DIVISIBILITY CONDITIONS
IN COMMUTATIVE RINGS
WITH ZERODIVISORS

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ABSTRACT

Let $R$ be a commutative ring. In this paper, we give several divisibility and ring-theoretic conditions for $R$ or $T(R)$ to be either zero-dimensional or von Neumann regular. We also consider divisibility conditions related to $R$ being completely integrally closed and study several closedness conditions which hold with respect to units of $T(R)$.

INTRODUCTION

In this paper, we continue our investigation begun in Ref. [1–3, 4] of extending ring-theoretic properties in integral domains to the context of commutative rings with zerodivisors by replacing conditions on elements of the total quotient ring $T(R)$ with internal divisibility conditions on elements.
of the ring $R$. In the first section, we consider divisibility conditions on $R$ which are equivalent to $R$ or $T(R)$ being either zero-dimensional or von Neumann regular. In the second section, we give some additional conditions on $R$ for $T(R)$ to be either zero-dimensional or von Neumann regular and relate this to ideas used in, Refs. [1,2] for subrings of a direct product of integral domains. The two main results (Theorems 2.2 and 2.3) are that $T(R)$ is zero-dimensional (resp., von Neumann regular) if and only if for each $x \in R$, there is a $y \in R$ such that $xy \in \text{nil}(R)$ (resp., $xy = 0$) and $x + y$ is a regular element of $R$. In the third section, we investigate divisibility conditions related to $R$ being completely integrally closed. In the fourth section, we consider several “closedness” properties which hold with respect to units of $T(R)$ and answer some questions raised in Refs. [1,2]. We show in Theorem 4.2 that if $R$ is a Marot ring, then $R$ satisfies these “closedness” properties with respect to $T(R)$ if and only if $R$ satisfies them with respect to units of $T(R)$. In the final section, we briefly consider some conditions related to seminormality from Ref. [1].

Throughout, $R$ is a commutative ring with $1 \neq 0$, group of units $U(R)$, nilpotent elements, $\text{nil}(R)$ its set of nilpotent elements, $Z(R)$ its set of zerodivisors, Spec($R$) its set of prime ideals, and total quotient ring $T(R) = RS$, where $S = R - Z(R)$. As usual, an $x \in R - Z(R)$ is called a regular element of $R$. When we write $x/y \in T(R)$, we will always mean that $x, y \in R$ with $y$ a regular element of $R$. For any undefined terminology, see Refs. [5,6] or [7]. For an excellent survey of recent work on zero-dimensional commutative rings, see Ref. [8].

1 DIVISIBILITY CONDITIONS AND VON NEUMANN REGULAR RINGS

In this section, we give several “divisibility” conditions for a commutative ring $R$ to be either von Neumann regular or a total quotient ring. Recall that a commutative ring $R$ is von Neumann regular if for each $x \in R$, there is a $y \in R$ such that $x = x^2y$; equivalently, $R$ is reduced and zero-dimensional.[6, Theorem 3.1] Of course, the most obvious such divisibility condition is that $R$ is von Neumann regular if and only if $x^2 | x$ for all $x \in R$. Our first result gives several less obvious divisibility conditions on $R$ which are equivalent to $R$ being von Neumann regular (the equivalence of conditions (4)–(6) is well known).

Proposition 1.1. The following statements are equivalent for a commutative ring $R$.

1. Let $m \geq 2$ be a fixed integer. If $x | y^m$ for $x, y \in R$, then $x | y$. 

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(2) Let $x, y \in R$. If $x \mid y^n$ for some integer $n \geq 1$, then $x \mid y$.

(3) Let $x, y \in R$. If $y^n = xd$ for some integer $n \geq 1$ with $d \in R$ a non-unit, then $x \mid y$.

(4) All ideals of $R$ are radical ideals.

(5) All principal ideals of $R$ are radical ideals.

(6) $R$ is von Neumann regular.

Proof. (1) $\Rightarrow$ (2) Suppose that $x \mid y^n$. If $n \leq m$, then also $x \mid y^m$; so $x \mid y$ by hypothesis. If $n > m$, then also $x \mid y^k$ with $k < n$. Thus $x \mid y^k$ by hypothesis, and hence $x \mid y$ by induction on $n$.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (4) Let $I$ be a proper ideal of $R$. Suppose that $x^n = i \in I$ for some $x \in R$ and integer $n \geq 1$. Thus $x^{2n} = i^2$, and hence $i \mid x$ by hypothesis. Then $x \in I$; so $I$ is a radical ideal of $R$.

(4) $\Rightarrow$ (5) Clear.

(5) $\Rightarrow$ (6) Let $x \in R$. Then $(x^2)$ is a radical ideal of $R$ by hypothesis. Since $x^2 \in (x^2)$, we have $x \in (x^2)$, and hence $x = x^2 y$ for some $y \in R$. Thus $R$ is von Neumann regular.

(6) $\Rightarrow$ (1) Let $x, y \in R$ with $x \mid y^m$; say $y^m = xd$ with $d \in R$. Since $R$ is von Neumann regular, $y = ue$ with $u \in U(R)$ and $e \in R$ idempotent. \cite[Corollary 3.3]{6} Thus $y = ue = u^{-m}(ue)^m = u^{-m}y^m = (u^{-m}d)x$; so $x \mid y$. $\square$

Proposition 1.1 yields our next result on when $T(R)$ is von Neumann regular (also, see Theorem 2.3). In a similar manner, one may obtain criteria for $T(R)$ to be either strongly root closed or strongly (2,3)-closed (see Sec. 4 for the definitions), zero-dimensional (see Sec. 2), or semi-normal (see Sec. 5).

**Proposition 1.2.** The following statements are equivalent for a commutative ring $R$.

1. $T(R)$ is von Neumann regular.
2. Let $x, y \in R$. If $x \mid y^n$ for some integer $n \geq 1$, then $x \mid sy$ for some regular element $s \in R$.

Proof. (1) $\Rightarrow$ (2) Suppose that $T(R)$ is von Neumann regular and $x \mid y^n$ in $R$. By Proposition 1.1, $x \mid y$ in $T(R)$; so $y = zx$ for some $z = d/s \in T(R)$. Then $x \mid sy$ in $R$ with $s$ a regular element of $R$.

(2) $\Rightarrow$ (1) By Proposition 1.1, we need only show that if $x \mid y^n$ in $T(R)$ for some integer $n \geq 1$ and $x, y \in T(R)$, then $x \mid y$ in $T(R)$. Write $x = a/c$ and $y = b/d$. Then $x \mid y^n$ yields that $d^n ae = cfb^n$ for some $e \in R$ and regular element $f \in R$. Thus $a(d^n ae) = cfb^n$ in $R$. By hypothesis, $a|s(cfb)^n$ in $R$ for some regular element $s \in R$. Hence $x \mid y$ in $T(R)$. $\square$
By restricting the divisibility conditions in Proposition 1.1 to regular elements, we obtain several equivalent conditions for \( R \) to be a total quotient ring (i.e., \( R = T(R) \), equivalently, each regular element of \( R \) is a unit of \( R \)). However, we lose the fact that \( R \) is reduced.

**Proposition 1.3.** The following statements are equivalent for a commutative ring \( R \).

1. Let \( x, y \in R \) with \( x \) a regular element of \( R \). If \( x \mid y^n \) for some integer \( n \geq 1 \), then \( x \mid y \).
2. Let \( x, y \in R \) with \( y \) a regular element of \( R \). If \( x \mid y^n \) for some integer \( n \geq 1 \), then \( x \mid y \).
3. Let \( x, y \in R \) with \( x, y \) regular elements of \( R \). If \( x \mid y^n \) for some integer \( n \geq 1 \), then \( x \mid y \).
4. \( R = T(R) \).

**Proof.** We first show that any of conditions (1) – (3) implies condition (4). Let \( x \in R \) be a regular element of \( R \). Then \( x^2 \mid x^2 \) implies that \( x^2 \mid x \), and hence \( x \) is a unit of \( R \). Thus condition (4) holds.

Conversely, suppose that condition (4) holds and that \( x \mid y^n \). If either \( x \) or \( y \) is a regular element of \( R \), then \( x \) is a unit of \( R \) by hypothesis, and hence \( x \mid y \). Thus conditions (1) – (3) all hold.

\[ \square \]

**Example 1.4.** The ring \( \mathbb{Z}/4\mathbb{Z} \) satisfies the equivalent conditions of Proposition 1.3, but not those of Proposition 1.1. Note that \( \mathbb{Z}/4\mathbb{Z} \) is a total quotient ring, but it is not reduced.

### 2 ZERO-DIMENSIONAL TOTAL QUOTIENT RINGS

In this section, we give several conditions on \( R \) for \( T(R) \) to be either zero-dimensional or von Neumann regular. We then show that a condition introduced in Ref. [1], and further used in Ref. [2], on a subring \( R \) of a direct product of integral domains is equivalent to \( T(R) \) being von Neumann regular.

Recall that a commutative ring \( R \) is called \( \pi \)-regular if for each \( x \in R \), there is a \( y \in R \) and an integer \( n \geq 1 \) such that \( x^{2n}y = x^n \), i.e., \( x^{2n} \mid x^n \). Then \( R \) is \( \pi \)-regular if and only if \( R \) is zero-dimensional. Thus, in the spirit of Proposition 1.2, one may easily show that \( T(R) \) is zero-dimensional if and only if for each \( x \in R \), there is an integer \( n \geq 1 \) and a regular element \( s \in R \) such that \( x^{2n} \mid sx^n \) in \( R \). We give a much more interesting internal characterization of when \( T(R) \) is zero-dimensional in Theorem 2.2, but first a lemma.
Lemma 2.1. Let \( R \) be a commutative ring and \( x, y \in R \).

1. Suppose that \( xy \in \text{nil}(R) \) and let \( n \geq 1 \) be an integer. Then \( x + y \) is a regular element of \( R \) if and only if \( x^n + y^n \) is a regular element of \( R \).

2. Suppose that \( xy = 0 \). If \( ax + by \) is a regular element of \( R \) for some \( a, b \in R \), then \( x + y \) is also a regular element of \( R \). (Thus the ideal \( (x, y) \) contains a regular element of \( R \) if and only if \( x + y \) is a regular element of \( R \).)

3. Suppose that \( xy = 0 \). Then \( x + y \) is a regular element of \( R \) if and only if \( x^n + y^n \) is a regular element of \( R \) for some integers \( m, n \geq 1 \).

Proof. (1) Let \( xy \in \text{nil}(R) \). By the Binomial Theorem, \( (x + y)^n = x^n + y^n + z \) with \( z \in \text{nil}(R) \). Thus \( (x + y)^n \) is a unit in \( T(R) \) if and only if \( x^n + y^n \) is a unit in \( T(R) \). Hence \( x + y \) is a regular element of \( R \) if and only if \( x^n + y^n \) is a regular element of \( R \).

(2) Suppose that \( xy = 0, ax + by \) is a regular element of \( R \), and that \( (x + y)d = 0 \) for some \( 0 \neq d \in R \). If \( yd = 0 \), then also \( xd = 0 \); and hence \( (ax + by)d = 0 \), a contradiction. Thus we may assume that \( yd \neq 0 \). Then \( (x + y)yd = 0 \) yields \( y^2d = 0 \) since \( xy = 0 \). Hence \( (ax + by)yd = 0 \), a contradiction. Thus \( x + y \) is a regular element of \( R \).

(3) This follows easily from part (2) above. \( \square \)

Theorem 2.2. The following statements are equivalent for a commutative ring \( R \).

1. \( T(R) \) is zero-dimensional.
2. For each \( x \in R \), there is a \( y \in R \) and an integer \( n \geq 1 \) such that \( x^n y = 0 \) and \( x^n + y \) is a regular element of \( R \).
3. For each \( x \in R \), there is a \( y \in R \) such that \( xy \in \text{nil}(R) \) and \( x + y \) is a regular element of \( R \).

Proof. (1) \( \Rightarrow \) (2) Suppose that \( T(R) \) is zero-dimensional, and let \( x \in R \). Since \( T(R) \) is \( \pi \)-regular, \( [6, \text{Theorem 3.1}] \) there is a \( z/s \in T(R) \) and an integer \( n \geq 1 \) such that \( x^{2n}(z/s) = x^n \). Thus \( x^n(s - x^n z) = 0 \). Let \( y = s - x^n z \). Then \( x^n y = 0 \) and \( z x^n + y = s \) is a regular element of \( R \). Hence \( x^n + y \) is a regular element of \( R \) by Lemma 2.1(2).

(2) \( \Rightarrow \) (3) Suppose that \( R \) satisfies condition (2), and let \( x \in R \). Then there is a \( y \in R \) and an integer \( n \geq 1 \) such that \( x^n y = 0 \) and \( x^n + y \) is a regular element of \( R \). Thus \( xy \in \text{nil}(R) \). Since \( x^n y = 0 \) and \( x^n + y \) is a regular element of \( R \), also \( (x^n)^n + y^n = (x^{n^n}) x^n + y^n \) is a regular element of \( R \) by Lemma 2.1(1). Hence \( x^n + y^n \) is a regular element of \( R \) by Lemma 2.1(2), and thus \( x + y \) is a regular element of \( R \) by Lemma 2.1(1) again.

(3) \( \Rightarrow \) (1) Suppose that \( R \) satisfies condition (3). To show that \( T(R) \) is zero-dimensional, it is sufficient to show that each non-minimal prime ideal
Q of R contains a regular element of R. Let P ⊂ Q be distinct prime ideals of R, and choose an x ∈ Q − P. By hypothesis, there is a y ∈ R such that xy ∈ nil(R) and x + y is a regular element of R. Thus y ∈ P ⊂ Q, and hence x + y ∈ Q. Thus Q contains a regular element of R.

In the next result, a slight modification of the conditions in Theorem 2.2 forces R to be reduced, and hence von Neumann regular.

**Theorem 2.3.** The following statements are equivalent for a commutative ring R.

1. T(R) is von Neumann regular.
2. For each x ∈ R, there is a y ∈ R such that xy = 0 and x + y is a regular element of R.

**Proof.** (1) ⇒ (2) Suppose that T(R) is von Neumann regular. Then T(R), and hence R, is reduced. Thus (2) follows from Theorem 2.2 since nil(R) = {0}.

(2) ⇒ (1) Suppose that R satisfies condition (2). We first show that R is reduced. Let x ∈ nil(R); say x^n = 0 for some integer n ≥ 1. Then by hypothesis, there is a y ∈ R such that xy = 0 and x + y is a regular element of R. Thus y^n = x^n + y^n is a regular element of R by Lemma 2.1(1). Hence y is also a regular element of R, so x = 0. Thus R, and hence T(R), is reduced. It follows from Theorem 2.2 that T(R) is zero-dimensional, and hence von Neumann regular. [6, Theorem 3.1]

**Remark 2.4.**

1. Theorems 2.2 and 2.3 also follow from Ref. [6, Theorem 3.2 and Corollary 3.3] via Lemma 2.1 and (for Theorem 2.3) the observation in the proof of (2) ⇒ (1) of Theorem 2.3 that R is reduced. The above two results in Ref. [6] are from Ref. [9].

2. There are many other characterizations in terms of R of when T(R) is von Neumann regular. For example, see Ref. [6, Theorem 4.5] for conditions concerning when Min(R), the set of minimal prime ideals of R, is compact.

We end this section by showing that condition (2) in Theorem 2.3 is equivalent to a concept introduced in Ref. [1], and further used in Ref. [2], for studying subrings of a direct product of integral domains.

Let R be a subring of the direct product ΠR_x of a family {R_x} of integral domains. As in Ref. [1], for x = (x_a), y = (y_a) ∈ R ⊂ ΠR_x, we say that y extends x, written yEx, if y_a = x_a whenever x_a ≠ 0. We say that x extends to a regular element of R if there is a regular element y ∈ R such that
For any commutative ring $S$, define an ordering on $S$ by $a \leq b$ if either $a = 0$ or $a = b$. For a family $\{R_z\}$ of commutative rings, the induced product order on $\Pi R_z$ is then $(x_z) \leq (y_z)$ if either $x_z = 0$ or $x_z = y_z$ for each $z$, i.e., $(y_z)$ extends $(x_z)$ in $\Pi R_z$. This ordering restricts to the ordering $E$ defined above on any subring $R$ of $\Pi R_z$.

Recall that a commutative ring $R$ is reduced if and only if $R$ is a subring of the direct product $\Pi R_z$ of some family $\{R_z\}$ of integral domains, and that $R$ is a subring of the product of a finite number of integral domains if and only if $R$ is reduced with only a finite number of minimal prime ideals. Thus much of our earlier work in Refs. [1,2] for reduced rings was set in the context of subrings of a direct product of integral domains, and the concept of “extending to a regular element” played a key role. Our next result shows that the concept of “extending to a regular element” is independent of the embedding of the reduced ring $R$ in a direct product of integral domains. Corollary 2.6 then relates this concept to $T(R)$ being von Neumann regular.

**Proposition 2.5.** Let $R$ be a subring of a direct product of integral domains. Then $x \in R$ extends to a regular element of $R$ if and only if there is a $y \in R$ such that $xy = 0$ and $x + y$ is a regular element of $R$.

**Proof.** ($\Rightarrow$) Let $x \in R$ and $zEx$ with $z \in R$ regular. Let $y = z - x$. Then $xy = 0$ and $x + y = z$ is a regular element of $R$.

($\Leftarrow$) Let $x \in R$. By hypothesis, there is a $y \in R$ such that $xy = 0$ and $x + y$ is a regular element of $R$. Clearly $x + y$ extends $x$ since $xy = 0$. 

**Corollary 2.6.** Let $R$ be a commutative ring which is a subring of a direct product of integral domains. Then each $x \in R$ extends to a regular element of $R$ if and only if $T(R)$ is von Neumann regular.

**Proof.** This follows immediately from Theorem 2.3 and Proposition 2.5.

Thus, in several results in Refs. [1,2], the hypothesis “every element of $R$ can be extended to a regular element of $R$” may be replaced by “$T(R)$ is von Neumann regular” (also, see Sec. 4 and 5).

The next corollary generalizes the well-known fact that a reduced commutative ring with only a finite number of minimal prime ideals (in particular, a reduced commutative Noetherian ring) has von Neumann regular total quotient ring.

**Corollary 2.7.** Let $R$ be a reduced commutative ring such that each nonzero zerodivisor of $R$ is contained in only a finite number of minimal prime ideals of $R$. Then $T(R)$ is von Neumann regular.
Proof. We may view $R$ as a subring of $\Pi(R/P_a)$, where $\{P_a\}$ is the set of minimal prime ideals of $R$. By hypothesis, each $0 \neq (r_a) \in Z(R)$ has only a finite number of zero entries, and hence can be extended to a regular element of $R$ by an argument similar to that in the proof of Ref. [1, Lemma 2.5]. Thus $T(R)$ is von Neumann regular by Corollary 2.6.

3 STRONGLY COMPLETELY INTEGRALLY CLOSED RINGS

Let $R$ be a commutative ring. Recall that $x \in T(R)$ is almost integral over $R$ if there is a regular element $s \in R$ such that $sx^n \in R$ for all integers $n \geq 1$. As usual, $R$ is called completely integrally closed (CIC) if whenever $x \in T(R)$ is almost integral over $R$, then $x \in R$ (equivalently, if $b^n|sa^n$ for all integers $n \geq 1$ with $a \in R$ and $b, s \in R$ regular elements of $R$, then $b|a$). In the spirit of Ref. [2], we define $R$ to be strongly completely integrally closed (SCIC) if whenever $b^n|sa^n$ for all integers $n \geq 1$ with $a, b \in R$ and $s \in R$ a regular element of $R$, then $b|a$. (Here we have replaced $a/b \in R$, where $a \in R, b \in R - Z(R)$, by $b|a$ in $R$, and we allow $b \in Z(R)$.) Clearly, if a ring $R$ is SCIC, then $R$ is also CIC (but not conversely, see Example 3.6).

Our first result gives a trivial case when $R$ is always SCIC.

Proposition 3.1. Let $R$ be a commutative ring. If $R = T(R)$, then $R$ is SCIC. In particular, a zero-dimensional commutative ring is SCIC.

Proof. The first part is clear. For the second part, just recall that $R = T(R)$ when $R$ is zero-dimensional.

As in Ref. [10], a commutative ring $R$ is called additively regular if for each $x \in T(R)$, there is a $y \in R$ such that $x + y$ is a regular element (unit) of $T(R)$; equivalently, $R$ is additively regular if and only if for all $x, y \in R$ with $y$ a regular element of $R$, there is an $a \in R$ such that $x + ay$ is a regular element of $R$. If either $T(R)$ is zero-dimensional or $Z(R)$ is a finite union of prime ideals, then $R$ is additively regular [6, Theorems 7.4 and 7.2]. In particular, von Neumann regular rings, Noetherian rings, and reduced rings with only a finite number of minimal prime ideals are additively regular.

We define $R$ to be strongly additively regular if for each $x \in R$, there is a $y \in R$ such that $xy = 0$ and $x + y$ is a regular element of $R$. Of course, by Theorem 2.3, $R$ is strongly additively regular if and only if $T(R)$ is von Neumann regular. Thus a strongly additively regular ring is additively regular (this was also observed in a different context in Ref. [2, Proposition 2.5]). However, the converse is false since any zero-dimensional ring is additively regular; for a reduced example, see Example 3.7.
We next show that a strongly additively regular commutative ring $R$ is SCIC if and only if it is CIC (i.e., when $T(R)$ is von Neumann regular). Example 3.6 shows that it is not enough to just assume that $R$ is additively regular.

**Proposition 3.2.** Let $R$ be a strongly additively regular commutative ring (equivalently, $T(R)$ is von Neumann regular). Then $R$ is SCIC if and only if $R$ is CIC.

**Proof.** We need only show that $R$ is SCIC if $R$ is CIC. Suppose that $R$ is CIC, and let $a, b, s \in R$ with $s$ a regular element of $R$ such that $bn = san$ for all integers $n \geq 1$; say $dnb^n = san$ with each $dn \in R$. Since $R$ is strongly additively regular, there is a $z \in R$ such that $bz = 0$ and $b + z$ is a regular element of $R$. Then also $az = 0$ since $s$ is regular and $R$ is reduced by the proof of Theorem 2.3. Thus $(dnb^n)(b + z) = dnb^{n+1} = bsan = [(b + z)s]a^n$ for all integers $n \geq 1$. Hence $(b + z)s][(b + z)s]a^n$ for all integers $n \geq 1$ with $b + z$, $(b + z)s \in R$ regular. Thus $a = (b + z)w$ for some $w \in R$ since $R$ is CIC. Note that $az = bz = 0$ yields $wz^2 = 0$; so $wz = 0$ since $R$ is reduced. Hence $a = bw$, and thus $R$ is SCIC. □

**Corollary 3.3.** Let $R$ be a reduced commutative ring such that each nonzero zerodivisor of $R$ is contained in only a finite number of minimal prime ideals of $R$. Then $R$ is SCIC if and only if $R$ is CIC. In particular, this holds if $R$ is a reduced commutative Noetherian ring.

**Proof.** This follows directly from Proposition 3.2 and Corollary 2.7. □

Let $M$ be an $R$-module. Then the idealization of $R$ and $M$ is the ring $R(+)M$ with underlying set $R \times M$ under coordinatewise addition, and multiplication given by $(r, x)(s, y) = (rs, ry + sx)$. In the next several results, we will use the facts that $U(R(+)M) = U(R)(+)M$ (6, Theorem 25.1(6)) and $Z(R(+)M) = A(+)M$, where $A = Z(R) \cup Z(M)$ (6, Theorem 25.3). See Ref. [6, Sec. 25] for more details on idealization. We next determine when certain idealizations are SCIC.

**Proposition 3.4.** Let $R$ be a subring of a commutative ring $B$ such that each regular element of $R$ is also a regular element of $B$ (for example, if $B \subset T(R)$). Then $R(+)B$ is SCIC if and only if $R = T(R)$.

**Proof.** Let $A = R(+)B$. First suppose that $R = T(R)$. Then each element of $R$ is either a unit or a zerodivisor, and hence each element of $A$ is either a unit or a zerodivisor. Thus $A = T(A)$, and hence $A$ is SCIC by Proposition 3.1. Conversely, suppose that $R \neq T(R)$. Let $x \in R$ be a nonunit regular element. Define $s = (x, 0), a = (0, 1)$, and $b = (0, x)$ in $A$. Then $s$ is a
regular element of \( A \) and \( b^n \mid sa^n \) in \( A \) for all integers \( n \geq 1 \). However, \( b \nmid a \) in \( A \) since \( x \) is not a unit of \( R \). Thus \( A \) is not SCIC.

**Proposition 3.5.** Let \( R \) be a CIC commutative ring with \( R \neq T(R) \). Then \( R(+)T(R) \) is CIC, but not SCIC.

**Proof.** By Proposition 3.4, \( A = R(+)T(R) \) is not SCIC. We next show that \( A \) is CIC. Suppose that \( (b, y)^n \mid (s, z)(a, x)^n \) for all integers \( n \geq 1 \) and \( (a, x), (b, y), (s, z) \in A \) with \( (b, y), (s, z) \) regular elements of \( A \). Then \( b^n \mid sa^n \) in \( R \) for all integers \( n \geq 1 \) with \( b, s \in R \) regular. Thus \( a = bc \) for some \( c \in R \) since \( R \) is CIC. Hence \( (a, x) = (b, y)(c, w) \) in \( A \) with \( w = (x - cy)/b \in T(R) \). Thus \( A \) is CIC.

**Example 3.6.** By Proposition 3.5, \( R = \mathbb{Z}(+)\mathbb{Q} \) is CIC, but not SCIC. Note that \( R \) is additively regular, but not strongly additively regular since \( R \) is not reduced.

**Example 3.7.** (A reduced commutative ring \( R \) which is additively regular, but not strongly additively regular.) Let \( K \) be a field, and let \( A = K[X_1, X_2, \ldots] = K[\{X_n \mid n \in \mathbb{N}\}] \). Then \( I = (\{X_iX_j \mid i, j \in \mathbb{N}, i \neq j\}) \) is a radical ideal of \( A \) contained in the maximal ideal \( N = (\{X_n \mid n \in \mathbb{N}\}) \). Thus \( R = A_N/I_N \) is a quasilocal reduced ring with maximal ideal \( M = N_N/I_N \). Clearly \( Z(R) = M \), and thus \( T(R) = R \). Hence \( R \) is additively regular.

However, for any \( 0 \neq x \in M \), there is no \( y \in R \) such that \( xy = 0 \) and \( x + y \) is a regular element of \( R \). Thus \( R \) is not strongly additively regular.

**Question 3.8.** Is there a reduced commutative ring \( R \) which is CIC, but not SCIC?

### 4 CLOSEDNESS WITH RESPECT TO UNITS OF \( T(R) \)

In this section, we continue our investigation from Refs. [1,2] of when \( R \) satisfies certain “closedness” properties with respect to units of \( T(R) \), i.e., if \( x \in U(T(R)) \) (equivalently, \( x = a/b \) with \( a, b \) regular elements of \( R \)) satisfies a given closedness property, then \( x \in R \) (equivalently, \( b \mid a \) in \( R \)).

The “closedness” properties we consider here are completely integrally closed, integrally closed, root closed, and (2,3)-closed. It is clear that these four “closedness” properties with respect to units of \( T(R) \) are inherited by direct products, intersections of overrings in \( T(R) \), and (except for the CIC case) localizations in \( T(R) \). We also briefly discuss rings such that \( T(R) - R \subset U(T(R)) \).

Recall that a commutative ring \( R \) is root closed (resp., (2,3)-closed) if whenever \( x^n \in R \) for some integer \( n \geq 1 \) (resp., \( x^2, x^3 \in R \)) and \( x \in T(R) \),
then \( x \in R \). As in Ref. [2], we say that \( R \) is **strongly root closed** (resp., **strongly (2,3)-closed**) if whenever \( b^n | a^m \) for some integer \( n \geq 1 \) (resp., \( b^2 | a^2, b^3 | a^3 \)) and \( a, b \in R \), then \( b | a \) in \( R \). (Here we have replaced \( a/b \in R \), where \( a \in R \) and \( b \in R - Z(R) \), by \( b | a \) in \( R \), and we allow \( b \in Z(R) \).) Note that \( R \) is strongly (2,3)-closed (resp., (2,3)-closed) if and only if whenever \( b^n | a^m \) for all sufficiently large integers \( n \geq 1 \) and \( a, b \in R \) (resp., \( a \in R \) and \( b \in R - Z(R) \)), then \( b | a \) in \( R \). Clearly a strongly root closed (resp., strongly (2,3)-closed) ring is root closed (resp., (2,3)-closed). Examples in Ref. [2] show that the converse is false.

Our first result is the CIC analog of Ref. [2, Propositions 2.2–2.4], i.e., if \( R \) is root closed (resp., (2,3)-closed, integrally closed) with respect to units of \( T(R) \) and \( R \) is additively regular (in particular, if \( T(R) \) is von Neumann regular), then \( R \) is root closed (resp., (2,3)-closed, integrally closed). A stronger result is given in Theorem 4.2.

**Proposition 4.1.** The following statements are equivalent for an additively regular commutative ring \( R \).

1. \( R \) is completely integrally closed.
2. If \( x \in U(T(R)) \) is almost integral over \( R \), then \( x \in R \).

**Proof.** Clearly (1) \( \Rightarrow \) (2). Conversely, suppose that condition (2) holds, and let \( x \in T(R) \) be almost integral over \( R \). Then there is a regular element \( s \in R \) such that \( sx^n \in R \) for all integers \( n \geq 1 \). Since \( R \) is additively regular, there is a \( y \in R \) such that \( x + y \in U(T(R)) \). Then \( s(x + y)^n \in R \) for all integers \( n \geq 1 \). Thus \( x + y \in R \) by hypothesis, and hence \( x \in R \). Thus \( R \) is completely integrally closed. \( \square \)

In the spirit of Refs. [1,2], and Sec. 3, we consider the following four conditions related to root closedness on a commutative ring \( R \) with \( a, b \in R \) and \( n \geq 1 \) an integer. (Similar conditions can be given for (2,3)-closedness, integral closedness, and complete integral closedness; we leave the specific details to the interested reader.)

1. If \( b^n | a^m \), then \( b | a \) (i.e., \( R \) is strongly root closed).
2. If \( b^n | a^m \) with \( b \) regular, then \( b | a \) (i.e., \( R \) is root closed).
3. If \( b^n | a^m \) with \( a, b \) regular, then \( b | a \) (i.e., \( R \) is root closed with respect to units of \( T(R) \)).
4. If \( b^n | a^m \) with a regular, then \( b | a \).

Clearly conditions (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3), and conditions (3) \( \iff \) (4). By Ref. [2, Example 3.1], condition (2) does not imply condition (1).

We always have: \( R \) is CIC wrt units of \( T(R) \) \( \Rightarrow \) \( R \) is integrally closed wrt units of \( T(R) \) \( \Rightarrow \) \( R \) is root closed wrt units of \( T(R) \) \( \Rightarrow \) \( R \) is (2,3)-closed.
wrt units of \( T(R) \). Although a CIC ring is always root closed, an SCIC ring need not be strongly root closed (cf. Proposition 3.1).

In Refs. [2, Question 2.7], we asked if \( R \) is integrally closed (resp., (2, 3)-closed, root closed) with respect to units of \( T(R) \) implies that \( R \) is integrally closed (resp., (2, 3)-closed, root closed). In Refs. [2, Propositions 2.2–2.4], we showed that this is true if \( R \) is additively regular. We next generalize this to the class of Marot rings. As in Ref. [6], a commutative ring \( R \) is called a Marot ring if every ideal of \( R \) which contains a regular element is generated by regular elements. Examples of Marot rings include integral domains, Noetherian rings, rings such that \( Z(R) \) is a finite union of prime ideals, polynomial rings, and rings with zero-dimensional total quotient ring.\[6, Theorems 7.2, 7.4, and 7.5\] An additively regular ring is a Marot ring, \[6, Theorem 7.2\] but not conversely (see Refs. [6, Example 12, p. 185] or Ref. [11]).

**Theorem 4.2.** Let \( R \) be a commutative Marot ring. Then \( R \) is root closed (resp., (2, 3)-closed, integrally closed, completely integrally closed) if and only if \( R \) is root closed (resp., (2, 3)-closed, integrally closed, completely integrally closed) with respect to units of \( T(R) \).

**Proof.** Suppose that \( R \) is a Marot ring which is root closed with respect to units of \( T(R) \). Let \( a, b \in R \) with \( b \) regular, and suppose that \( (a/b)^n \in R \), i.e., \( b^n | a^n \), for some integer \( n \geq 1 \). Let \( I = (a, b^n) \), and let \( x \) be a regular element of \( I \). Then \( x = ca + db^n \) for some \( c, d \in R \). Using the Binomial Theorem, one can easily show that \( b^n | x^n \) since \( b^n | a^n \), and hence \( b | x \) by hypothesis. Since \( R \) is a Marot ring, \( a \) is a linear combination of regular elements of \( I \), and thus \( b | a \). Hence \( a/b \in R \); so \( R \) is root closed. The converse is clear. The proofs for the other three closedness properties are similar. For the (2, 3)-closed case, let \( I = (a, b^3) \); for the integrally closed case, let \( I = (a, b) \); and for the completely integrally closed case, let \( I = (a, b) \). Details are left to the reader. \( \square \)

We next give an example of a reduced commutative ring \( R \) which satisfies the four closedness conditions with respect to units of \( T(R) \), but not with respect to \( T(R) \). This answers questions raised in Ref. [2, Question 2.7] and Refs. [1, Sec. 3]. We would like to thank Thomas G. Lucas for suggesting this example.

**Example 4.3.** (A reduced commutative ring \( R \) which is root closed (resp., (2, 3)-closed, integrally closed, completely integrally closed) with respect to units of \( T(R) \), but \( R \) is not root closed (resp., (2, 3)-closed, integrally closed, completely integrally closed).) We employ the “\( A + B \)” construction, see Refs. [6, Section 26] and [12] for more details. Let \( K \) be a field, \( D = K[X^2, X^3, XY, Y] \), and let \( \mathcal{P} = \{ P \in \text{Spec}(R) \mid htP = 1 \text{ and } Y \notin P \} \). Let \( \mathcal{A} \) be an indexing set for \( \mathcal{P} \), and let \( I = \mathcal{A} \times \mathbb{N} \). For each \( i = (a, n) \in I \), let
Let \( R \) be a commutative ring.

**Proposition 4.4.** If a commutative ring \( R \) is (2, 3)-closed with respect to units of \( T(R) \), then \( \text{nil}(T(R)) = \text{nil}(R) \).

**Proof.** Clearly \( \text{nil}(R) \subset \text{nil}(T(R)) \). Conversely, suppose that \( (x/y)^n = 0 \) for some integer \( n \geq 1 \). Then \( (x + y^n)/y = (x/y) + y^{n-1} \) is a unit in \( T(R) \) with \( [(x + y^n)/y]^m \in R \) for all integers \( m \geq n \). Thus \( (x + y^n)/y \in R \) by hypothesis, and hence \( x/y \in R \).

**Corollary 4.5.** Let \( R \) be a commutative ring.

1. If \( R \) is (2, 3)-closed with respect to units of \( T(R) \), then \( \text{nil}(R) \subset sR \) for each regular element \( s \in R \).
2. Suppose that \( \text{nil}(R) = Z(R) \). Then \( R \) is (2, 3)-closed (resp., root closed, integrally closed, completely integrally closed) if and only if \( R \) is (2, 3)-closed (resp., root closed, integrally closed, completely integrally closed) with respect to units of \( T(R) \).

**Proof.** (1) This follows directly from Proposition 4.4.

(2) We show that if \( R \) is (2, 3)-closed with respect to units of \( T(R) \) and \( \text{nil}(R) = Z(R) \), then \( R \) is (2, 3)-closed; the proofs of the other cases are left to the reader. Let \( x = a/b \in T(R) \) with \( x^2, x^3 \in R \). If \( a \in R - Z(R) \), then \( x \in R \) by hypothesis. If \( a \in Z(R) = \text{nil}(R) \), then \( x \in \text{nil}(T(R)) = \text{nil}(R) \subset R \). Thus \( R \) is (2, 3)-closed.

If \( T(R) - R \subset U(T(R)) \), then clearly \( R \) is (2, 3)-closed (resp., root closed, integrally closed, completely integrally closed) if and only if \( R \) is (2, 3)-closed (resp., root closed, integrally closed, completely integrally closed) with respect to units of \( T(R) \). Thus it is of interest to characterize the commutative rings \( R \) such that \( T(R) - R \subset U(T(R)) \). Recall that an ideal of a commutative ring \( R \) is said to be divided if it is comparable to every other ideal of \( R \) (equivalently, to every principal ideal of \( R \)). Note that if \( Z(R) \) is a
divided ideal (and hence necessarily prime), then \( T(R) - R \subseteq U(T(R)) \). We next show that the converse is also true except in trivial cases (note that if \( R = T(R) \), then \( Z(R) = R - U(R) \); so, in this case, \( Z(R) \) is an ideal of \( R \) if and only if \( R \) is quasilocal with maximal ideal \( Z(R) \).

**Proposition 4.6.** Let \( R \) be a commutative ring which contains a nonunit regular element (i.e., \( R \neq T(R) \)). Then \( T(R) - R \subseteq U(T(R)) \) if and only if \( Z(R) \) is a divided prime ideal of \( R \).

**Proof.** We have already observed that \( T(R) - R \subseteq U(T(R)) \) if \( Z(R) \) is a divided (prime) ideal of \( R \). Conversely, suppose that \( R \) contains a nonunit regular element and \( T(R) - R \subseteq U(T(R)) \). Thus \( Z(R) \subseteq sR \) for each regular element \( s \) of \( R \). We need only show that \( Z(R) \) is an ideal of \( R \). Let \( I = \cap \{xR \mid x \text{ is a regular element of } R\} \). By hypothesis, \( I \) is a proper ideal of \( R \) containing \( Z(R) \). We show that \( I = Z(R) \). If not, then there is an \( x \in I - Z(R) \). Thus \( Z(R) \subseteq xR \subseteq I \subseteq xR \), and hence \( I = xR \). Similarly, \( I = x^2R \). Thus \( x \in U(R) \), a contradiction.

We next consider several classes of commutative rings \( R \) such that \( Z(R) \) is a divided prime ideal of \( R \). Note that such rings are Marot rings. As in Ref. [4], we say that a commutative ring \( R \) is a pseudo-valuation ring (PVR) if \( aP \) and \( bR \) are comparable for all \( a, b \in R \) and prime ideals \( P \) of \( R \). A PVR is necessarily quasilocal (for this and other results about PVRs, see Refs. [3,4]). We say that a commutative ring \( R \) is a \( \Phi \)-pseudo-valuation ring (\( \Phi \)-PVR) if \( \text{nil}(R) \) is a divided prime ideal of \( R \) and for each prime ideal \( P \neq \text{nil}(R) \) of \( R \), \( aP \) and \( bR \) are comparable for all \( a, b \in R - \text{nil}(R) \) (cf. [13, Corollary 7] and [14, Proposition 1.1(6)]); equivalently, \( \text{nil}(R) \) is a divided prime ideal of \( R \) and \( R/\text{nil}(R) \) is a PVR. [15, Proposition 2.9, 16, Theorem 3.1] Also, see [16] for some other generalizations of PVRs.

**Corollary 4.7.** Let \( R \) be a commutative ring such that \( Z(R) \) is a divided prime ideal of \( R \). Then \( R \) is \((2,3)\)-closed (resp., root closed, integrally closed, completely integrally closed) if and only if \( R \) is \((2,3)\)-closed (resp., root closed, integrally closed, completely integrally closed) with respect to units of \( T(R) \). In particular, the above statement holds for a PVR or a \( \Phi \)-PVR.

**Proof.** The first statement is clear since in this case \( T(R) - R \subseteq U(T(R)) \) by Proposition 4.6. For the “in particular” statement, just note that a PVR or a \( \Phi \)-PVR clearly satisfies the given hypothesis.

### 5 SEMINORMAL RINGS

Following Swan [17], we say that a (necessarily reduced) commutative ring \( R \) is seminormal if whenever \( a^2 = b^3 \) for \( a, b \in R \), then \( a = c^3 \) and \( b = c^2 \).
for some \( c \in R \). The importance of seminormality is that \( \operatorname{Pic}(R[X]) = \operatorname{Pic}(R) \) if and only if \( R/\operatorname{nil}(R) \) is seminormal.\cite{17, Theorem 1} In Ref. \cite{1}, we studied several variants of seminormality. In this section, we briefly continue that investigation. The following two conditions were conditions (2) and (6) in Ref. \cite{1}, respectively.

\[ \begin{align*}
(5) & \quad \text{If } a^2 = b^3 \text{ with } a, b \in R \text{, then } b \mid a \text{ in } R. \\
(6) & \quad \text{If } a^2 = b^3 \text{ with } a, b \in R \text{ regular, then } b \mid a \text{ in } R. 
\end{align*} \]

Clearly condition (5) implies condition (6) and any seminormal ring satisfies conditions (5) and (6). However, condition (6) is not equivalent to \( R \) being seminormal since any total quotient ring satisfies (6). The ring \( R \) in Example 4.3 also satisfies (6), but \( R \) is not seminormal. In Ref. \cite{1, Example 2.7(a)}, an example was given of a reduced commutative ring which satisfies condition (5), but is not seminormal. In fact, \( R \) is seminormal if and only if \( R \) is \((2, 3)\)-closed and \( T(R) \) is seminormal.\cite{1, Theorem 3.1(a)} In the spirit of Proposition 1.2, one may easily show that \( T(R) \) is seminormal if and only if whenever \( a^2 = b^3 \) for \( a, b \in R \), there are \( c, s \in R \) with \( s \) regular such that \( s^3 a = c^3 \) and \( s^2 b = c^2 \).

We next give several other conditions equivalent to condition (6).

The \( \Leftrightarrow \) equivalence of Proposition 5.1 is also in Ref. \cite[Theorem 3.1(c)]{1}.

**Proposition 5.1.** The following statements are equivalent for a commutative ring \( R \).

\[ \begin{align*}
(1) & \quad \text{If } a^2 = b^3 \text{ with } a, b \in R \text{ regular, then } a = c^3 \text{ and } b = c^2 \text{ for some (regular) } c \in R \text{ (i.e., } R \text{ is seminormal with respect to regular elements of } R). \\
(2) & \quad \text{If } a^2 = b^3 \text{ with } a, b \in R \text{ regular, then } b \mid a \text{ (i.e., } R \text{ satisfies condition } (6)). \\
(3) & \quad \text{If } b^2 \mid a^2 \text{ and } b^3 \mid a^3 \text{ with } a, b \in R \text{ regular, then } b \mid a \text{ (i.e., } R \text{ is } (2, 3)\text{-closed with respect to units of } T(R)).
\end{align*} \]

**Proof.** (1) \( \Rightarrow \) (2) Clear.

(2) \( \Rightarrow \) (3) Suppose that \( b^2 \mid a^2 \) and \( b^3 \mid a^3 \) with \( a, b \in R \) regular. Then \( \beta = a^2/b^2 \) and \( z = a^3/b^3 \) are regular elements of \( R \) which satisfy \( \beta^2 = \beta^3 \), and hence \( \beta \mid z \) by hypothesis. Thus \( b \mid a \).

(3) \( \Rightarrow \) (1) Suppose that \( a^2 = b^3 \) with \( a, b \in R \) regular. Then \( b^2 \mid a^2 \) and \( b^3 \mid a^3 \), and hence \( b \mid a \) by hypothesis. Let \( c = a/b \in R \). Then \( a = c^3 \) and \( b = c^2 \). \( \square \)

Our next result is a slight generalization of Ref. \cite[Theorem 3.3]{1} via Corollary 2.6. As a special case, if \( T(R) \) is von Neumann regular, then \( R \) is
seminormal if and only if $R$ satisfies any of the three equivalent conditions in Proposition 5.1 (recall that a von Neumann regular ring is seminormal).

**Proposition 5.2.** Let $R$ be a commutative Marot ring such that $T(R)$ is seminormal. Then $R$ is seminormal if and only if $R$ satisfies condition (6).

**Proof.** By Theorem 4.2 and Proposition 5.1 $R$ satisfies condition (6) if and only if $R$ is $(2, 3)$-closed. As mentioned above, $R$ is seminormal if and only if $T(R)$ is seminormal and $R$ is $(2, 3)$-closed. The result follows. □

**Corollary 5.3.** (cf. [1, Theorem 3.3]) Let $R$ be a strongly regular commutative ring (i.e., $T(R)$ is von Neumann regular). Then $R$ is seminormal if and only if $R$ satisfies condition (6). □

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Received April 2001