemma 2.5]*) Let $R \in \mathcal{H}$ and let P be a prime ideal of R. Then R/P is ring-isomorphic to $\phi(R)/\phi(P)$.*

In particular, $R/Nil(R)$ is ring-isomorphic to $\phi(R)/Nil(\phi(R))$, and thus $\dim \phi(R) = \dim R$.

Theorem 2.5. *Let $R \in \mathcal{H}$. Then R is a ϕ-Dedekind ring if and only if $R/Nil(R)$ is a Dedekind domain.*

Proof. Suppose that R is a ϕ-Dedekind ring. Since $\phi(R) \in \mathcal{H}$, $Nil(\phi(R)) = Z(\phi(R))$, and every nonnil ideal of $\phi(R)$ is invertible, we conclude that every nonzero ideal of $\phi(R)/Nil(\phi(R))$ is invertible by Lemma 2.3. Since $Nil(\phi(R)) = \phi(Nil(R))$ and $R/Nil(R)$ is ring-isomorphic to $\phi(R)/Nil(\phi(R))$ by Lemma 2.4, we conclude that $R/Nil(R)$ is a Dedekind domain.

Conversely, suppose that $R/Nil(R)$ is a Dedekind domain. Hence, once again, by Lemma 2.4 we conclude that $\phi(R)/Nil(\phi(R))$ is a Dedekind domain. Since $\phi(R) \in \mathcal{H}$ and $Nil(\phi(R)) = Z(\phi(R))$, we conclude that every nonnil ideal of $\phi(R)$ is invertible by Lemma 2.3. Hence R is a ϕ-Dedekind ring by Proposition 2.1. ☐

Marco Fontana has asked the second-named author if this type of ring can be characterized as a pullback of a Dedekind domain. In light of Therem 2.5, we see that the answer is “yes.”A similar pullback holds for ϕ-Prüfer rings.

Theorem 2.6. *Let $R \in \mathcal{H}$. Then R is a ϕ-Dedekind ring if and only if $\phi(R)$ is ring-isomorphic to a ring A obtained from the following pullback diagram:*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A/M \arrow[d] \\
T \arrow[r] & T/M
\end{tikzcd}$$

where T is a zero-dimensional quasilocal ring with maximal ideal M, A/M is a Dedekind subring of T/M, the verical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

Proof. Suppose $\phi(R)$ is ring-isomorphic to a ring A obtained from the given diagram. Then $A \in \mathcal{H}$ and $Nil(A) = Z(A) = M$. Since A/M is a Dedekind domain, A is a ϕ-Dedekind ring by Theorem 2.5, and thus R is a ϕ-Dedekind ring.

Conversely, suppose that R is a ϕ-Dedekind ring. Then, letting $T = R_{Nil(R)}$, $M = Nil(R_{Nil(R)})$, and $A = \phi(R)$ yields the desired pullback diagram. □

Our non-domain examples of ϕ-Dedekind rings are provided by the idealization construction $R(+)B$ arising from a ring R and an R-module B as in [14, Chapter VI]. We recall this construction. Let $R(+)B = R \times B$, and define:

(1) $(r, b) + (s, c) = (r + s, b + c)$.
(2) $(r, b)(s, c) = (rs, sb + rc)$.

Under these definitions, $R(+)B$ becomes a commutative ring with identity.

Example 2.7. *Let D be a Dedekind domain with quotient field K, and let L be an extension ring of K. Set $R = D(+)L$. Then $R \in \mathcal{H}$ and R is a ϕ-Dedekind ring which is not a Dedekind domain.*

Proof. First, $Nil(R) = \{0\}(+)L$ is a divided prime ideal of R. For let $(0, y) \in Nil(R)$ and $(a, x) \in R \setminus Nil(R)$; then $(0, y) = (a, x)(0, y/a)$. Thus $R \in \mathcal{H}$. Since $R/Nil(R)$ is ring-isomorphic to D, we conclude that R is a ϕ-Dedekind ring by Theorem 2.5. □

Remark 1. *Let D be an integral domain and M a D-module. Then $R = D(+)M$ has $Nil(R) = \{0\}(+)M$, and $Nil(R)$ is a prime ideal of R. It is easily verified that $Nil(R)$ is a divided prime ideal of R if and only if M is divisible as a D-module. Moreover, $Nil(R)$ is a divided prime ideal and $Nil(R) = Z(R)$ if and only if M is torsionfree and divisible as a D-module.*

For a ring R, let R' denote the integral closure of R in $T(R)$, and let $c(R)$ denote the complete integral closure of R in $T(R)$. Recall that a ring $R \in \mathcal{H}$ is called ϕ-(completely) integrally closed if $\phi(R)$ is (completely) integrally closed in $T(\phi(R)) = R_{Nil(R)}$.

Lemma 2.8. *Let $R \in \mathcal{H}$ and set $D = \phi(R)/Nil(\phi(R))$. Then $D' = \phi(R)'/Nil(\phi(R))$ and $c(D) = c(\phi(R))/Nil(\phi(R))$. In particular, R is ϕ-(completely) integrally closed if and only if D is (completely) integrally closed, if and only if $R/Nil(R)$ is (completely) integrally closed.*

Proof. The proof relies on the following three facts: 1) $Nil(\phi(R))$ is a divided prime ideal of $\phi(R)$, 2) $T(D) = T(\phi(R))/Nil(\phi(R)) = R_{Nil(R)}/Nil(\phi(R))$, and 3) D is ring-isomorphic to $R/Nil(R)$. We leave the details of the proof to the reader. □

Recall from [6] that a ring $R \in \mathcal{H}$ is called a nonnil-Noetherian ring if every nonnil ideal of R is finitely generated. It was shown [6, Theorem 2.2] that a ring $R \in \mathcal{H}$ is a nonnil-Noetherian ring if and only if $R/Nil(R)$ is a Noetherian domain. We recall that a ring $R \in \mathcal{H}$ is called a discrete ϕ-chained ring if R is a ϕ-chained ring with at most one nonnil prime ideal and every nonnil ideal of R is principal.

We leave the proof of the following lemma to the reader.

Lemma 2.9. *Let $R \in \mathcal{H}$. Then R is a discrete ϕ-chained ring if and only if $R/Nil(R)$ is a discrete valuation domain.*

The following characterization of ϕ-Dedekind rings resembles that of Dedekind domains as in [15, Theorem 96].

Theorem 2.10. *Let* $R\in\mathcal{H}$*. Then the following statements are equivalent:*

1. R *is* ϕ*-Dedekind;*
2. R *is nonnil-Noetherian,* ϕ*-integrally closed, and of dimension* ≤ 1*;*
3. R *is nonnil-Noetherian and* R_M *is a discrete* ϕ*-chained ring for each maximal ideal* M *of* R*.*

Proof. Let $D=R/Nil(R)$. Observe that each maximal ideal of D is of the form $M/Nil(R)$ for some maximal ideal M of R, $R_M\in\mathcal{H}$ for each maximal ideal M of R, $Nil(R_M)=Nil(R)_M$, and $D_{M/Nil(R)}=R_M/Nil(R_M)$ for each maximal ideal M of R.

(1) $\Longrightarrow$ **(2)**. Since D is a Dedekind domain by Theorem 2.5, we conclude that D is Noetherian, integrally closed, and of dimension ≤ 1 by [15, Theorem 96]. Hence R is nonnil-Noetherian by [6, Theorem 2.2], ϕ-integrally closed by Lemma 2.8, and it is clear that R has dimension ≤ 1.

(2) $\Longrightarrow$ **(3)**. Since R is nonnil-Noetherian, ϕ-integrally closed, and of dimension ≤ 1, we conclude that D is Noetherian by [6, Theorem 2.2], integrally closed by Lemma 2.8, and of dimension ≤ 1. Thus D is Noetherian and $D_{M/Nil(R)}=R_M/Nil(R_M)$ is a discrete valuation domain for each maximal ideal M of R by [15, Theorem 96]. Thus R is nonnil-Noetherian and R_M is a discrete ϕ-chained ring for each maximal ideal M of R by Lemma 2.9.

(3) $\Longrightarrow$ **(1)**. Since R is nonnil-Noetherian, we conclude that D is Noetherian (again) by [6, Theorem 2.2]. Let M be a maximal ideal of R. Since R_M is a discrete ϕ-chained ring, $D_{M/Nil(R)}=R_M/Nil(R_M)$ is a discrete valuation domain by Lemma 2.9. Thus D is a Dedekind domain by [15, Theorem 96], and hence R is ϕ-Dedekind by Theorem 2.5. □

Recall that a ring $R\in\mathcal{H}$ is called a ϕ-Prüfer ring if every finitely generated nonnil ideal of R is ϕ-invertible. Also, recall from [14] that a ring R is called a *Prüfer ring* if every finitely generated regular ideal of R is invertible. Hence we have the following two results.

Proposition 2.11. *Let* $R\in\mathcal{H}$ *be a nonnil-Noetherian ring. Then* R *is a* ϕ*-Dedekind ring if and only if* R *is a* ϕ*-Prüfer ring.*

Theorem 2.12. *Let* $R\in\mathcal{H}$ *be a* ϕ*-Dedekind ring. Then* R *is a Dedekind ring.*

Proof. Since R is a nonnil-Noetherian ring by Theorem 2.10, we conclude that R is a ϕ-Prüfer ring by Proposition 2.11. Hence R is a Prüfer ring by [1, Theorem 2.14]. Since R is a nonnil-Noetherian Prüfer ring, we conclude that R is a Dedekind ring (i.e., every regular ideal of R is invertible). □

The following is an example of a ring $R\in\mathcal{H}$ which is a Dedekind ring but not a ϕ-Dedekind ring.

Example 2.13. *Let* D *be a non-Dedekind domain with (proper) quotient field* K*. Set* $R=D(+)K/D$*. Then* $R\in\mathcal{H}$ *and* $R=T(R)$*. Hence* R *is a Dedekind ring. Since* $R/Nil(R)$ *is ring-isomorphic to* D*,* R *is not a* ϕ*-Dedekind ring by Theorem 2.5.*

In light of Corollary 2.2 and Theorem 2.12, we have the following result; we omit its proof.

Theorem 2.14. *Let* $R\in\mathcal{H}$ *such that* $Nil(R)=Z(R)$*. Then* R *is a Dedekind ring if and only if* R *is a* ϕ*-Dedekind ring.*

It is well-known that an integral domain R is a Dedekind domain iff every nonzero proper ideal of R is (uniquely) a product of prime ideals of R. We have the following result.

Theorem 2.15. *Let* $R \in \mathcal{H}$. *Then* R *is a* ϕ*-Dedekind ring if and only if every nonnil proper ideal of* R *is (uniquely) a product of nonnil prime ideals of* R.

Proof. Suppose that R is ϕ-Dedekind. Then $D = R/Nil(R)$ is a Dedekind domain by Theorem 2.5. Let I be a nonnil proper ideal of R. Since D is a Dedekind domain, $I/Nil(R) = (P_1/Nil(R))(P_2/Nil(R)) \cdots (P_n/Nil(R))$ for some nonnil prime ideals $P_1, \ldots, P_n$ of R. Let $Q = P_1P_2 \cdots P_n$. We claim that $I = Q$. This follows since $Nil(R) \subset Q$ because $Nil(R) \subset P_i$ for each i and $Nil(R)$ is a divided prime ideal of R. For the uniqueness, just observe that $P_1/Nil(R) = P_2/Nil(R)$ in D for prime ideals P_1 and P_2 of R if and only if $P_1 = P_2$.

Conversely, if each nonnil proper ideal of R is a product of nonnil prime ideals of R, then each proper nonzero ideal of D is a product of prime ideals of D. Thus D is a Dedekind domain, and hence R is a ϕ-Dedekind ring by Theorem 2.5. □

Recently, Brewer and Heinzer [10, Theorem 9] gave the following characterization of Dedekind domains.

Theorem ([10, Theorem 9]). Let R be an integral domain. Then the following statements are equivalent:

(1) R is a Dedekind domain;
(2) Each nonzero proper principal ideal aR can be written in the form $aR = Q_1Q_2 \cdots Q_n$, where each Q_i is a power of a prime ideal of R and the Q_i's are pairwise comaximal;
(3) Each nonzero proper ideal I of R can be written in the form $I = Q_1Q_2 \cdots Q_n$, where each Q_i is a power of a prime ideal of R and the Q_i's are pairwise comaximal.

For a ring $R \in \mathcal{H}$, we have the following analog of the above theorem; we omit its proof.

Theorem 2.16. *Let* $R \in \mathcal{H}$. *Then the following statements are equivalent:*

(1) R *is a* ϕ*-Dedekind ring;*
(2) *Each nonnil proper principal ideal* aR *can be written in the form* $aR = Q_1Q_2 \cdots Q_n$*, where each* Q_i *is a power of a nonnil prime ideal of* R *and the* Q_i*'s are pairwise comaximal;*
(3) *Each nonnil proper ideal* I *of* R *can be written in the form* $I = Q_1Q_2 \cdots Q_n$*, where each* Q_i *is a power of a nonnil prime ideal of* R *and the* Q_i*'s are pairwise comaximal.*

Recall from [13] that a ring R is called a *ZPI-ring* if every nonzero proper ideal of R is uniquely a product of prime ideals of R, and R is called a *general ZPI-ring* if every nonzero proper ideal of R is a product of prime ideals of R. We say that a ring $R \in \mathcal{H}$ is a *nonnil-ZPI-ring* if every nonnil proper ideal of R is uniquely a product of (nonnil) prime ideals of R, and we say that R is a *general* *nonnil-ZPI-ring* if every nonnil proper ideal of R is a product of (nonnil) prime ideals of R. In view of Theorem 2.15, we have the following result.

Corollary 2.17. *Let $R \in \mathcal{H}$. Then the following statements are equivalent:*

1. *R is a ϕ-Dedekind ring;*
2. *R is a nonnil-ZPI-ring;*
3. *R is a general nonnil-ZPI-ring.*

Theorem 2.18. *Let $R \in \mathcal{H}$ be a ϕ-Dedekind ring and let I be an ideal of R. Then:*

1. *If $I \subseteq Nil(R)$, then R/I is a ϕ-Dedekind ring.*
2. *If I is a nonnil ideal of R, then R/I is a general ZPI-ring.*

Proof. **(1)**. Suppose that $I \subset Nil(R)$, and set $A = R/I$. Then $Nil(A) = Nil(R)/I$ is a divided prime ideal of A. Hence $A \in \mathcal{H}$. Since $A/Nil(A)$ is ring-isomorphic to $D = R/Nil(R)$ and D is a Dedekind domain, we conclude that $A = R/I$ is a ϕ-Dedekind ring.

(2). Suppose that I is a nonnil ideal of R. Since $J = I/Nil(R)$ is a nonzero proper ideal of the Dedekind domain $D = R/Nil(R)$, we conclude that D/J is a general ZPI-ring by [13, Chapter 39, page 469]. Since D/J is ring-isomorphic to R/I, we conclude that R/I is a general ZPI-ring. □

The following characterization of ϕ-Dedekind domains resembles that of general ZPI-rings as in [13, Theorem 39.2, page 470].

Theorem 2.19. *Let $R \in \mathcal{H}$. Then the following statements are equivalent:*

1. *R is a ϕ-Dedekind ring;*
2. *R is a nonnil-Noetherian ring and there are no ideals properly between M and M^2 for each nonnil maximal ideal M of R.*

Proof. Set $D = R/Nil(R)$.

(1) $\Longrightarrow$ (2). Since D is a Dedekind domain (general ZPI-ring) by Theorem 2.5, we conclude that D is a Noetherian domain and there are no ideals properly between J and J^2 for each maximal ideal J of D by [13, Theorem 39.2, page 470]. Hence R is a nonnil-Noetherian ring by [6, Theorem 2.2], and it is clear that there are no ideals properly between M and M^2 for each nonnil maximal ideal M of R.

(2) $\Longrightarrow$ (1). Since D is Noetherian by [6, Theorem 2.2] and there are no ideals properly between J and J^2 for each maximal ideal J of D, D is a Dedekind domain by [13, Theorem 39.2, page 470]. Hence R is a ϕ-Dedekind ring by Theorem 2.5. □

It is well-known [15, Problems 11 and 12, page 73] that an integral domain R is a Dedekind domain iff every nonzero prime ideal of R is invertible, iff R is Noetherian and every nonzero maximal ideal of R is invertible. Hence, in light of Theorem 2.5 and [15, Problems 11 and 12, page 73], we have the following result which will not be proved here.

Theorem 2.20. *Let $R \in \mathcal{H}$. Then the following statements are equivalent:*

1. *R is a ϕ-Dedekind ring;*
2. *Each nonnil prime ideal of R is ϕ-invertible;*

(3) *R is a nonnil-Noetherian ring and each nonnil maximal ideal of R is ϕ-invertible.*

It is well-known [13, Problem 4, page 475] that a principal ideal ring is a general ZPI-ring. We call a ring $R \in \mathcal{H}$ a *nonnil-principal ideal ring* if every nonnil ideal of R is principal. It is easy to prove the following result.

Theorem 2.21. *Let $R \in \mathcal{H}$. Then R is a nonnil-principal ideal ring if and only if $R/Nil(R)$ is a principal ideal domain.*

Theorem 2.22. *Let $R \in \mathcal{H}$ be a nonnil-principal ideal ring. Then R is a ϕ-Dedekind ring.*

Proof. Set $D = R/Nil(R)$. Then D is a principal ideal domain by Theorem 2.21. Hence D is a Dedekind domain, and thus R is a ϕ-Dedekind ring by Theorem 2.5. □

Recall that a ring B is called an overring of a ring R if $R \subseteq B \subseteq T(R)$. It is well-known [13, Theorem 40.1, page 477] that an overring of a Dedekind domain is a Dedekind domain. We end this section with the following result.

Theorem 2.23. *Let $R \in \mathcal{H}$ be a ϕ-Dedekind ring. Then every overring of R is a ϕ-Dedekind ring.*

Proof. Let S be an overring of R. Then $S \in \mathcal{H}$, $Nil(S) = Nil(R)$, and $S/Nil(R)$ is an overring of $R/Nil(R)$. Since D is a Dedekind domain and $S/Nil(R)$ is an overring of $R/Nil(R)$, we conclude that $S/Nil(R)$ is a Dedekind domain by [13, Theorem 40.1, page 477]. Hence S is a ϕ-Dedekind ring by Theorem 2.5. □

3. On ϕ-Krull Rings

Recall that a ring $R \in \mathcal{H}$ is said to be a ϕ-Krull ring if $\phi(R) = \cap V_i$, where each V_i is a discrete ϕ-chained overring of $\phi(R)$, and for every nonnilpotent element $x \in R$, $\phi(x)$ is a unit in all but finitely many V_i. We begin this section with the Krull domain analog of Theorem 2.5, Theorem 2.6, Lemma 2.9, and Theorem 2.21.

Theorem 3.1. *Let $R \in \mathcal{H}$. Then R is a ϕ-Krull ring if and only if $R/Nil(R)$ is a Krull domain.*

Proof. Suppose that R is a ϕ-Krull ring. Then $\phi(R) = \cap V_i$, where each V_i is a discrete ϕ-chained overring of $\phi(R)$, and for every nonnilpotent element $x \in R$, $\phi(x)$ is a unit in all but finitely many V_i. Since each V_i is a discrete ϕ-chained overring of $\phi(R)$ and $T(\phi(R)/Nil(\phi(R)) = T(\phi(R))/Nil(\phi(R)) = R_{Nil(R)}/Nil(\phi(R))$, we conclude that each $V_i/Nil(\phi(R))$ is a discrete valuation overring of $\phi(R)/Nil(\phi(R))$ by Lemma 2.9. Hence $\phi(R)/Nil(\phi(R)) = \cap V_i/Nil(\phi(R))$ and every nonzero element of $\phi(R)/Nil(\phi(R))$ is a unit in all but finitely many $V_i/Nil(\phi(R))$. Thus $\phi(R)/Nil(\phi(R))$ is a Krull domain. Since $\phi(R)/Nil(\phi(R))$ is ring-isomorphic to $R/Nil(R)$ by Lemma 2.4, $R/Nil(R)$ is a Krull domain.

Conversely, suppose that $R/Nil(R)$ is a Krull domain. Since $R/Nil(R)$ is ring-isomorphic to $\phi(R)/Nil(\phi(R))$ by Lemma 2.4, we conclude that $\phi(R)/Nil(\phi(R))$ is a Krull domain. Since a ring $A \in \mathcal{H}$ is a discrete ϕ-chained ring if and only if $A/Nil(A)$ is a discrete valuation ring by Lemma 2.4 and $T(\phi(R)/Nil(\phi(R)) = T(\phi(R))/Nil(\phi(R)) = R_{Nil(R)}/Nil(\phi(R))$, we conclude that $\phi(R)/Nil(\phi(R)) =$

$\cap V_i/Nil(\phi(R))$, where each V_i is a discrete ϕ-chained overring of $\phi(R)$. Hence $\phi(R) = \cap V_i$. Since for every nonnilpotent element $x \in R$, $\phi(x) + Nil(\phi(R))$ is a unit in all but finitely many $V_i/Nil(\phi(R))$, we conclude that $\phi(x)$ is a unit in all but finitely many V_i. Hence R is a ϕ-Krull ring. $\square$

We have the following pullback characterization of ϕ-Krull rings.

Theorem 3.2. *Let $R \in \mathcal{H}$. Then R is a ϕ-Krull ring if and only if $\phi(R)$ is ring-isomorphic to a ring A obtained from the following pullback diagram:*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A/M \arrow[d] \\
T \arrow[r] & T/M
\end{tikzcd}$$

where T is a zero-dimensional quasilocal ring with maximal ideal M, A/M is a Krull subring of T/M, the verical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

Proof. Suppose $\phi(R)$ is ring-isomorphic to a ring A obtained from the given diagram. Then $A \in \mathcal{H}$ and $Nil(A) = Z(A) = M$. Since A/M is a Krull domain, A is a ϕ-Krull ring by Theorem 3.1, and thus R is a ϕ-Krull ring.

Conversely, suppose that R is a ϕ-Krull ring. Then, letting $T = R_{Nil(R)}$, $M = Nil(R_{Nil(R)})$, and $A = \phi(R)$ yields the desired pullback diagram. $\square$

Example 3.3. *Let D be a Krull domain with quotient field K, and let L be a ring extension of K. Set $R = D(+)L$. Then $R \in \mathcal{H}$ and R is a ϕ-Krull ring which is not a Krull domain.*

Proof. As in Example 2.7, $Nil(R) = \{0\}(+)L$ is a divided prime ideal of R. Thus $R \in \mathcal{H}$. Since $R/Nil(R)$ is ring-isomorphic to D, we conclude that R is a ϕ-Krull ring by Theorem 3.1. $\square$

It is well-known [12, Theorem 3.6] that an integral domain R is a Krull domain if and only if R is a completely integrally closed Mori domain. We have a similar characterization for ϕ-Krull rings.

Theorem 3.4. *Let $R \in \mathcal{H}$. Then R is a ϕ-Krull ring if and only if R is a ϕ-completely integrally closed ϕ-Mori ring.*

Proof. Set $D = R/Nil(R)$. Suppose that R is a ϕ-Krull ring. Then D is a Krull domain by Theorem 3.1. Hence D is a completely integrally closed Mori domain. Thus R is a ϕ-completely integrally closed ϕ-Mori ring by Lemma 2.8 and [8], respectively.

Conversely, suppose that R is a ϕ-completely integrally closed ϕ-Mori ring. Then D is a completely integrally closed Mori domain by Lemma 2.9 and [8]. Hence D is a Krull domain, and thus R is a ϕ-Krull ring by Theorem 3.1. $\square$

It is known [13, Theorem 43.16, page 536] that a Krull domain R which is not a field is a Prüfer domain iff R is a Dedekind domain, iff R is one-dimensional. We have the following analogous result for ϕ-Krull rings.

Theorem 3.5. *Let $R \in \mathcal{H}$ be a ϕ-Krull ring which is not zero-dimensional. Then the following statements are equivalent:*

(1) *R is a ϕ-Prüfer ring;*

(2) *R is a ϕ-Dedekind ring;*

(3) R *is one-dimensional.*

Proof. Set $D = R/Nil(R)$. Then D is a Krull domain by Theorem 3.1, and it is clear that D is not a field.

(1) $\Longrightarrow$ (2). Since D is a Prüfer domain by [1, Theorem 2.6], D is a Dedekind domain by [13, Theorem 43.16, page 536], and hence R is a ϕ-Dedekind ring by Theorem 2.5.

(2) $\Longrightarrow$ (3). Since D is a Dedekind domain by Theorem 2.5, we conclude that D is one-dimensional by [13, Theorem 43.16, page 536], and thus R is one-dimensional.

(3) $\Longrightarrow$ (1). Since D is one-dimensional, D is a Prüfer domain again by [13, Theorem 43.16, page 536], and hence R is a ϕ-Prüfer ring by [1, Theorem 2.6]. $\square$

It is well-known that if R is a Noetherian domain, then R' is a Krull domain. In particular, an integrally closed Noetherian domain is a Krull domain. We have the following analogous result for nonnil-Noetherian rings.

Theorem 3.6. *Let $R \in \mathcal{H}$ be a nonnil-Noetherian ring. Then $\phi(R)'$ is a ϕ-Krull ring. In particular, if R is a ϕ-integrally closed nonnil-Noetherian ring, then R is a ϕ-Krull ring.*

Proof. Set $D = \phi(R)/Nil(\phi(R))$. Since $R/Nil(R)$ is a Noetherian domain by [6, Theorem 2.2] and $R/Nil(R)$ is ring-isomorphic to D by Lemma 2.4, we conclude that D is a Noetherian domain. Since $D' = \phi(R)'/Nil(\phi(R))$ by Lemma 2.8 and D' is a Krull domain, we conclude that $\phi(R)'$ is a ϕ-Krull ring by Theorem 3.1. The "in particular" statement is now clear. $\square$

It is known [15, Problem 8, page 83] that if R is a Krull domain in which all prime ideals of height ≥ 2 are finitely generated, then R is a Noetherian domain. We have the following analogous result for nonnil-Noetherian rings.

Theorem 3.7. *Let $R \in \mathcal{H}$ be a ϕ-Krull ring in which all prime ideals of R with height ≥ 2 are finitely generated. Then R is a nonnil-Noetherian ring.*

Proof. Since $R/Nil(R)$ is a Krull domain in which all prime ideals of height ≥ 2 are finitely generated, we conclude that $R/Nil(R)$ is a Noetherian domain by [15, Problem 8, page 83]. Hence R is a nonnil-Noetherian ring by [6, Theorem 2.2]. $\square$

For a ring $R \in \mathcal{H}$, let ϕ_R denotes the ring-homomorphism $\phi : T(R) \longrightarrow R_{Nil(R)}$. We have the following lemma.

Lemma 3.8. *Let $R \in \mathcal{H}$ and let P be a nonnil prime ideal of R. Then $\phi_{R_P}(R_P) = \phi_R(R)_{\phi_R(P)}$ is an overring of $\phi_R(R)$.*

Proof. Since $(R_P)_{Nil(R_P)} = R_{Nil(R)} = T(\phi_R(R))$, we conclude that $\phi_{R_P}(R_P) \subseteq R_{Nil(R)} = T(\phi_R(R))$. Let $y \in R$. Then $y/1 \in R_P$, and hence $\phi_{R_P}(y/1) = \phi_R(y)$. Also, suppose that $y \in R \setminus P$. Then $\phi_{R_P}(y/y) = \phi_{R_P}(1/y)\phi_{R_P}(y/1) = \phi_{R_P}(1/y)\phi_R(y) = 1$, and thus $\phi_{R_P}(1/y) = 1/\phi_R(y)$. Hence let $x = a/b \in R_P$ for some $a \in R$ and $b \in R \setminus P$. Then $\phi_{R_P}(a/b) = \phi_R(a)/\phi_R(b)$, and thus $\phi_{R_P}(R_P) \subseteq \phi_R(R)_{\phi_R(P)}$. Conversely, suppose that $x \in \phi_R(R)_{\phi_R(P)}$. Then $x = \phi_R(a)/\phi_R(b)$ for some $a \in R$ and $b \in R \setminus P$. Hence $x = \phi_R(a)/\phi_R(b) = \phi_{R_P}(a/b) \in \phi_{R_P}(R_P)$, and thus $\phi_R(R)_{\phi_R(P)} \subseteq \phi_{R_P}(R_P)$. $\square$

It is well-known [12, Proposition 1.9, page 8] that an integral domain R is a Krull domain if and only if R satisfies the following three conditions:

(1) R_P is a discrete valuation domain for every height-one prime ideal P of R;
(2) $R = \cap R_P$, the intersection being taken over all height-one prime ideals P of R;
(3) Each nonzero element of R is in only a finite number of height-one prime ideals of R, i.e., each nonzero element of R is a unit in all but finitely many R_P, where P is a height-one prime ideal of R.

We have the following result which is an analog of [12, Proposition 1.9, page 8].

Theorem 3.9. *Let* $R \in \mathcal{H}$ *with* $\dim R \geq 1$. *Then* R *is a* ϕ*-Krull ring if and only if* R *satisfies the following three conditions:*

(1) R_P *is a discrete* ϕ*-chained ring for every height-one prime ideal* P *of* R*;*
(2) $\phi_R(R) = \cap \phi_{R_P}(R_P)$*, the intersection being taken over all height-one prime ideals* P *of* R*;*
(3) *Each nonnilpotent element of* R *lies in only a finite number of height-one prime ideals of* R*, i.e., each nonnilpotent element of* R *is a unit in all but finitely many* R_P*, where* P *is a height-one prime ideal of* R.

Proof. First observe that $Nil(\phi_{R_P}(R_P)) = Nil(\phi_R(R))$. Suppose that R is a ϕ-Krull ring. Set $D = R/Nil(R)$, and let P be a height-one prime ideal of R. Since D is a Krull domain by Theorem 3.1, $D_{P/Nil(R)}$ is a discrete valuation domain. Since $D_{P/Nil(R)}$ is ring-isomorphic to $R_P/Nil(R_P)$, we conclude that R_P is a discrete ϕ-chained ring by Lemma 2.9. Since $R_P/Nil(R_P)$ is ring-isomorphic to $\phi_{R_P}(R_P)/Nil(\phi_{R_P}(R_P))$, we conclude that $\phi_{R_P}(R_P)$ is a discrete ϕ-chained ring by Lemma 2.9. Hence $\phi_R(R)_{\phi_R(P)}$ is a discrete ϕ-chained ring by Lemma 3.8. Now, set $F = \phi_R(R)/Nil(\phi_R(R))$. Since D is a Krull domain by Theorem 3.1 and D is ring-isomorphic to F by Lemma 2.4, we conclude that F is a Krull domain. Hence $F = \phi_R(R)/Nil(\phi_R(R)) = \cap \phi_R(R)_{\phi_R(P)}/Nil(\phi_R(R)) = \cap \phi_{R_P}(R_P)/Nil(\phi_R(R))$, the intersection being taken over all height-one prime ideals P of R. Thus it is easily verified that $\phi_R(R) = \cap \phi_{R_P}(R_P)$, the intersection being taken over all height-one prime ideals P of R. Since for each nonnilpotent element x of R, $\phi_R(x) + Nil(\phi_R(R))$ lies in only a finite number of height-one prime ideals of F, we conclude that each nonnilpotent element of R lies in only a finite number of height-one prime ideals of R.

The converse is clear by the definition of ϕ-Krull rings. □

Recall that a ring R is called a *Marot ring* if each regular ideal of R is generated by its set of regular elements. A Marot ring is called a *Krull ring* in the sense of [14, page 37] if either $R = T(R)$ or if there exists a family $\{V_i\}$ of discrete rank one valuation rings such that:

(1) R is the intersection of the valuation rings $\{V_i\}$.
(2) Each regular element of $T(R)$ is a unit in all but finitely many V_i.

The following is an example of a discrete ϕ-chained ring which is not a discrete rank one valuation ring in the sense of [14].

Example 3.10. *Let* D *be a discrete valuation domain with maximal ideal* M *and quotient field K. Set* $R = D(+)K/D$. *Then* $R \in \mathcal{H}$ *and* $R = T(R)$. *Hence* R *is not a discrete rank one valuation by* [14, Lemma 8.1(1), page 37]. *Since* $R/Nil(R)$ *is ring-isomorphic to* D, R *is a discrete* ϕ*-chained ring by Lemma 2.9.*

Observe that the ring R in the above example is a Krull ring since $R = T(R)$. We have the following result which is the ϕ-Krull analog of Theorem 2.14.

Theorem 3.11. *Let* $R \in \mathcal{H}$ *such that* $Nil(R) = Z(R)$*. Then* R *is a Krull ring if and only if* R *is a* ϕ*-Krull ring.*

Proof. Since $Z(R)$ is a prime ideal of R, R is a Marot ring by [14, Theorem 7.2, page 32]. It is easily verified that for each nonnil prime ideal P of R, R_P is a discrete rank one valuation ring if and only if R_P is a discrete ϕ-chained ring. Hence the claim is now clear by Theorem 3.9. □

The following is an example of a ring $R \in \mathcal{H}$ which is a Krull ring but not a ϕ-Krull ring.

Example 3.12. *Let* D *be a non-Krull domain with (proper) quotient field* K*. Set* $R = D(+)K/D$*. Then* $R \in \mathcal{H}$ *and* $R = T(R)$*. Hence* R *is a Krull ring. Since* $R/Nil(R)$ *is ring-isomorphic to* D*,* R *is not a* ϕ*-Krull ring by Theorem 3.1.*

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