ON DOMAINS WHICH HAVE PRIME IDEALS THAT ARE LINEARLY ORDERED

Ayman Badawi
Department of Mathematics
Emory & Henry College
Emory, VA 24327

Introduction. Throughout this paper the letter $R$ denotes a commutative integral domain with identity and quotient field $K$. If $I$ is an ideal of a ring $A$, then $\text{Rad}(I)$ denotes the radical of $I$. A domain $R$ is a valuation domain if and only if for every $a, b \in R$, either $a \mid b$ or $b \mid a$. Recall that an integral domain $R$ is a GCD domain if any two elements in $R$ have a greatest common divisor. It is well-known that a GCD domain $R$ which has prime ideals that are linearly ordered is a valuation domain. The purpose of this paper is to provide an alternative proof of this fact. Furthermore, we will give a characterization of divided domains and another characterization of pseudo-valuation domains that are somewhat analogous to the characterization of valuation domains given above.

We start by recalling the following definitions

**Definition 1.** A domain $R$ is called a divided domain in the sense of [6] if every prime ideal of $R$ is comparable to every principal ideal of $R$.

**Definition 2.** A prime ideal $P$ of $R$ is called strongly prime in the sense of [7] if whenever $x, y \in K$
and \( xy \in P \), then \( x \in P \) or \( y \in P \). If every prime ideal of \( R \) is strongly prime, then \( R \) is called a pseudo-
valuation domain [abbreviated PVD].

We start with the following Theorem:

**Theorem 1.** The following statements are equivalent for a commutative ring \( A \) with identity.

1. The prime ideals of \( A \) are linearly ordered.
2. The radical ideals of \( A \) are linearly ordered.
3. Each proper radical ideal of \( A \) is prime.
4. The radical ideals of principal ideals of \( A \) are linearly ordered.
5. For each \( a, b \in A \), there is an \( n \geq 1 \) such that either \( a \mid b^n \) or \( b \mid a^n \).

**Proof.** (1) \( \Rightarrow \) (2). Let \( I \) be a proper ideal of \( A \) and \( P \) be the minimum prime ideal of \( A \) over \( I \). Then \( \text{Rad}(I) = P \). (2) \( \Rightarrow \) (1). This requires no comment. (2) \( \Rightarrow \) (3). Let \( I \) be a proper ideal of \( A \) and \( P \) be the minimum prime ideal of \( A \) over \( I \). Then \( \text{Rad}(I) = P \). (3) \( \Rightarrow \) (1). Suppose that \( P, Q \) are two distinct prime ideals of \( A \). Let \( I = P \cap Q \). Then \( \text{Rad}(I) = I \) is a prime ideal of \( A \). But this is possible only if \( P \subseteq Q \) or \( Q \subseteq P \). (2) \( \Rightarrow \) (4). Clear. (4) \( \Rightarrow \) (5). This is clear by the definition of radical ideals. (5) \( \Rightarrow \) (1). Suppose that \( P, Q \) are two distinct prime ideals of \( A \). Now, suppose that there is a \( p \in P - Q \). Then for every \( q \in Q \) there is an \( n \geq 1 \) such that \( p \mid q^n \). Therefore \( q \in P \).

In view of the above Theorem, we have:

**Corollary 1.** Suppose the prime ideals of a commutative ring \( A \) with identity are linearly ordered and \( a, b \) are nonzero nonunit elements of \( A \). Let \( P \) be the minimum prime ideal of \( A \) that contains \( a \) and \( Q \) be the minimum prime ideal of \( A \) that contains \( b \).
Then $P = Q$ if and only if there exist $n \geq 1$, $m \geq 1$ such that $a \mid b^n$ and $b \mid a^m$.

**Proof.** This is just the observation that $P = \text{Rad}(\langle a \rangle) = \text{Rad}(\langle b \rangle) = Q$. 

The following result appeared in [9, Theorem 1], [5, Corollary 4.3], [10, Corollary 3.8], and [11, Proposition A]. In view of the above Theorem, we give a different proof of it.

**Proposition 1.** A GCD domain $R$ which has prime ideals that are linearly ordered is a valuation domain.

**Proof.** Let $a, b$ be nonzero nonunit elements of $R$, and let $f = \gcd(a, b)$. Suppose that $f$ is associated in $R$ to neither $a$ nor $b$. Let $d = a/f$, and $g = b/f$. Then neither $d$ nor $g$ is a unit of $R$. Thus, by Theorem 1 there exists $m \geq 1$ such that either $d \mid g^n$ or $g \mid d^n$. But it is well-known that $\gcd(d, g) = 1$ and therefore for every $n \geq 1$ $\gcd(d, g^n) = \gcd(g, d^n) = 1$ (see [8, Theorem 49]). Hence, $d$ or $g$ is a unit of $R$, which is a contradiction. Thus, the assumption that $f$ is associated in $R$ to neither $a$ nor $b$ is invalid. Hence, $a \mid b$ or $b \mid a$. Therefore, $R$ is a valuation domain.

The following Proposition gives a characterization of divided domains in the sense of [6].

**Proposition 2.** The following statements are equivalent for an integral domain $R$.

1. $R$ is a divided domain.
2. For every pair of proper ideals $I, J$ of $R$, the ideals $I$ and $\text{Rad}(J)$ are comparable.
3. For every $a, b \in R$, the ideals $(a)$ and $\text{Rad}(\langle b \rangle)$ are comparable.
4. For every $a, b \in R$, either $a \mid b$ or $b \mid a^n$ for some $n \geq 1$. 
Proof. (1) ⇒ (2). Suppose \( R \) is a divided domain. Then by the definition of divided domains, the prime ideals of \( R \) are linearly ordered. Let \( I, J \) be two proper ideals of \( R \). Since \( R \) is divided, \( \text{Rad}(J) = \mathfrak{p} \) is prime by Theorem 1 above. Thus, either \( (a) \subseteq \mathfrak{p} \) or \( \mathfrak{p} \subseteq (a) \) for every \( a \in I \) since \( R \) is divided. Hence, the ideals \( I, \text{Rad}(J) \) are comparable. (2) ⇒ (3). This requires no comment. (3) ⇒ (4). Clear. (4) ⇒ (1). Suppose that for every \( a, b \in R \), either \( a|b^n \) for some \( n \geq 1 \) or \( b|a \). Let \( \mathfrak{p} \) be a prime ideal of \( R \) and \( s \in R-\mathfrak{p} \) and \( p \in \mathfrak{p} \). Since for every \( n \geq 1 \) \( p \) does not divide \( s^n \), \( s|p \). Hence, \( \mathfrak{p} \) is comparable to every principal ideal of \( R \). Therefore \( R \) is a divided domain. \( \blacksquare \)

Let \( R \) be a PVD, and \( a, b \) be nonzero nonunit elements of \( R \). Suppose that \( a \) does not divide \( b \) and \( b \) does not divide \( a^2 \). Then \( c=b/a \) and \( g=a^2/b \) are elements in \( K-R \). Let \( M \) be a maximal ideal of \( R \) that contains \( a \). Then \( cg = a \in M \). But neither \( c \) nor \( g \) is an element of \( M \). A contradiction, since \( M \) is strongly prime. Thus, \( a|b \) or \( b|a^2 \). Hence, by Theorem 1 the prime ideals of \( R \) are linearly ordered. In particular, \( R \) is quasilocal. This argument provides an alternative proof of \([7, \text{Corollary 1.3}]\). Anderson [1, Proposition 3.1] proved that a quasilocal domain \( R \) with maximal ideal \( M \) is a PVD if and only if for every \( x \in K \), either \( xR \subseteq M \) or \( M \subseteq xR \), that is, if for every \( a, b \in R \), either \( aM \subseteq bR \) or \( bR \subseteq aM \). In view of [1, Proposition 3.1] and Theorem 1, we now give several other Characterizations of pseudovaluation domains.

Proposition 3. Let \( N \) be the set of all nonunit elements of an integral domain \( R \). The following statements are equivalent.
(1) $R$ is a PVD with the maximal ideal $N$.

(2) For each pair $I, J$ of ideals of $R$, either $J \subseteq I$ or $IB \subseteq J$ for every proper ideal $B$ of $R$.

(3) For every $a, b \in R$, either $bR \subseteq aR$ or $acR \subseteq bR$ for every nonunit $c \in R$.

(4) For every $a, b \in R$, either $a|b$ or $b|ac$ for every nonunit $c \in R$.

(5) For every $a, b \in R$, either $bR \subseteq aR$ or $aN \subseteq bR$.

(6) For every $a, b \in R$, either $bN \subseteq aR$ or $aR \subseteq bN$.

Proof. (1) $\Rightarrow$ (2). Let $I, J$ be ideals of $R$ and $B$ be a proper ideal of $R$. Suppose that $J$ is not a subset of $I$ and $BI$ is not a subset of $J$. Then there exist $j \in J - I$ and $ib \in IB$ for some $i \in I$ and $b \in B$ such that $j/i \in K - R$ and $ib/j \in K - R$. But $(j/i)(bi/j) = b \in N$ and neither $j/i \in N$ nor $ib/j \in N$, which is a contradiction. (2) $\Rightarrow$ (3). Clear. (3) $\Rightarrow$ (4). Clear. (4) $\Rightarrow$ (1). Suppose that for every $a, b \in R$ and any nonunit $c$ in $R$, either $a|b$ or $b|ac$. Let $a$ be any nonunit element of $R$ and $b \in R$. Then either $a|b$ or $b|a^2$. Hence, the prime ideals of $R$ are linearly ordered by Theorem 1. In particular, $R$ is quasilocal with maximal ideal $N$. By [2, Proposition 4.8] (see also [4, Proposition 2]), it suffices to show that $N$ is strongly prime. Suppose that $xy \in N$ for some $x, y \in K$. If $x \in R$ or $y \in R$, then it is easy to see that $x \in N$ or $y \in N$. Hence, suppose that $x, y \in K - R$. Write $x = b/a$ and $y = c/d$ for some $a, b, c, d \in R$. Since $x = b/a \in K - R$ and $xy = bc/ad \in N$, $b|a(bc/ad)$. Thus, $y = c/d \in R$, which is a contradiction. Therefore, if $xy \in N$ for some $x, y \in K$, then $x \in N$ or $y \in N$. (4) $\iff$ (5). Clear. (6) $\Rightarrow$ (1). Let $a, b \in R$. Then either $a|b$ or $b|a^2$. 
Hence, the prime ideals of \( R \) are linearly ordered. In particular, \( R \) is quasilocal with the maximal ideal \( N \). Hence, \( R \) is a PVD by [1, Proposition 3.1]. (1) \( \Rightarrow \) (6). Again, This is just a restatement of [1, Proposition 3.1].

An immediate consequence of the above Proposition is [7, Proposition 1.1]. We state it here as a corollary.

**Corollary 2.** Every valuation domain is a PVD.

**RELATED RESULTS**

Throughout this section the letter \( N \) denotes the set of all nonunit elements of \( R \), and \( G \) denotes the group of divisibility of \( R \). If \( I \) is an ideal of \( R \), then \( I : I = \{ x \in K : xI \subseteq I \} \).

Anderson [1, Proposition 3.10] proved the following result.

**Fact 1 [1, Proposition 3.10].** The following statements are equivalent for a quasilocal domain \( R \) with maximal ideal \( M \).

1. For every \( a, b \in R \), either \( aM \subseteq bR \) or \( bM \subseteq aR \).
2. For every \( a, b \in R \), either \( aM \subseteq bM \) or \( bM \subseteq aM \).

In view of Fact 1 above and Theorem 1, we have the following result.

**Proposition 4.** The following statements are equivalent for an integral domain \( R \). Furthermore, if \( R \) satisfies any of the following conditions, then \( R \) is quasilocal with the maximal ideal \( N \) and \( N : N \) is a valuation domain.

1. For every \( a, b \in R \), either \( aN \subseteq bR \) or \( bN \subseteq aR \).
(2) For every $a, b \in R$, either $aN \subseteq bN$ or $aN \subset bN$.

**Proof.** Suppose that $R$ satisfies (1) or (2) above. Let $a, b \in R$. Then $a|b^2$ or $b|a^2$. Hence, the prime ideals of $R$ are linearly ordered by Theorem 1. In particular, $R$ is quasilocal with the maximal ideal $N$. Thus, (1) and (2) are equivalent by Fact 1 above. Now, if $R$ satisfies (1) or (2), then $N : N$ is a valuation domain by [1, Corollary 3.4].

In light of Proposition 4 above, we have the following result.

**Proposition 5.** The following statements are equivalent for an integral domain $R$.

(1) $R$ is quasilocal with the maximal ideal $N$ such that $N : N$ is a valuation domain.

(2) For every $a, b \in R$, either $aN \subseteq bR$ or $bN \subseteq aR$.

(3) For every $a, b \in R$, $a|bc$ for every nonunit $c \in R$ or $b|ac$ for every nonunit $c \in R$.

(4) For every $a, b \in R$, either $aN \subseteq bN$ or $bN \subseteq aN$.

**Proof.** Clearly, (3) is a restatement of (2). By Proposition 4 above, we now only need show that (1) $\Rightarrow$ (2). Hence, suppose that $R$ is quasilocal with the maximal ideal $N$ and $N : N$ is a valuation domain. For nonzero $a, b \in R$, either $a/bN \subset N$ or $b/aN \subset N$ since $N : N$ is a valuation domain. Thus, $aN \subseteq bR$ or $bN \subseteq aR$.

**Remark 1.** It is well-known that a quasilocal domain $R$ with maximal ideal $M$ is a PVD iff $M : M$ is a valuation domain with maximal ideal $M$ (see [3, Proposition 2.5]). So it is natural to ask whether the condition $aN \subseteq bR$ or $bN \subseteq aR$ implies that the domain in
the above Proposition is a PVD. The answer is negative and for a counter-example see [1, Example 3.2].

Combining [1, Proposition 3.10, Corollary 3.4, Corollary 3.8, Proposition 3.11 (b), Proposition 3.12, Proposition 4.3, and Proposition 5.2] with Proposition 5, we arrive at the following Corollary.

**Corollary 3.** The following statements are equivalent for an integral domain \( R \).

1. \( R \) is quasilocal with maximal ideal \( M \) such that \( M:M \) is a valuation domain.
2. For each nonzero prime ideal \( P \) of \( R \), \( P:P \) is a valuation domain.
3. The prime ideals of \( R \) are linearly ordered and if \( M \) is the maximal ideal of \( R \), then \( M:M \) is a valuation domain.
4. For every \( a,b \in R \), either \( aN \subseteq bR \) or \( bN \subseteq aR \).
5. For every \( a,b \in R \), either \( aN \subseteq bN \) or \( bN \subseteq aN \).
6. For every \( a,b \in R \), either \( a|bc \) for every nonunit \( c \in R \) or \( b|ac \) for every nonunit \( c \in R \).
7. For each \( g \in G \), either \( g > h \) for all \( h \in G \) with \( h < 0 \) or \( g < h \) for all \( h \in G \) with \( h > 0 \).
8. There is a valuation overring \( V \) of \( R \) and a maximal ideal \( J \) of \( R \) which is also an ideal of \( V \).
9. For each \( x \in K \) and maximal ideal \( M \) of \( R \), \( xM \) and \( M \) are comparable.
10. \( R \) is quasilocal with maximal ideal \( M \) such that for every \( a,b \in R \), either \( aM \subseteq bR \) or \( bM \subseteq aR \).
11. \( R \) is quasilocal with maximal ideal \( M \) such that for every \( a,b \in R \), either \( aM \subseteq bM \) or \( bM \subseteq aM \).
12. For some maximal ideal \( M \) of \( R \), \( xM \) and \( M \) are comparable for each \( x \in K \).
13. For each \( x \in K \), there is a maximal ideal \( M \) of \( R \) so that \( xM \) and \( M \) are comparable.
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REFERENCES


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