ON ABELIAN $\pi$-REGULAR RINGS

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INTRODUCTION

Throughout this paper the letter $R$ denotes an associative ring with 1, $\text{Id}(R)$ denotes the set of all idempotent elements of $R$, $\mathcal{C}(R)$ denotes the center of $R$, and $\text{Nil}(R)$ denotes the set of all nilpotent elements of $R$. A $\pi$-regular ring $R$ is called an abelian $\pi$-regular ring if $\text{Id}(R)$ is a subset of $\mathcal{C}(R)$. Recall that a ring $R$ is called strongly $\pi$-regular if for every $x \in R$ there exist $n \geq 1$ and $y \in R$ such that $x^{2^n} y = x^n$. It is easy to see that an abelian $\pi$-regular ring is strongly $\pi$-regular. In [14, Theorem 2, (2)], Ohori showed that in an abelian $\pi$-regular ring $R$, the $\text{Nil}(R)$ is a two-sided ideal of $R$ and $R/\text{Nil}(R)$ is regular. His proof relies on [1, Lemma 1] and [4, Remark]. The purpose of this paper is to give an
alternative proof of this fact and in Theorem 3 we prove the converse of this fact. Also, we show that every element in an abelian $\pi$-regular ring $R$ is a sum of two units if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of $R$. Recall an element $x$ of $R$ is called regular (unit regular) if there exists $y \in R$ (a unit $u$ in $R$) such that $xyx = x$ ($xux = x$).

We start with the following lemma:

**Lemma 1.** Let $x \in R$. If $x$ is unit regular, then $x = eu$ for some $e \in \text{Id}(R)$ and $u \in U(R)$, where $U(R)$ denotes the set of all units of $R$.

**Proof.** Suppose $x$ is unit regular. Then for some $v \in U(R)$ we have $xvx = x$. Let $e = xv \in \text{Id}(R)$ and $u = v^{-1}$. Then $x = eu$.

The following fact is needed in the proof of Theorem 1.

**Fact 1.** [2, Theorem 2]. Suppose $\text{Id}(R) \subseteq C(R)$. Let $x \in R$. If $x$ is regular, then $x$ is unit regular.

The following theorem gives a characterization of all $\pi$-regular elements in a ring $R$ such that $\text{Id}(R) \subseteq C(R)$.
**Theorem 1.** Suppose $\text{Id}(R) \subseteq C(R)$. Let $x \in R$. Then $x$ is $\pi$-regular if and only if there exists $e \in \text{Id}(R)$ such that $ex$ is regular and $(1-e)x \in \text{Nil}(R)$.

**Proof.** Since $x$ is $\pi$-regular, for some $n \geq 1$, $x^n$ is regular. Hence, by Fact 1 and Lemma 1, we have $x^n = eu$ for some $e \in \text{Id}(R)$ and $u \in U(R)$. Then $ex(x^{n-1}u^{-1})ex = (ex^n u^{-1})ex = (e uu^{-1})ex = ex$. Hence, $ex$ is regular. Now, $[(1-e)x]^n = (1-e)x^n = (1-e)eu = 0$, since $(1-e) \in C(R)$ and $x^n = eu$.

For the converse, suppose for some $e \in \text{Id}(R)$, $ex$ is regular and $(1-e)x \in \text{Nil}(R)$. Then for some $n \geq 1$, $[(1-e)x]^n = (1-e)x^n = 0$. Hence, $(*)$ $ex^n = x^n$. Since $ex$ is regular, by Lemma 1, $ex = cu$ for some $c \in \text{Id}(R)$ and $u \in U(R)$. Hence, $(ex)^n = (cu)^n = cu^n$, since $c \in C(R)$. But $(ex)^n = ex^n = x^n$ by $(*)$. Thus $x^n = cu^n$. Let $y = cu^n$. Then $x^nyx^n = x^n$ and hence $x$ is $\pi$-regular. □

Suppose $\text{Id}(R) \subseteq C(R)$ and $x \in R$ such that $x$ is $\pi$-regular. Then by the proof of the above theorem for some $e \in \text{Id}(R)$, $v \in U(R)$ and $m \geq 1$, we have $x^n = ev$ and $ex$ is regular. Hence, by Fact 1 and Lemma 1, $ex = cw$ for some $c \in \text{Id}(R)$ and $w \in U(R)$. In fact, $e = c$. For, $e(ex) = e(cw)$. But $e(ex) = ex = cw$. Thus, $ecw = cw$ and therefore $(**)$ $ec = c$.

Since $e, c \in C(R)$, we have $(ex)^m = ex^m = cw^m$. Since $x^m = ev$, $ex^m = ev = cw^m$. Hence, $e = cw^m v^{-1}$. Thus $ec = cw^m v^{-1} c = cw^m v^{-1}$, since $c \in C(R)$. Hence, $ec = e$. Since $ec = c$ by $(**)$ and $ec = e$, $e = c$. Thus, $ex = ew$.
In light of the above argument and Theorem 1, we have

**Lemma 2.** Suppose \( \text{Id}(R) \subset C(R) \). Let \( x \in R \) such that \( x \) is \( \pi \)-regular. Then for some \( e \in \text{Id}(R) \) and \( u \in U(R) \) we have \( ex = eu \).

**MAJOR RESULTS**

Now, we state the first major result in this paper.

**Theorem 2.** Suppose \( R \) is abelian \( \pi \)-regular. Then \( \text{Nil}(R) \) is a two-sided ideal of \( R \).

**Proof.** Let \( w \in \text{Nil}(R) \) and \( r \in R \). Suppose \( rw \) is not in \( \text{Nil}(R) \). By Lemma 2, there exists \( e \in \text{Id}(R) \) and \( u \in U(R) \) such that \( erw = rew = eu \). Observe that \( e \neq 0 \). For, if \( e = 0 \) then \( (1-e)rw = rwe \text{Nil}(R) \) by Theorem 1 and this contradicts the assumption that \( rw \) is not in \( \text{Nil}(R) \). Since \( ew \in \text{Nil}(R) \), let \( n \) be the smallest integer such that \( (ew)^n = 0 \). Then \( n \geq 2 \), since \( e \neq 0 \). Thus, \( 0 = rew(ew)^{n-1} = eu(ew)^{n-1} = u(ew)^{n-1} \). Hence, \( (ew)^{n-1} = 0 \), a contradiction. Thus, for any \( w \in \text{Nil}(R) \) and \( r \in R \), we have \( rw \in \text{Nil}(R) \).

A similar argument will show that for any \( w \in \text{Nil}(R) \) and \( r \in R \), we have \( wr \in \text{Nil}(R) \). Now, let \( w, z \in \text{Nil}(R) \) and suppose \( w + z \) is not in \( \text{Nil}(R) \). Then, once again, there exist \( c \in \text{Id}(R) \), \( c \neq 0 \), and \( u \in U(R) \) such that \( c(w + z) = cu \). Hence, \( cw = cv - cz = cv(1 - v^{-1}z) \). Since \( -v^{-1}z \in \text{Nil}(R) \),
\[ 1 - v^1 z = u \in U(R). \] Thus, \( cw = cvu \). But \( cw \in \text{Nil}(R) \) and \( cvu \) is not in \( \text{Nil}(R) \). Hence, \( w + z \in \text{Nil}(R) \). Thus, \( \text{Nil}(R) \) is a two-sided ideal of \( R \).

Before stating the second major result, the following two well-known lemmas are needed.

**Lemma 3.** Let \( R \) be a ring with 1 and \( I \) be a two-sided nil ideal of \( R \). If \( [c] \in \text{Id}(R/I) \), then there exists \( e \in \text{Id}(R) \) such that \( [e] = [c] \) in \( R/I \).

**Lemma 4.** Let \( I \) be a two-sided nil ideal of \( R \), \( K = R/I \) and \( u \in R \). Then \( [u] \in U(K) \) if and only if \( u \in U(R) \).

**Theorem 3.** Suppose \( \text{Id}(R) \subset C(R) \). Then \( R \) is \( \pi \)-regular if and only if \( \text{Nil}(R) \) is a two-sided ideal of \( R \) and \( R/\text{Nil}(R) \) is regular.

**Proof.** Suppose \( R \) is \( \pi \)-regular. By Theorem 2, \( \text{Nil}(R) \) is a two-sided ideal of \( R \). Let \( [x] \in R/\text{Nil}(R) \). Then for some \( y \in R \) and \( n \geq 1 \), \( x^n y x^n = x^n \). Thus, \( e = x^n y \in \text{Id}(R) \) and therefore \( 1 - e \in \text{Id}(R) \). Since \( 1 - e \in C(R) \), \( ((1 - e)x)^n = (1 - e)x^n = (1 - x^n y)x^n = 0 \). Thus, \( (1 - e)x = (1 - x^n y)x \in \text{Nil}(R) \).

Thus, \([x] [x^{n-1} y] [x] = [x^n y] [x] = [x] \).

Suppose \( \text{Nil}(R) \) is a two-sided ideal of \( R \) and \( K = R/\text{Nil}(R) \) is regular. Let \( x \in R \). By Fact 1, \([x] \) is unit regular in \( K \) and, by Lemma 1, \([x] = [c] [u] \) for some \([c] \in \text{Id}(R) \).
Id(K) and \([u] \in U(K)\). By Lemma 3, there exists \(e \in Id(R)\) such that \([c] = [e]\) and by Lemma 4, \(u \in U(R)\). Thus, \(x = eu + w\) for some \(w \in Nil(R)\). Now, \(ex = e(u + w)\). Since \(w \in J(R)\), \(u + w \in U(R)\), where \(J(R)\) denotes the Jacobson radical of \(R\). Thus, \(ex\) is regular. Further, \((1 - e)x = x - ex = (eu + w) - (eu + ew) = w - ew \in Nil(R)\). Hence, \((1 - e)x \in Nil(R)\).

Thus, by Theorem 1, \(x\) is \(\pi\)-regular.

Suppose a ring \(R\) is an abelian \(\pi\)-regular ring. Since \(Nil(R)\) is a two-sided ideal of \(R\), \(Nil(R) \subset J(R)\). Since \(R/Nil(R)\) is regular by Theorem 3 and the Jacobson radical of any regular ring is 0, we have \(J(R) = Nil(R)\).

**Lemma 5.** Suppose \(R\) is abelian \(\pi\)-regular. Then \(J(R) = Nil(R)\).

The following result follows from Theorem 3 and Lemma 1.

**Corollary 1.** A ring \(R\) is abelian \(\pi\)-regular if and only if \(Id(R) \subset C(R)\), \(Nil(R)\) is a two-sided ideal of \(R\), and for every \(x \in R\) there exist \(e \in Id(R)\), \(u \in U(R)\), and \(w \in Nil(R)\) such that \(x = eu + w\).

In light of Theorems 1 and 3, we have:
Theorem 4. Suppose \( \text{Id}(R) \) is a subset of \( \mathcal{C}(R) \). Then \( R \) is \( \pi \)-regular if and only if for some two-sided nil ideal \( I \) of \( R \), \( K = R/I \) is \( \pi \)-regular.

**Proof.** Suppose \( R \) is \( \pi \)-regular. By Theorem 2, \( I = \text{Nil}(R) \) is a two-sided ideal of \( R \), and by Theorem 3, \( K = R/I \) is regular and hence \( \pi \)-regular.

For the converse, assume that \( R/I \) is \( \pi \)-regular for some two-sided nil ideal \( I \) of \( R \). Then \( \text{Nil}(R/I) = \text{Nil}(R)/I \) is a two-sided ideal of \( R/I \) by Theorem 3. So \( \text{Nil}(R) \) is a two-sided ideal of \( R \). Since \( R/I \) is \( \pi \)-regular, so is \( R/\text{Nil}(R) \). Therefore by Theorem 3, \( R \) is \( \pi \)-regular.

A consequence of the above theorem is the following corollary

**Corollary 2.** Suppose \( \text{Id}(R) \) is a subset of \( \mathcal{C}(R) \). Then \( R \) is \( \pi \)-regular if and only if \( R/\text{N}(R) \) is \( \pi \)-regular where \( \text{N}(R) \) is the prime radical of \( R \).

**RELATED RESULTS**

Recall, a prime ideal \( P \) of a ring \( R \) is called completely prime iff \( R/P \) is domain. It is well-known that if \( \text{Id}(R) \subset \mathcal{C}(R) \) and \( R \) is regular and \( I \) is a prime ideal of \( R \), then \( R/I \) is a division ring. However, the above fact is not always true for an abelian \( \pi \)-regular
ring $R$. The referee provided us with a counterexample, see [13, Proposition 1.11] and [3, example 3.3]. But, we are able to state the following result:

**Theorem 5.** Suppose $R$ is abelian $\pi$-regular and let $P$ be a prime ideal of $R$, then every element in $K = R/P$ is either a nilpotent element of $K$ or a unit element of $K$. In particular, if $P$ is a prime ideal of $R$ containing $\text{Nil}(R)$ (e.g., a left or right primitive ideal of $R$), then $K$ is a division ring.

**Proof.** Let $x \in R$ such that $x \notin P$. Then for some $e \in \text{Id}(R)$ and $u \in U(R)$ and $n \geq 1$, we have $x^n = eu$ by Lemma 1. Now, if $e \in P$, then $x \notin \text{Nil}(K)$. Hence, suppose that $e \notin P$. Thus, $eu \notin P$. Since $(1 - e)Re \subset P$ and $e \notin P$, $(1 - e)e \in P$. Thus $[e] = [1]$ in $R/P$. Thus $[x^n] = [eu] = [u]$ in $R/P$. But $[x^n] = [u]$ in $R/P$ implies $[x^n]$ is a unit in $R/P$ and therefore $[x]$ is a unit in $R/P$.

By Theorem 3, $\text{Nil}(R)$ is a two-sided ideal and $R/\text{Nil}(R)$ is a reduced regular ring. Thus every prime factor of $R/\text{Nil}(R)$ is a division ring. Let $P$ be a prime ideal of $R$ containing $\text{Nil}(R)$. Then $K = R/P$ is a prime factor ring of $R/\text{Nil}(R)$ and so $K$ is a division ring. Particularly, if $P$ is a left (or right) primitive ideal of $R$, then note that $\text{Nil}(R) = J(R)$ by Lemma 5 and so $\text{Nil}(R) \subset P$. Thus the ring $K$ is a division ring.
**Remark.** Let $K$ and $P$ as in the above theorem. It is easy to see that $K = R/P$ is a division ring iff $R/P$ is domain iff $P$ is completely prime.

Ehrlich [5] showed that if $R$ is a unit regular ring, then every element in $R$ is a sum of two units. A ring $R$ is called an $(s,2)$-ring [11], see also [7], if every element in $R$ is a sum of two units of $R$. The following theorem gives a characterization of all abelian $\pi$-regular $(s,2)$-rings.

**Theorem 6.** Suppose $R$ is abelian $\pi$-regular. Then $R$ is an $(s,2)$-ring if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of $R$.

**Proof.** Suppose $R$ is an $(s,2)$-ring and $\mathbb{Z}/2\mathbb{Z}$ is a homomorphic image of $R$. Then $1 \in R$ cannot be a sum of two units. Hence, $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of $R$.

Conversely, suppose $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of $R$. By Theorem 5, every primitive factor of $R$ is a division ring and hence Artinian. Thus, by [7, Theorem 2] $R$ is an $(s,2)$-ring.

From Theorem 6, we have the following corollaries:

**Corollary 3.** Let $R$ be an abelian $\pi$-regular ring such that $2 = (1+1)\epsilon U(R)$. Then $R$ is an $(s,2)$-ring.
Corollary 4. Let \( R \) be an abelian \( \pi \)-regular ring. Then \( R \) is an \((s,2)\)-ring if and only if for some \( d \in \text{U}(R) \), \( 1+d \in \text{U}(R) \).

If 2 is a nonnilpotent element in an abelian \( \pi \)-regular ring \( R \), then we have

Theorem 7. Suppose \( R \) is abelian \( \pi \)-regular and 2 is a nonnilpotent element of \( R \). Then there exists \( e \in \text{Id}(R) \) such that \( e \neq 0 \), and every element in \( eR \) is a sum of two units of \( R \).

Proof. Since 2 is \( \pi \)-regular, by Lemma 2 we have \( e2 = eu \) for some \( e \in \text{Id}(R) \) and \( u \in \text{U}(R) \). Since 2 is not nilpotent, we see that \( e \neq 0 \) and \((1-e)2\) is nilpotent by Theorem 1 and the proof of Theorem 2. Now, let \( x \in eR \).

By Corollary 1, there exist \( c \in \text{Id}(R) \), \( v \in \text{U}(R) \) and \( w \in \text{Nil}(R) \) such that \( x = cv + w \). Since \( ex = x \), we have \( x = ex = ecv + ew \). On the other hand, since \((1-e)2 = 2-2e\) is nilpotent, \( 1-(2-2e) = -1 + 2e \in \text{U}(R) \) and so \( 1-2e \in \text{U}(R) \). If \( c = 0 \), then \( 1-2ec = 1 \in \text{U}(R) \). If \( c \neq 0 \), then \( c(1-2e) = c-2ec \in \text{U}(cR) = \text{U}(cRc) \) and thus there is \( a \in cR \) such that \( (c-2ec)a = a(c-2ec) = c \).

Therefore \((1-2ec)(a+1-c) = (c-2ec+1-c)(a+1-c) = 1\) and similarly \((a+1-c)(1-2ec) = 1\). Thus, \( 1-2ec \in \text{U}(R) \).

Since \( 2e = eu \), we have \( 1-uec \in \text{U}(R) \). Now, \( 1-uec = (u^1 - ec)u \in \text{U}(R) \) and \( u \in \text{U}(R) \). So \( u^1 - ec \in \text{U}(R) \).
U(R) and hence \(-u^{-1} + ec \in U(R)\). Therefore \(ec = (-u^{-1} + ec) + u^{-1}\) with \(-u^{-1} + ec \in U(R)\) and \(u^{-1} \in U(R)\). Now for our convenience, let \(z = -u^{-1} + ec\) and \(d = u^{-1}\). Hence, \(x = (z+d)v + ew = zv + (dv+ew)\). Since \(ew \in \text{Nil}(R)\) and \(\text{Nil}(R) = J(R)\), \((dv+ew) \in U(R)\). Thus, \(x\) is a sum of two units of \(R\).

Observe that if 2 is a nonnilpotent element of \(R\), then this does not imply that \(R\) is an \((s,2)\)-ring. For example, \(R = \mathbb{Z}_6\) is abelian \(\pi\)-regular and 2 is a nonnilpotent element of \(R\), but \(R\) is not an \((s,2)\)-ring. However, 4 \(\in \text{Id}(R)\) and every element in 4\(R\) is a sum of two units.

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