American University of Sharjah

Department of Mathematics and Statistics

PLACEMENT EXAM MANUAL

for

Architecture, Engineering, and Sciences

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Contents

Chapter

1 *EQUATIONS AND INEQUALITIES*

In this chapter, we give a summary of the algebraic techniques and the graphical approach for solving equations and inequalities. The following types of equations and inequalities will be discussed:

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- *Quadratic Equations.*
- *Radical Equations.*
- *Equations Involving Absolute Value.*
- *Fractional Equations.*
- *Exponential Equations.*
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- *Trigonometric Equations.*
- *Linear Inequalities.*
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- *Absolute Value Inequalities.*
- *Graphical Solution of Equations and Inequalities.*

LINEAR EQUATIONS

The general form of the *linear equation*, or *first-degree equation*, is

$$ax + b = cx + d$$

where a, b, c and d are constants. Consider the following examples of linear equations and their solution.

Example 1. Solve the equation

$$(x-2)(x-5) = x^2 - 6(4x+7) + 18$$

Solution. Expanding the terms by multiplication yields

$$x^2 - 7x + 10 = x^2 - 24x - 42 + 18$$

Note that the x^2 term cancel, hence the resulting equation is linear.

$$-7x + 10 = -24x - 42 + 18$$

Grouping similar terms on both sides we obtain

$$-7x + 10 = -24x - 24$$

We now isolate the terms involving x on one side. First we subtract 10 from both sides, then add $24x$ to both sides.

$$24x - 7x = -24 - 10$$

$$17x = -34$$

By dividing both sides by the coefficient of x, which is 17, we get the solution:

$$x = -2$$

Example 2. Solve the equation

$$\frac{4x-7}{4}+\frac{5}{6}=3x-\frac{x}{3}$$

Solution. Multiply both sides by the *lowest common denominator (LCD)* of 4, 6, and 3, which is $LCD = 12$.

$$12\left(\frac{4x-7}{4}+\frac{5}{6}\right)=12\left(3x-\frac{x}{3}\right)$$

Expand the terms by multiplication.

$$12\left(\frac{4x-7}{4}\right)+12\left(\frac{5}{6}\right)=12(3x)-12\left(\frac{x}{3}\right)$$

After simplifying

$$3(4x-7)+10=36x-4x$$

$$12x-21+10=36x-4x$$

Group similar terms on both sides.

$$12x-11=32x$$

Isolate the terms involving x on one side.

$$12x-32x=11$$

$$-20x=11$$

Divide both sides by the coefficient of x, which is -20, yields the solution

$$x=-\frac{11}{20}$$

QUADRATIC EQUATIONS

A *quadratic equation* in one variable can be written in the *standard form*

$$ax^2+bx+c=0, \qquad a\neq 0$$

where a, b, and c are constants. The solutions of this quadratic equation are given by the *quadratic formula*:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

where b^2-4ac is called the *discriminant*. We have the following three cases:

1. If $b^2-4ac>0$, then the equation has two real roots.
2. If $b^2-4ac=0$, then the equation has one root (a double root).
3. If $b^2-4ac<0$, then the equation has two imaginary (complex) roots.

Example 1. Solve the equation

$$6x^2=13x+5$$

Solution. Rewrite the equation in standard form.

$$6x^2-13x-5=0$$

Factor the left side.

$$(3x+1)(2x-5)=0$$

So either

$$3x+1=0 \qquad \text{or} \qquad 2x-5=0$$

The first equation gives $x=-\dfrac{1}{3}$, while the second equation yields $x=\dfrac{5}{2}$. The equation has the two solutions:

$$x=-\frac{1}{3},\ \frac{5}{2}$$

Example 2. Solve the equation

$$\frac{x^2}{3}=\frac{x}{2}$$

Solution. Multiply both sides of the equation by the $LCD=6$, in order to get rid of the denominators.

$$2x^2=3x$$

$$2x^2-3x=0$$

$$x(2x-3)=0$$

so either

$$x=0 \qquad \text{or} \qquad 2x-3=0$$

The second equation gives $x=\dfrac{3}{2}$. The equation has the two solutions:

$$x=0,\ \frac{3}{2}$$

Example 3. Solve the equation

$$1+\frac{8}{x^2}=\frac{4}{x}$$

Solution. Multiply both sides of the equation by the $LCD=x^2$, in order to eliminate the denominators.

$$x^2\left(1+\frac{8}{x^2}\right)=x^2\left(\frac{4}{x}\right)$$

$$x^2+8=4x$$

$$x^2-4x+8=0$$

The left side can't be factored, so instead we use the quadratic formula with $a=1,\ b=-4$, and $c=8$. This gives $b^2-4ac=16-32=-16$, i.e. we have complex roots.

$$x=\frac{4\pm\sqrt{-16}}{2}=\frac{4\pm 4i}{2}=2\pm 2i$$

where $i=\sqrt{-1}$. The equation has the two complex solutions:

$$x=2+2i,\ 2-2i$$

Example 4. Solve the equation

$$(x+1)(2x-3)+1=x(x-3)+3$$

Solution. Expand the terms on both sides by multiplication.

$$2x^2 - x - 3 + 1 = x^2 - 3x + 3$$

Rewrite the equation in standard from.

$$x^2 + 2x - 5 = 0$$

Apply the quadratic formula with $a = 1,\ b = 2$, and $c = -5$ to obtain

$$x = \frac{-2 \pm \sqrt{4 + 20}}{2} = \frac{-2 \pm \sqrt{24}}{2} = \frac{-2 \pm 2\sqrt{6}}{2} = -1 \pm \sqrt{6}$$

Thus, the equation has the two real solutions:

$$x = -1 + \sqrt{6}, -1 - \sqrt{6}$$

RADICAL EQUATIONS

In this section, we deal with equations involving radicals, and equations involving rational exponents.

Example 1. Solve the radical equation

$$x = \sqrt{x + 6}$$

Solution. To remove the radical, square both sides of the equation.

$$x^2 = \left(\sqrt{x + 6}\right)^2$$

$$x^2 = x + 6$$

Subtract x and 6 from both sides.

$$x^2 - x - 6 = 0$$

Upon factoring we get

$$(x - 3)(x + 2) = 0$$

This gives

$$x = -2, \qquad x = 3$$

Check these x values by substituting them into the original equation. We find that $x = 3$ is the only solution. However, $x = -2$ is not a solution, which is called *extraneous solution* meaning it is an extra answer.

Example 2. Solve the radical equation

$$\sqrt{2x - 3} + x = 1$$

Solution. First, we need to isolate the radical. This is achieved by subtracting x from both sides.

$$\sqrt{2x - 3} = 1 - x$$

Square both sides to get rid of the radical.

$$\left(\sqrt{2x - 3}\right)^2 = (1 - x)^2$$

Expanding we get

$$2x - 3 = 1 - 2x + x^2$$

Subtract $2x$ then add 3, to both sides of the equation.

$$0 = x^2 - 4x + 4$$

$$0 = (x-2)(x-2)$$

Therefore, $x = 2$ is the only solution, which can be checked by substituting it into the original equation.

Example 3. Solve the radical equation

$$\sqrt{2x-1} - \sqrt{x-5} = 3$$

Solution. To solve an equation containing two radicals we first isolate one of them. Thus, add $\sqrt{x-5}$ to both sides.

$$\sqrt{2x-1} = \sqrt{x-5} + 3$$

Square both sides.

$$\left(\sqrt{2x-1}\right)^2 = \left(\sqrt{x-5}+3\right)^2$$

Expand both sides.

$$2x - 1 = x - 5 + 6\sqrt{x-5} + 9$$

After simplifying we get

$$2x - 1 = x + 4 + 6\sqrt{x-5}$$

We got rid of one radical. In order to remove the second root, namely $\sqrt{x-5}$, we isolate it by subtracting x and 4 from both sides of the equation.

$$x - 5 = 6\sqrt{x-5}$$

Square both sides.

$$(x-5)^2 = \left(6\sqrt{x-5}\right)^2$$

Expand each term.

$$x^2 - 10x + 25 = 36(x-5)$$

$$x^2 - 10x + 25 = 36x - 180$$

Rewrite the equation in standard form.

$$x^2 - 46x + 205 = 0$$

Factor the left-hand side.

$$(x-5)(x-41) = 0$$

Therefore, we obtain the two solutions

$$x = 5, \qquad x = 41$$

These can be checked by substitution into the original equation.

Example 5. Solve the following equation with rational exponents.

$$x^{1/3} - x^{1/6} - 2 = 0$$

Solution. This is an equation involving radical powers. If we substitute

$$u = x^{1/6}$$

then we get a quadratic equation in u.

$$u^2 - u - 2 = 0$$

Factor the left-hand side.

$$(u-2)(u+1) = 0$$

This gives

$$u = -1 \qquad \text{or} \qquad u = 2$$

so

$$x^{1/6} = -1 \qquad \text{or} \qquad x^{1/6} = 2$$

The first equation gives

$$x = (-1)^6 = 1$$

while the second results

$$x = 2^6 = 64$$

Checking both answers, we find that $x = 1$ is not a solution, but $x = 64$ is the only solution.

ABSOLUTE VALUE EQUATIONS

We consider the solution of equations involving the absolute value. It is important to remark that

$$|u| = a \qquad \text{is equivalent to} \qquad u = a \text{ or } u = -a$$

Example 1. Solve the absolute value equation

$$|2x - 5| = 3$$

Solution. The equation is equivalent to

$$2x - 5 = 3 \qquad \text{or} \qquad 2x - 5 = -3$$

To solve the first equation $2x - 5 = 3$ add 5 to both sides.

$$2x = 8$$

Divide both sides by the coefficient of x, which is 2.

$$x = 2$$

As for the second equation $2x - 5 = -3$ if we add 5 we obtain

$$2x = -8$$

$$x = -4$$

Therefore, the equation has the two solutions $x = 2$ and $x = 4$.

Example 2. Solve the absolute value equation

$$5\,|2x - 1| - 6 = 14$$

Solution. First, isolate the absolute value term by adding 6 to both sides of the equation.

$$5\,|2x - 1| = 20$$

Divide by 5.

$$|2x - 1| = 4$$

This implies that

$$2x - 1 = 4 \qquad \text{or} \qquad 2x - 1 = -4$$

Solving the first equation $2x - 1 = 4$, we get $x = \frac{5}{2}$. The second equation $2x - 1 = -4$ yields $x = -\frac{3}{2}$. Thus, the equation has the two solutions

$$x = \frac{5}{2}, \qquad x = -\frac{3}{2}$$

Example 3. Solve the absolute value equation

$$|7 - 10x| = -3\,|4x + 3|$$

Solution. It is important to note that

$$|u| = |v| \qquad \text{is equivalent to} \qquad u = v \quad \text{or} \quad u = -v$$

Therefore, for our equation

$$7 - 10x = -3(4x + 3) \qquad \text{or} \qquad 7 - 10x = -3(-4x - 3)$$

To solve the first equation $7 - 10x = -3(4x + 3)$:

$$7 - 10x = -12x - 9$$

Isolating the term involving x yields

$$2x = -16$$

Hence $x = 8$. For the second equation $7 - 10x = -3(-4x - 3)$:

$$7 - 10x = 12x + 9$$

Isolate the variable x.

$$-22x = 2$$

Hence the second solution is $x = -\frac{1}{11}$. Therefore, the equation has the two solutions:

$$x = -8, \qquad x = -\frac{1}{11}$$

FRACTIONAL EQUATIONS

Example 1. Solve the fractional equation

$$\frac{x+1}{2x-1} = \frac{3x}{6x-7}$$

Solution. Recall the *cross multiplication*:

$$\frac{a}{b} = \frac{c}{d} \qquad \text{is equivalent to} \qquad ad = bc$$

Therefore, by cross-multiplication we have

$$(x + 1)(6x - 7) = 3x(2x - 1)$$

Expand both sides by multiplying.

$$6x^2 - 7x + 6x - 7 = 6x^2 - 3x$$

Note that the term $6x^2$ cancels from both sides. Collecting similar terms gives

$$-x - 7 = -3x$$

$$-7 = -2x$$

Hence, the solution is $x = \dfrac{-7}{-2} = 3.5$.

Example 2. Solve the fractional equation

$$\frac{x}{x-2} + \frac{3}{x+2} = \frac{8}{x^2-4}$$

Solution. Multiply both sides of the equation by the $LCD = x^2 - 4 = (x-2)(x+2)$.

$$(x-2)(x+2)\left[\frac{x}{x-2} + \frac{3}{x+2}\right] = (x-2)(x+2)\left[\frac{8}{(x-2)(x+2)}\right]$$

which is equal to

$$(x-2)(x+2)\left(\frac{x}{x-2}\right) + (x-2)(x+2)\left(\frac{3}{x+2}\right) = 8$$

$$x(x+2) + 3(x-2) = 8$$

Multiply the two expressions on the left-hand side.

$$x^2 + 2x + 3x - 6 = 8$$

Rewrite the quadratic equation in standard form.

$$x^2 + 5x - 14 = 0$$

Factor the quadratic equation.

$$(x+7)(x-2) = 0$$

Thus $x = -7$ or $x = 2$. It can be easily checked that $x = -7$ is a solution of the equation. However, if we substitute $x = 2$ into the equation, then the first fraction becomes $\dfrac{2}{0}$, which is not valid since we can't divide by 0. So we must discard the extraneous solution $x = 2$. Therefore, $x = -7$ is the only solution.

EXPONENTIAL EQUATIONS

Example 1. Solve the exponential equation

$$4^{3x+1} = \frac{1}{16}$$

Solution. We make the base of the two expressions on both sides of the equation the same.

$$\left(2^2\right)^{3x+1} = \frac{1}{2^4}$$

$$2^{6x+2} = 2^{-4}$$

Note that

$$a^x = a^y \qquad \text{implies} \qquad x = y$$

Hence

$$6x + 2 = -4$$

Solving this linear equation gives the solution $x = -1$.

Example 2. Solve the exponential equation

$$3^{5x-4} = 5$$

Solution. Unlike example 1, the bases (3 and 5) of the two expressions on both sides cannot be made the same. Instead, to solve the equation we take the logarithm of both sides.

$$\log\left(3^{5x-4}\right) = \log 5$$

Using the properties of logs we get.

$$(5x-4)\log 3 = \log 5$$

$$5x - 4 = \frac{\log 5}{\log 3}$$

so

$$x = \frac{1}{5}\left(4 + \frac{\log 5}{\log 3}\right) \approx 1.092994704$$

Example 3. Solve the exponential equation

$$2e^{3x} - 5 = 11$$

Solution. Isolate the term involving the exponential expression. Add 5 to both sides.

$$2e^{3x} = 16$$

Divide by 2.

$$e^{3x} = 8$$

Take the natural logarithm of both sides, i.e. $\ln x = \log_e x$.

$$\ln\left(e^{3x}\right) = \ln 8$$

The properties of logs give

$$3x \ln e = \ln 8$$

since $\ln e = 1$ so

$$3x = \ln 8$$

The solution is $x = \dfrac{1}{3}\ln 8$.

LOGARITHMIC EQUATIONS

Example 1. Solve the logarithmic equation

$$\log_2(7 - 3x) = 4$$

Solution. Recall that

$$\log_a x = y \qquad \text{is equivalent to} \qquad a^y = x$$

It is important to remark that $\log_a x$ is defined for $x > 0$, which is the *domain* of $\log_a x$.

The definition of logs yields

$$7 - 3x = 2^4 = 16$$

Isolate the x term.

$$-3x = 9$$

Thus, the only solution is $x = -3$.

Example 2. Solve the logarithmic equation

$$1 + 5\ln(3x) = 11$$

Solution. Note that the natural logarithm is defined by $\ln x = \log_e x$. Isolate the expression involving ln.

$$5\ln(3x) = 10$$

$$\ln(3x) = 2$$

which is equivalent to

$$e^2 = 3x$$

The solution is $x = \frac{1}{3}e^2$.

Example 3. Solve the logarithmic equation

$$\log_2(x+2) + \log_2(x-1) = 2$$

Solution. Combine the left side using the property:

$$\log_a M + \log_a N = \log_a MN$$

We have

$$\log_2[(x+2)(x-1)] = 2$$

which is equivalent to

$$2^2 = (x+2)(x-1)$$

$$4 = x^2 + x - 2$$

$$0 = x^2 + x - 6$$

$$0 = (x+3)(x-2)$$

Hence, $x = 2$ or $x = -3$. Note that $x = -3$ is not a solution because if we substitute it into the *original equation* we get the term $\log_2(-1)$, which is not defined (since $\log x$ is defined for $x > 0$). Thus, $x = 2$ is the only solution.

Example 4. Solve the logarithmic equation

$$\log x = \log(2x+1) - \log(x+2)$$

Solution. Combine the right side using

$$\log_a M - \log_a N = \log_a\left(\frac{M}{N}\right)$$

We have

$$\log x = \log\left(\frac{2x+1}{x+2}\right)$$

which implies that

$$x = \frac{2x+1}{x+2}$$

$$x(x+2) = 2x+1$$

$$x^2 + 2x = 2x + 1$$

$$x^2 = 1$$

so $x = -1$ and $x = 1$. Note that $x = -1$ in not a solution, since when substituted in the equation the first term becomes $\log(-1)$ which is not defined. So $x = 1$ is the only solution.

TRIGONOMETRIC EQUATIONS

Example 1. Solve the trigonometric equation

$$2\sin^2 x - \sin x = 0$$

Solution. Factor out $\sin x$.

$$\sin x(2\sin x - 1) = 0$$

Since $ab = 0$ if $a = 0$ or $b = 0$, so

$$\sin x = 0 \qquad \text{or} \qquad \sin x = \frac{1}{2}$$

First, we solve the two equations over one period $[0, 2\pi)$. For the first equation the solutions are $x = 0$, and $x = \pi$. As for the second equation, the solutions are $x = \frac{\pi}{6}$ and $\frac{5\pi}{6}$. Since the sine function is periodic, all solutions are given by

$$x = \begin{cases} n\pi \\ \pi/6 + 2n\pi \\ 5\pi/6 + 2n\pi \end{cases}$$

for $n = 0, \pm 1, \pm 2, ...$

Example 2. Solve the trigonometric equation

$$8\sin^2 x = 5 - 10\cos x$$

Solution. Move all the terms to the left side.

$$8\sin^2 x + 10\cos x - 5 = 0$$

Express the left side in terms of $\cos x$ by using the identity $\sin^2 x + \cos^2 x = 1$.

$$8\left(1 - \cos^2 x\right) + 10\cos x - 5 = 0$$

$$8 - 8\cos^2 x + 10\cos x - 5 = 0$$

$$3 - 8\cos^2 x + 10\cos x = 0$$

Multiply both sides by -1.

$$8\cos^2 x - 10\cos x - 3 = 0$$

$$(4\cos x + 1)(2\cos x - 3) = 0$$

this implies that

$$\cos x = -\frac{1}{4} \qquad \text{or} \qquad \cos x = \frac{3}{2}$$

First, we solve each equation over one period $[0, 2\pi)$. The second equation $\cos x = \frac{3}{2}$ has no solution since $-1 \le \cos x \le 1$. For the first equation, the solution is $x = \cos^{-1}(-1/4) \approx 1.82348$ and $x = \pi + 1.82348 = 4.4597$. The two answers are angles in the second and third quadrants since the cosine is negative there. Since the cosine function is periodic, all solutions are given by

$$x = \begin{cases} 1.82348 + 2n\pi \\ 4.4597 + 2n\pi \end{cases}$$

for $n = 0, \pm 1, \pm 2, ...$

Example 3. Solve the trigonometric equation

$$\cos 2x = \sin 2x - 1, \qquad 0 \leq x < 2\pi$$

Solution. We use the following identities:

$$\sin 2x = 2 \sin x \cos x, \qquad \cos 2x = 2\cos^2 x - 1$$

The equation becomes

$$2\cos^2 x - 1 = 2 \sin x \cos x - 1$$

$$2\cos^2 x - 2 \sin x \cos x = 0$$

$$2 \cos x(\cos x - \sin x) = 0$$

This implies that

$$\cos x = 0 \qquad \text{or} \qquad \cos x = \sin x$$

The first equation gives $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$. As for the second equation $\sin x = \cos x$, it is equivalent to $\tan x = 1$ which gives $x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$. Thus, the complete set of solutions is

$$x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}, \frac{3\pi}{2}$$

Example 4. Solve the trigonometric equation

$$\sec^2 x - 2 \tan x = 4$$

Solution. We convert all the trigonometric terms to one trigonometric function. If we replace the $\sec^2 x$ term with $\tan^2 x + 1$ using the identity $\tan^2 x + 1 = \sec^2 x$, all the trigonometric terms will be tangent terms.

$$1 + \tan^2 x - 2 \tan x = 4$$

$$\tan^2 x - 2 \tan x - 3 = 0$$

$$(\tan x - 3)(\tan x + 1) = 0$$

This implies

$$\tan x = 3 \qquad \text{or} \qquad \tan x = -1$$

First, we restrict the domain so the function is one-to-one. The graph of the tangent function is one-to-one on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. On that interval, the first equation gives $x = \tan^{-1}(3) \approx 1.249045772$. The second equation yields $x = \tan^{-1}(-1) = -\dfrac{\pi}{4} \approx -.7853981634$. Since $\tan x$ has period π, therefore the complete set of solutions is

$$x = \begin{cases} 1.249045772 + n\pi \\ -.7853981634 + n\pi \end{cases}$$

INEQUALITIES

In the next sections, we consider different kinds of inequalities and the procedures for solving them. To solve them, we need to make use of the basic properties of inequalities which are listed below.

Properties of Inequalities.

Let a, b, c, and d be real numbers.

1. *Transitive Property*

$$a < b \quad \text{and} \quad b < c \qquad \Longrightarrow \qquad a < c$$

2. *Addition of a Constant*

$$a < b \qquad \Longrightarrow \qquad a + c < b + c$$

3. *Subtraction of a Constant*

$$a < b \qquad \Longrightarrow \qquad a - c < b - c$$

5. *Multiplication Property*

$$\text{For } c > 0,\ a < b \qquad \Longrightarrow \qquad ac < bc$$

$$\text{For } c < 0,\ a < b \qquad \Longrightarrow \qquad ac > bc$$

5. *Division Property*

$$\text{For } c > 0,\ a < b \qquad \Longrightarrow \qquad \frac{a}{c} < \frac{b}{c}$$

$$\text{For } c < 0,\ a < b \qquad \Longrightarrow \qquad \frac{a}{c} > \frac{b}{c}$$

LINEAR INEQUALITIES

Example 1. Solve the linear inequality

$$3x + 7 > 7x - 5$$

Solution. We isolate the variable x.

$$3x - 7x > -5 - 7$$

$$-4x > -12$$

Divide by -4. Since $-4 < 0$, so we need to change the direction of the inequality.

$$x < 3$$

Therefore, the solution consists of all real numbers less than 3. In interval notation, the solution is $(-\infty, 3)$. The solution set is sketched below in Fig 1.

Example 2. Solve the linear inequality

$$\frac{5 + 4x}{2} - \frac{1}{3} \geq x - \frac{7}{2}$$

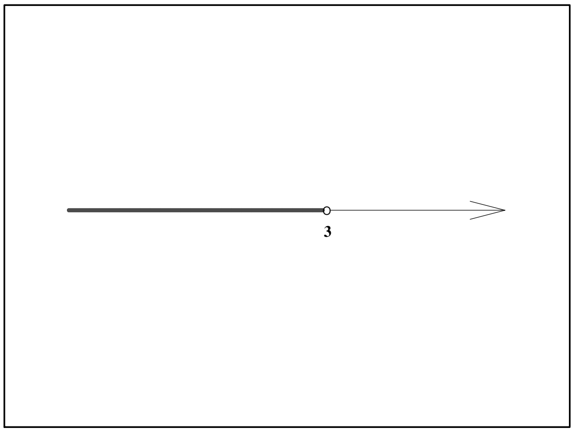


Figure 1:

Solution. Multiply both sides of the inequality by the $LCD = 6$.

$$6\left(\frac{5+4x}{2}\right) - 6\left(\frac{1}{3}\right) \geq 6x - 6\left(\frac{7}{2}\right)$$

$$3(5+4x) - 2 \geq 6x - 21$$

$$15 + 12x - 2 \geq 6x - 21$$

$$12x - 6x \geq -21 + 2 - 15$$

$$6x \geq -34$$

Finally, divide by the coefficient of x.

$$x \geq -\frac{34}{6}$$

The solution consists of all real numbers greater or equal to $-\dfrac{17}{3}$. The solution interval is $\left[-\dfrac{17}{3}, \infty\right)$. The solution set is sketched in Fig 2.

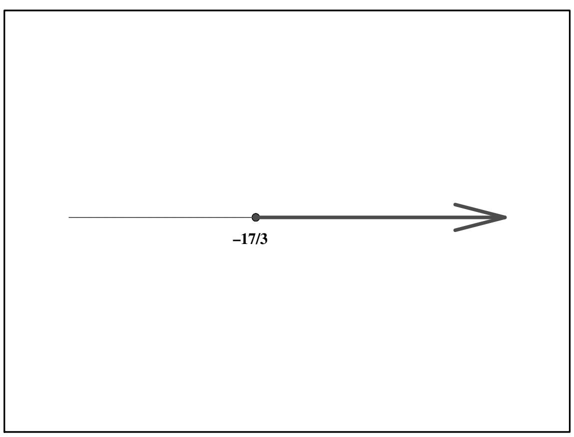


Figure 2:

Example 3. *(Double Inequality)* Solve the compound inequality

$$-3 \leq 4 - 7x < 18$$

Solution. We need to isolate the x in the middle. Subtract 4 from each member.

$$-3 - 4 \le -7x < 18 - 4$$

$$-7 \le -7x < 14$$

Divide each member by -7, i.e the coefficient of x. Since $-7 < 0$, so we need to reverse the direction of the inequality.

$$\frac{-7}{-7} \ge \frac{-7x}{-7} > \frac{14}{-7}$$

$$1 \ge x > -2$$

or equivalently

$$-2 < x \le 1$$

The solution interval is $(-2, 1]$. The solution set is sketched in Fig 3.

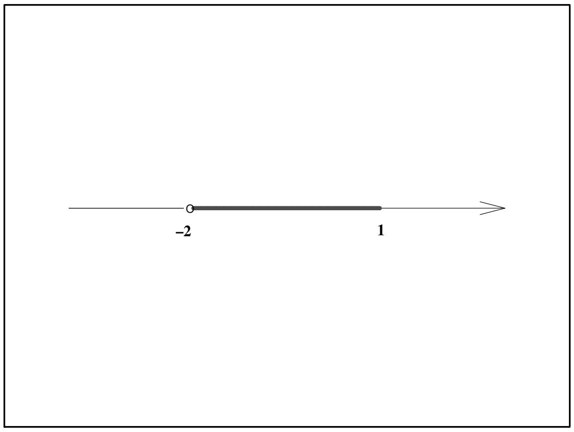


Figure 3:

NONLINEAR INEQUALITIES

Example 1. *(Quadratic Inequality)* Solve the polynomial inequality

$$x^2 + 2x \ge 15$$

Solution. First, we rewrite the inequality in standard form.

$$x^2 + 2x - 15 \ge 0$$

$$(x + 5)(x - 3) \ge 0$$

We find the critical points (i.e. the zeros) of the polynomial $P(x) = x^2 + 2x - 15$. They occur at $x = -5$ and $x = 3$, which divide the real number line into three test intervals: $(-\infty, -5)$, $(-5, 3)$, and $(3, \infty)$. To solve $x^2 + 2x - 15 \ge 0$, we need to choose only one x-value from each test interval and evaluate the polynomial at that value. If the answer is negative (positive), the polynomial will have negative (positive) values for every x-value in the interval. Table (1) summarizes the different cases.

The table shows that the polynomial has nonnegative values on the solution interval: $(-\infty, -5] \cup [3, \infty)$.

Test Interval	*Test Point*	*Value of Polynomial*	*Sign of Polynomial*
$(-\infty, -5)$	$x = -6$	$P(-6) = 9$	Positive
$(-5, 3)$	$x = 0$	$P(0) = -15$	Negative
$(3, \infty)$	$x = 4$	$P(4) = 9$	Positive

Table 1:

Example 2. Solve the quadratic inequality

$$x^2 + 4 < 4x$$

Solution. First, we rewrite the inequality in standard form.

$$x^2 - 4x + 4 < 0$$

$$(x - 2)(x - 2) < 0$$

$$(x - 2)^2 < 0$$

Note that there are no values of x that satisfy the inequality $(x - 2)^2 < 0$, since the quantity $(x - 2)^2$ is nonnegative. Thus, the solution set is empty, i.e. there are no values of x that satisfy the given inequality.

Example 3. Solve the inequality

$$\frac{x + 1}{x - 3} > -2$$

Solution. The $LCD = x - 3$, however the quantity $x - 3$ can have both negative and positive values. Therefore, we multiply both sides of the inequality by the nonnegative term $(x - 3)^2$.

$$(x - 3)^2 \left(\frac{x + 1}{x - 3}\right) < -2(x - 3)^2$$

$$(x - 3)(x + 1) < -2(x^2 - 6x + 9)$$

$$x^2 - 2x - 3 < -2x^2 + 12x - 18$$

$$3x^2 - 14x + 15 < 0$$

$$(3x - 5)(x - 3) < 0$$

The critical points of the polynomial $P(x) = 3x^2 - 14x + 15$ occur at $x = \dfrac{5}{3}$ and $x = 3$, which divide the real number line into three test intervals: $\left(-\infty, \dfrac{5}{3}\right)$, $\left(\dfrac{5}{3}, 3\right)$, and $(3, \infty)$. Table 2 tests these intervals.

Test Interval	*Test Point*	*Value of Polynomial*	*Sign of Polynomial*
$\left(-\infty, \dfrac{5}{3}\right)$	$x = 0$	$P(0) = 15$	Positive
$\left(\dfrac{5}{3}, 3\right)$	$x = 2$	$P(2) = -1$	Negative
$(3, \infty)$	$x = 4$	$P(4) = 7$	Positive

Table 2:

The table shows that the polynomial has negative values on the solution interval: $(\frac{5}{3}, 3)$.

ABSOLUTE VALUE INEQUALITIES

In this section, we show how to solve inequalities involving the absolute value. The basic properties of the absolute value inequalities are:

1. $|x| < a$ if and only if $-a < x < a$.
2. $|x| \leq a$ if and only if $-a \leq x \leq a$.
3. $|x| > a$ if and only if $x > a$ or $x < -a$.
4. $|x| \geq a$ if and only if $x \geq a$ or $x \leq -a$.

Example 1. Solve the following absolute value inequality

$$|2x - 5| < 3$$

Solution. The inequality is equivalent to

$$-3 < 2x - 5 < 3$$

Add 5 to each member, then divide all terms by 2.

$$2 < 2x < 8$$

$$1 < x < 4$$

The solution interval is: $(1, 4)$.

Example 2. Solve the following absolute value inequality

$$2\left|\frac{1}{2}x + 3\right| - 3 \geq 25$$

Solution. Add 3 to both sides.

$$2\left|\frac{1}{2}x + 3\right| \geq 28$$

$$\left|\frac{1}{2}x + 3\right| \geq 14$$

The resulting inequality is equivalent to

$$\frac{1}{2}x + 3 \geq 14 \qquad \text{or} \qquad \frac{1}{2}x + 3 \leq -14$$

Multiply both sides of each inequality by 2.

$$x + 6 \geq 28 \qquad \text{or} \qquad x + 6 \leq -28$$

$$x \geq 22 \qquad \text{or} \qquad x \leq -34$$

Thus, the solution interval is: $(-\infty, -34] \cup [22, \infty)$.

GRAPHICAL SOLUTION OF EQUATIONS AND INEQUALITIES

In this last section, we show how to approximate the solution of equations and inequalities graphically.

To approximate the solution of an equation graphically, follow the following steps:

1. Rewrite the equation in *standard form*, i.e. in the form $f(x) = 0$ with only zero on the right-hand side.

2. Use a graphing utility to graph the function $y = f(x)$.

3. Use the zoom and tracing features to find the x-intercepts of the graph of f, which give the approximate solutions of the given equation.

To approximate the solution of an inequality graphically, follow the following steps:

1. Rewrite the inequality in *standard form* $f(x) > 0$ (or $f(x) \geq 0$).

2. Use a graphing utility to graph the function $y = f(x)$.

3. Find the intervals over which the graph of $f(x)$ lies above the x-axis. These intervals are the solutions of the given inequality.

It is important to note that you can as well rewrite the inequality in the *standard form* $f(x) < 0$ (or $f(x) \leq 0$) instead of $f(x) > 0$ (or $f(x) \geq 0$). However, in this case you have to approximate the intervals over which the graph of $f(x)$ lies below the x-axis.

Example 1. Use the graphical technique to approximate the solution of the following equation.

$$x^2 = 3x + 4$$

Solution. First, rewrite the equation in standard form to obtain $x^2 - 3x - 4 = 0$. Then, we sketch the graph of the function $f(x) = x^2 - 3x - 4$. From the graph shown in Fig 4, the intercepts are: $x = -1$ and $x = 4$, which are the solutions of the equation.

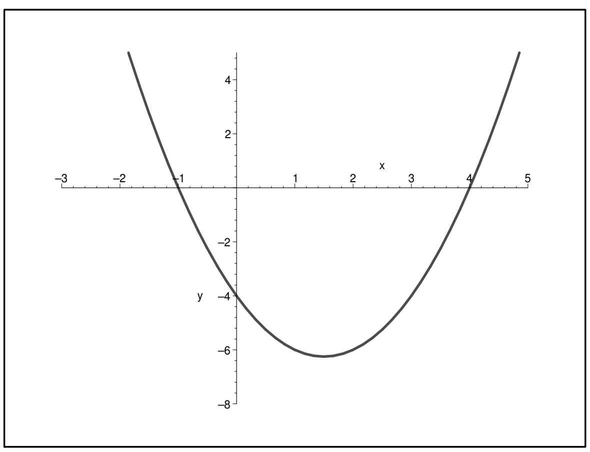


Figure 4:

Example 2. Use the graphical technique to approximate the solution of the following inequality.

$$2x^2 > 10 - x$$

Solution. Rewrite the inequality in standard form to obtain $2x^2 + x - 10 > 0$. Then, we sketch the graph of the function $f(x) = 2x^2 + x - 10$. From Fig 5, the graph is above the x-axis on the intervals: $\left(-\infty, \dfrac{-5}{2}\right) \cup (2, \infty)$, which is the solution interval of the inequality.

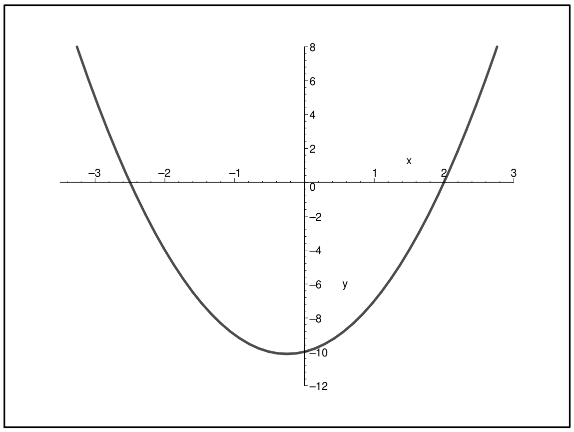


Figure 5:

Chapter

2 *FUNCTIONS*

In this chapter, we explore the concept of functions and how we manipulate them. The following ideas will be studied:

- *Prerequisites*
- *Functions*
- *Combinations of Functions*
- *Transformations of Functions*
- *Applied Functions: Variation*
- *Odd and Even Functions*
- *Increasing and Decreasing Functions*
- *Extreme Values of Functions*
- *Inverse Functions*
- *Intercepts*

PREREQUISITES

Distance Formula.

The *distance* d between the points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint Formula.

The *midpoint* of the line segment joining the points (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Standard Form of the Equation of a Circle.

The standard form of the equation of a circle whose *center* is at (h, k) and has *radius* r is

$$(x - h)^2 + (y - k)^2 = r^2$$

FUNCTIONS

Definition. A function f from a set A to a set B is a rule that assigns to each element x in the set A exactly one element $f(x)$ in the set B.

The ser A is the *domain* of f. The element $f(x)$ is called the *image of x under f*. The set of all images $f(x)$ as x varies throughout the domain is called the *range* of f.

Representations of Functions. There are three ways to represent a function.

1. *Numerically:* This is done through a table or a set of ordered pairs.
2. *Geometrically:* The function is described by a graph.
3. *Algebraically:* The function is represented by a formula.

To determine whether a given curve is the graph of a function we use the following rule:

The Vertical Line Test. A curve in the xy-plane is the graph of a function if and only if no vertical line intersects the curve more than one point.

COMBINATIONS OF FUNCTIONS

Two given functions f and g with overlapping domains A and B can be combined to form the new functions $f+g$, $f-g$, fg, and f/g. For all $x \in A \cap B$, these functions are defined as follows:

1. *Sum* : $(f+g)(x) = f(x) + g(x)$
2. *Difference* : $(f-g)(x) = f(x) - g(x)$
3. *Product* : $(fg)(x) = f(x) \cdot g(x)$
4. *Quotient* : $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$

Definition. The *composition* of the functions f and g, denoted by fog, is defined by

$$(fog)(x) = f(g(x))$$

The domain of fog is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

Piecewise Defined Functions. Functions are sometimes defined using different formulas on different parts of its domain. For example,

$$f(x) = \begin{cases} 3x-1, & x < 2 \\ x-5, & 2 \leq x < 4 \\ -1, & x = 4 \\ x+1, & x > 4 \end{cases}$$

Another example is the absolute value function.

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

These are referred to as *piecewise-defined* functions.

TRANSFORMATIONS OF FUNCTIONS

The common type of transformations are:

A. Horizontal and Vertical Shifts.

Suppose we know the graph of $y = f(x)$, then the following graphs are obtained as follows:

1. $y = f(x) + c$: Vertical shift upwards if $c > 0$, and vertical shift downward if $c < 0$.
2. $y = f(x - c)$: Horizontal shift right if $c > 0$, and horizontal shift left if $c < 0$.

B. Reflections.

Reflections of the graph of $y = f(x)$ through the two coordinate axis are represented as follows:

1. $y = -f(x)$: Reflection in the x-axis.
2. $y = f(-x)$: Reflection in the y-axis.

C. Vertical Stretching and Shrinking.

These nonrigid transformations of the graph of $y = f(x)$ are represented as follows:

1. $y = cf(x)$ $(c > 1)$: Vertical stretch by a factor of c.
2. $y = cf(x)$ $(0 < c < 1)$: Vertical shrink by a factor of c.

APPLIED FUNCTIONS: VARIATION

Definition. *(Proportionality)*

1. The quantity y is *directly proportional* to x if there exists a constant $k \neq 0$ (proportionality constant) such that
$$y = kx$$
2. The quantity y is *inversely proportional* to x if there exists a constant $k \neq 0$ (proportionality constant) such that
$$y = \frac{k}{x}$$
3. The quantity z is *jointly proportional* to x and y if there exists a constant $k \neq 0$ (proportionality constant) such that
$$z = kxy$$

EVEN AND ODD FUNCTIONS

Definition.

1. The function f is *even* if $f(-x) = f(x)$ for all x in the domain of f.
2. The function f is *odd* if $f(-x) = -f(x)$ for all x in the domain of f.

INCREASING AND DECREASING FUNCTIONS

Definition. *Increasing and Decreasing Functions*

1. The function f is *increasing* on an interval I if, for any x_1 and x_2 in I:
$$x_1 < x_2 \qquad \text{implies} \qquad f(x_1) < f(x_2)$$
2. The function f is *decreasing* on an interval I if, for any x_1 and x_2 in I:
$$x_1 < x_2 \qquad \text{implies} \qquad f(x_1) > f(x_2)$$

3. The function f is *constant* on an interval I if, for any x_1 and x_2 in I, $f(x_1) = f(x_2)$.

EXTREME VALUES OF FUNCTIONS

Definition. *(Extreme Values)*

1. The function f has an *absolute(global) maximum* on the domain D at a point c if
$$f(c) \geq f(x) \qquad \text{for all } x \in D$$
2. The function f has an *absolute(global) minimum* on the domain D at a point c if
$$f(c) \leq f(x) \qquad \text{for all } x \in D$$
3. The function f has an *relative(local) maximum* at a point c if there exists an interval (x_1, x_2) that contains c such that
$$f(c) \geq f(x) \qquad \text{for all } x \in (x_1, x_2)$$
4. The function f has an *relative(local) minimum* at a point c if there exists an interval (x_1, x_2) that contains c such that
$$f(c) \leq f(x) \qquad \text{for all } x \in (x_1, x_2)$$

INVERSE FUNCTIONS

Next, we will introduce the basic results related to inverse functions. First, we state the definition of an inverse function.

Definition. *(Inverse functions)* The functions f and g are inverse functions if and only if

$$\begin{aligned} g(f(x)) &= x \qquad && \text{for every } x \text{ in the domain of } f. \\ f(g(y)) &= y && \text{for every } y \text{ in the domain of } g. \end{aligned}$$

The inverse of $f(x)$ is denoted by $f^{-1}(x)$. Alternatively, if we use the notation f^{-1} (rather than g), then the defining conditions are

$$\begin{aligned} f^{-1}(f(x)) &= x, \qquad && \text{for every } x \text{ in the domain of } f \\ f(f^{-1}(x)) &= x, && \text{for every } x \text{ in the domain of } f^{-1} \end{aligned}$$

In other words, we have

$$f^{-1}(x) = y \qquad \Longleftrightarrow \qquad f(y) = x$$

for all x in the domain of f^{-1}.

We have the following relationships between f and f^{-1}.

$$\text{Domain of } f^{-1} = \text{Range of } f$$
$$\text{Domain of } f = \text{Range of } f^{-1}$$

The next results deal with the existence of inverse functions. We require the following definition.

Definition. *(One-to-one)* A function $f(x)$ is one-to-one if:

$$x_1 \neq x_2 \qquad \Longrightarrow \qquad f(x_1) \neq f(x_2)$$

or equivalently,

$$f(x_1) = f(x_2) \qquad \Longrightarrow \qquad x_1 = x_2$$

Horizontal Line Test. A function $f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

This leads to the following existence theorem.

Theorem. A function f has an inverse if and only if it is one-to-one.

It follows from the last theorem and the horizontal line test that:

Theorem. A function f has an inverse if and only if its graph is cut at most once by any horizontal line.

The next theorem states that the functions $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about the line $y = x$.

Theorem. If f has an inverse, then the graph of $y = f(x)$ is the mirror image of the graph of $y = f^{-1}(x)$ about the line $y = x$; i.e. the functions $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about the line $y = x$.

The following theorem is useful.

Theorem. If f is increasing or decreasing on its domain, then the function f has an inverse.

Finally, we give the steps needed to compute inverse functions.

How to find the inverse of a function f?

Step 1. Verify that the function f is one-to-one, to make sure that the inverse exists.

Step 2. Write $y = f(x)$.

Step 3. Solve (if possible) the equation $y = f(x)$ for x in terms of y.

Step 4. Interchange x with y in the equation found in step 3. The resulting equation is $y = f^{-1}(x)$.

INTERCEPTS

Definition. *(Intercepts of a Graph)*

1. *x-intercepts:* The x-coordinates of the points where the graph of an equation intersects the x-axis.
2. *y-intercepts:* The y-coordinates of the points where the graph of an equation intersects the y-axis.

How to find the Intercepts?

1. *x-intercepts:* Set $y = 0$ in the equation of the graph, then solve the resulting equation for x.
2. *y-intercepts:* Set $x = 0$ in the equation of the graph, then solve the resulting equation for y.

Example 1. Given the two points $(-3, -4)$ and $(-5, 2)$.

(a) Find the distance between the two points

(b) Find the midpoint of the line segment joining the two points.

Solution.

(a) Using the distance formula we get

$$\begin{aligned} d &= \sqrt{(-5-(-3))^2 + (2-(-4))^2} \\ &= \sqrt{(-2)^2 + (6)^2} = \sqrt{40} \end{aligned}$$

(b) By using the midpoint formula, we have

$$\left(\frac{-3-5}{2}, \frac{-4+2}{2}\right) = (-4, -1)$$

Example 2. Find the equation of the circle with center $(0, -3)$ and radius 3.

Solution. The equation of the circle is

$$\begin{aligned} (x-0)^2 + (y-(-3))^2 &= 3^2 \\ x^2 + (y+3)^2 &= 9 \end{aligned}$$

Example 3. Let

$$f(x) = -x^2 + 5, \qquad g(x) = \sqrt{5x+4}$$

Find the following:

(*a*) $f(-2)$	(*b*) $f(3t)$
(*c*) $g(0)$	(*d*) $\dfrac{f(a+h) - f(a)}{h}$
(*e*) $(f+g)(0)$	(*f*) $\left(\dfrac{f}{g}\right)(0)$
(*g*) $(gof)(-2)$	(*h*) $(fof)(x)$
(*i*) $(fog)(x)$	(*j*) $(gof)(x)$

Solution.

(a) $f(-2) = -(-2)^2 + 5 = -4 + 5 = 1$

(b) $f(3t) = -(3t)^2 + 5 = -9t^2 + 5$

(c) $g(0) = \sqrt{5(0)+4} = \sqrt{4} = 2$

(d) $\dfrac{f(a+h)-f(a)}{h} = \dfrac{\left[-(a+h)^2+5\right]-\left[-a^2+5\right]}{h}$

$= \dfrac{-a^2-2ah-h^2+5+a^2-5}{h} = \dfrac{-2ah-h^2}{h} = \dfrac{h(-2a-h)}{h} = -2a-h$

(e) $(f+g)(0) = f(0)+g(0) = 5+2 = 7$

(f) $\left(\dfrac{f}{g}\right)(0) = \dfrac{f(0)}{g(0)} = \dfrac{5}{2}$

(g) $(gof)(-2) = g(f(-2)) = g(1) = 3$

(h) $(fof)(x) = f(f(x)) = f(-x^2+5) = -(-x^2+5)^2+5$

$$= -\left(x^4-10x^2+25\right)+5 = -x^4+10x^2-20$$

(i) $(fog)(x) = f(g(x)) = f\left(\sqrt{5x+4}\right) = -\left(\sqrt{5x+4}\right)^2+5$
$= -(5x+4)+5 = -5x+1$

(j) $(gof)(x) = g(f(x)) = g\left(-x^2+5\right) = \sqrt{5(-x^2+5)+4} = \sqrt{29-5x^2}$

Example 4. Find the domain of each function.

(a) $f(x) = x^2+2x-3$

(b) $f(x) = \dfrac{2}{x^2-3x}$

(c) $f(x) = \dfrac{x}{x^2+1}$

(d) $f(x) = \sqrt{6-2x}$

(e) $f(x) = \sqrt{4-x^2}$

(f) $f(x) = \dfrac{|x|}{x}$

Solution.

(a) The function is defined for all values of x, so the domain is all real numbers, i.e. Domain $= (-\infty,\infty)$.

(b) The function is not defined when the denominator is 0. Since

$$x^2-3x = x(x-3) = 0$$

when $x = 0$, or $x = 3$, hence the domain $= (-\infty,\infty)/\{0,3\}$.

(c) The function is not defined when the denominator is 0. Note that the denominator $x^2+1 \neq 0$. Actually $x^2+1 > 0$ (since x^2 and 1 are both nonnegative numbers). Therefore, the domain $= (-\infty,\infty)$.

(d) We cannot take the square root of a negative number, so we require

$$6-2x \geq 0$$

$$6 \geq 2x$$

$$3 \geq x$$

Thus, the domain is $(-\infty,3]$.

(e) As in part (d) we must have

$$4-x^2 \geq 0$$

$$4 \geq x^2$$

$$2 \geq |x|$$

Thus, the domain is $[-2, 2]$.

(f) The absolute value function is defined for all x. We need the denominator, which is x, to be different than 0. Hence, Domain $= (-\infty, \infty)/\{0\}$.

Example 5. Consider the piecewise defined function

$$f(x) = \begin{cases} x^2 - 3x + 6, & x \leq 2 \\ 2x - 5, & x > 2 \end{cases}$$

Evaluate the function at -3, 2 and 4.

Solution.

The rule of this piecewise function states that if it happens that the number x is less or equal to 2, then the value of $f(x)$ is $x^2 - 3x + 6$. On the other hand, if $x > 2$, then the image of x is $2x - 5$.

(a) Since $-3 < 2$, hence $f(-3) = (-3)^2 - 3(-3) + 6 = 24$

(b) Since $2 \leq 2$, we have $f(2) = (2)^2 - 3(2) + 6 = 4$

(c) Since $4 > 2$, hence $f(4) = 2(4) - 5 = 3$

Example 6. Use the graph of $f(x) = x^2$, to sketch $g(x) = (x+2)^2 - 1$.

Solution. We start with the graph of $y = x^2$, then shift it horizontally 2 units to the left to get the graph of $y = (x+2)^2$. Then, we shift the resulting graph vertically downward 1 unit to get the graph of $g(x) = (x+2)^2 - 1$. See figures 1 and 2.

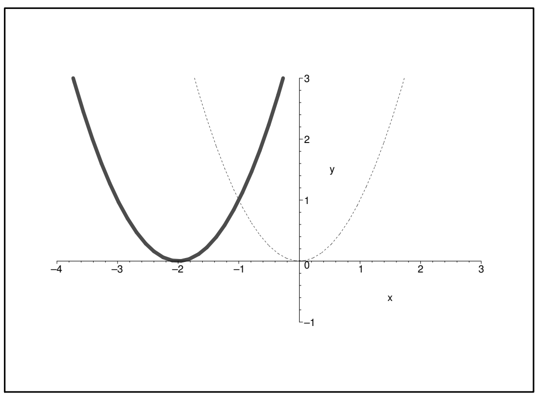


Figure 1: Graph of $y = x^2$ and $y = (x+2)^2$

Example 7. Use the graph of $f(x) = \sqrt{x}$ to sketch $g(x) = -2\sqrt{x}$.

Solution. We start with the graph of $y = \sqrt{x}$, then we stretch the graph vertically by a factor of 2 to obtain the graph of $y = 2\sqrt{x}$. Then, we reflect the resulting graph through the x-axis to get the graph of $g(x) = -2\sqrt{x}$. See figures 3 and 4.

Example 8. Sketch the graph of $f(x) = \sqrt{-x}$.

Solution. We start with the graph of $y = \sqrt{x}$, then reflect the graph through the y-axis to obtain $y = \sqrt{-x}$. See figure 5.

Example 9. Determine whether the curve, shown in Figure 6, is the graph of a function of x.

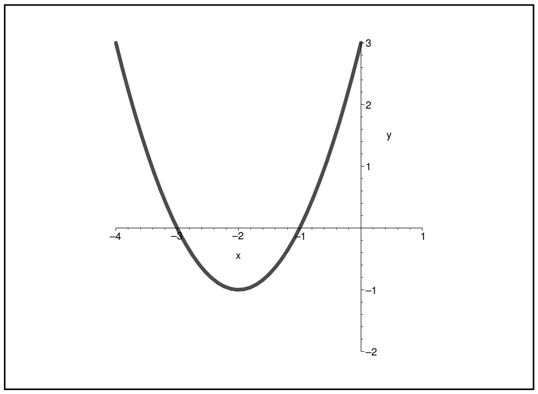


Figure 2: Graph of $y = (x+2)^2 - 1$

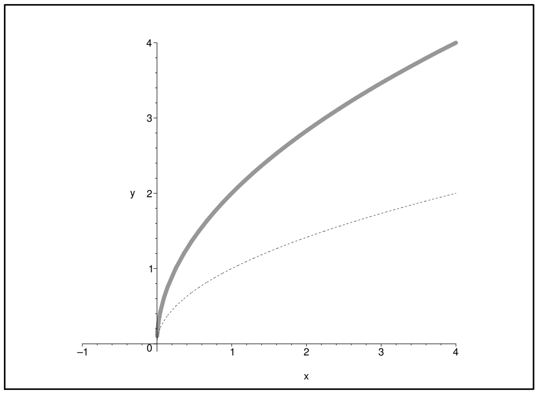


Figure 3: Graph of $y = \sqrt{x}$ and $y = 2\sqrt{x}$

Solution. Applying the Vertical Line Test: If we draw a vertical line, such as $x = 1$, it will cross the curve at two points, hence the curve is not the graph of a function. Alternatively, the point $x = 0$ has two images 3 and -3, i.e. $f(0) = \pm 3$. Hence, f is not a function.

Example 10. Use the given information to find the constant of proportionality.

(a) y is directly proportional to x. If $x = 3$, then $y = 24$.

(b) s is inversely proportional to the square of r. If $r = 6$, then $s = 14$.

(c) w is jointly proportional to x and y and inversely proportional to z. If $x = 2$, $y = 3$ and $z = 5$, then $w = 50$.

Solution.

(a) Since y is directly proportional to x, so there exists a $k \neq 0$ such that

$$y = kx$$

To find k we use the fact that $x = 3$ when $y = 24$.

$$24 = k.3$$

$$k = 8$$

Therefore, $\quad y = 8x$.

(b) Since s is inversely proportional to the square of r, so we have

$$s = \frac{k}{r^2}$$

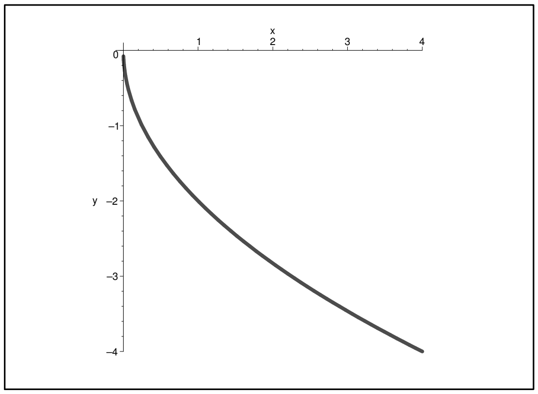


Figure 4: Graph of $y = -2\sqrt{x}$

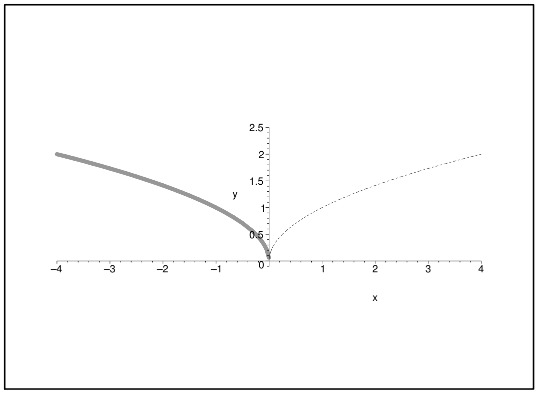


Figure 5: Graph of $y = \sqrt{x}$ and $y = \sqrt{-x}$

To find k we use the fact that $r = 6$ when $s = 14$.

$$14 = \frac{k}{6^2}$$

$$k = 14(36) = 504$$

Therefore, $\quad s = \dfrac{504}{r^2}.$

(c) Since w is jointly proportional to x and y and inversely proportional to z we have

$$w = \frac{kxy}{z}$$

To find k we use the fact that $x = 2$, $y = 3$ and $z = 5$ when $w = 50$.

$$50 = \frac{k(2)(3)}{5}$$

$$250 = 6k$$

$$k = \frac{125}{3}$$

Therefore, $\quad w = \dfrac{125xy}{3z}.$

Example 11. *Boyle's law* states that when a sample of gas is compressed at a constant temperature, the pressure of the gas is inversely proportional to the volume of the gas. Therefore,

$$P = \frac{k}{V}$$

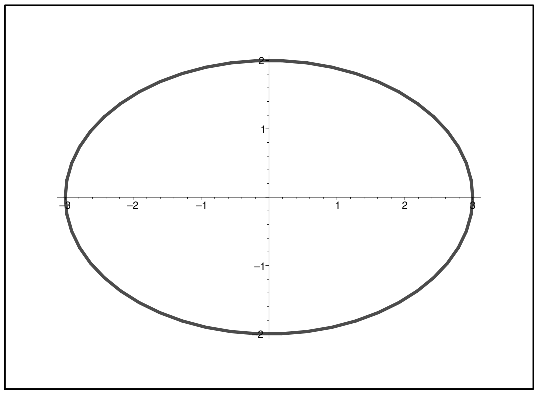


Figure 6:

or equivalently

$$PV = constant$$

The formula that can be used to calculate the affects of pressure changes on the volume of a gas at constant temperature is:

$$P_1V_1 = P_2V_2$$

Assume a sample of gas has volume 350 cm^3 exerts a pressure of 103 kPa. What would be the volume of this gas at 150 kPa of pressure?

Solution We have $P_1 = 103$ kPA, $V_1 = 350$ cm^3, $P_2 = 150$ kPa, we need to find V_2. Since $P_1V_1 = P_2V_2$ so

$$V_2 = \frac{P_1V_1}{P_2} = (103)(350)/(150) \approx 240\text{cm}^3$$

Example 12. Determine whether each function is even, odd, or neither.

(a) $f(x) = |x|$

(b) $f(x) = x^3 + 7x$

(c) $f(x) = \sin x$

(d) $f(x) = \cos x$

(e) $f(x) = x^4 - 4x^2 + 3$

(f) $f(x) = x^5 - 10$

Solution.

(a) The function is even because

$$f(-x) = |-x| = |-1||x| = |x| = f(x)$$

(b) The function is odd because

$$f(-x) = (-x)^3 + 7(-x) = -x^3 - 7x = -(x^3 + 7x) = -f(x)$$

(c) The sine function is odd because

$$f(-x) = \sin(-x) = -\sin x = -f(x)$$

where we used the trigonometric identity: $\sin(-x) = -\sin x$.

(d) The cosine function is even because

$$f(-x) = \cos(-x) = \cos x = f(x)$$

where we used the trigonometric identity: $\cos(-x) = \cos x$.

(e) The function is even because

$$f(-x) = (-x)^4 - 4(-x)^2 + 3 = x^4 - 4x^2 + 3 = f(x)$$

(f) The function is neither odd nor even because

$$f(-x) = (-x)^5 - 10 = -x^5 - 10$$

Note that $f(x) = x^5 - 10$ and $-f(x) = -x^5 + 10$, hence $f(-x) \neq \pm f(x)$.

Example 13. Consider

$$f(x) = x^2 - 4x + 3$$

(a) Plot f and determine whether the curve is the graph of a function of x.

(b) Find the extreme values of f, if any.

Solution.

(a) By the method of completing the square, we can rewrite f as

$$f(x) = x^2 - 4x + 4 - 1$$

$$f(x) = (x - 2)^2 - 1$$

Therefore, the graph of f is a parabola which is concave upward with vertex $(2, -1)$. See Figure 7.

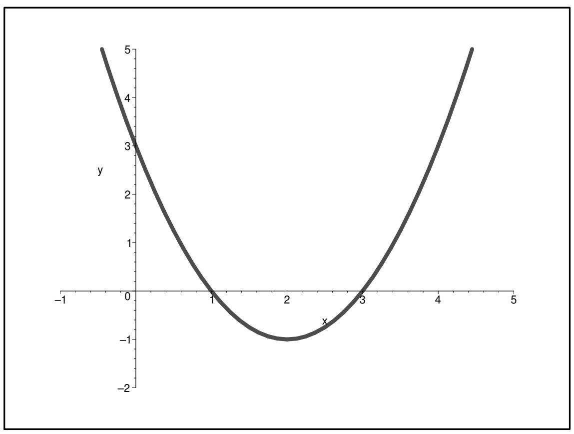


Figure 7:

(b) The function f has no absolute maximum, and has an absolute minimum at $x = 2$ whose value is -1 .

Clearly any vertical line will intersect the parabola at exactly one point, therefore the curve is the graph of a function of x.

Example 14. The graph of f is shown in Figure 8. Answer the following questions.

(a) Determine whether the curve of f is a function. Is f one-to-one?

(b) Find the domain and range of f.

(c) Find the relative maximum and relative minimum of f.

(d) Find the intervals on which f increases and decreases.

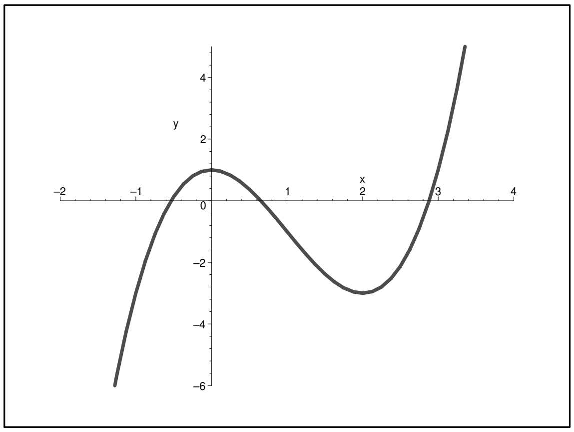


Figure 8:

Solution.

(a) Using the Vertical Line Test, it is clear that any vertical line intersects the curve only once. Hence, f is a function. Applying the Horizontal Line Test: If we draw the horizontal line $y = -1$, it will intersect the curve at three points, hence the function is not one-to-one.

(b) The domain is $(-\infty, \infty)$. The range is $(-\infty, \infty)$.

(c) The function has no absolute maximum and no absolute minimum. The function has a relative (local) minimum at $(2, -3)$. The function has a relative (local) maximum at $(0, 1)$.

(d) The function is increasing on $(-\infty, 0]$ and $[2, \infty)$. It is decreasing on $[0, 2]$.

Example 15. Show that the function $f(x) = 3x - 5$ is one-to-one, then find its inverse.

Solution. Suppose that there are numbers x_1, x_2 such that $f(x_1) = f(x_2)$. Then

$$3x_1 - 5 = 3x_2 - 5$$

$$3x_1 = 3x_2$$

$$x_1 = x_2$$

Hence, f is one-to-one and therefore has an inverse $f^{-1}(x)$.

To find the inverse, we first write $y = 3x - 5$ and solve the equation for x.

$$y + 5 = 3x$$

$$\frac{y+5}{3} = x$$

Finally, we interchange x and y to obtain $\dfrac{x+5}{3} = y$. Therefore, the inverse function is

$$f^{-1}(x) = \frac{x+5}{3}$$

Example 16. Find the inverse of the function $f(x) = \dfrac{2+5x}{4-3x}$.

Solution. We first write $y = \dfrac{2+5x}{4-3x}$, and then solve the equation for x. Multiply both sides by $(4-3x)$.

$$y(4-3x) = 2+5x$$

$$\begin{aligned} 4y - 3xy &= 2 + 5x \\ 4y - 2 &= 5x + 3xy \\ 4y - 2 &= x(5+3y) \\ \frac{4y-2}{5+3y} &= x \end{aligned}$$

Interchange x and y to get $\dfrac{4x-2}{3x+5} = y$. Therefore, the inverse function is

$$f^{-1}(x) = \frac{4x-2}{3x+5}$$

Example 17. Find the inverse of the function $f(x) = \dfrac{x^5+4}{3}$.

Solution. We first write $y = \dfrac{x^5+4}{3}$ and solve the equation for x. Multiply both sides by 3.

$$\begin{aligned} 3y &= x^5 + 4 \\ 3y - 4 &= x^5 \\ (3y-4)^{1/5} &= x \end{aligned}$$

Interchange x and y to get $y = (3x-4)^{1/5}$. Therefore, the inverse function is

$$f^{-1}(x) = \sqrt[5]{3x-4}$$

Example 18. Find the inverse of the function $f(x) = \sqrt{x-3}$.

Solution. We write $y = \sqrt{x-3}$, and solve for x. Note that $y \geq 0$.

$$\sqrt{x-3} = y$$

Square both sides.

$$\begin{aligned} x - 3 &= y^2 \\ x &= y^2 + 3 \end{aligned}$$

Note that since $y^2 + 3 > 0$, hence $x \geq 0$. Finally, we interchange x and y, to obtain $y = x^2 + 3, \quad x \geq 0$. Therefore, the inverse function is

$$f^{-1}(x) = x^2 + 3, \quad x \geq 0$$

Example 18. Find the x- and y-intercepts of the graph of the equation

$$x^2 + x = 3y + 6$$

Solution.

To find the x-intercepts: set $y = 0$ then solve for x. Therefore

$$x^2 + x = 0 + 6$$

$$x^2 + x - 6 = 0$$

$$(x+3)(x-2) = 0$$

This means the x-intercepts are $x = -3$ and $x = 2$.

To find the y-intercepts: set $x = 0$ then solve for y. Thus

$$0 + 0 = 3y + 6$$

$$-6 = 3y$$

This means the y-intercept is $x = \dfrac{-6}{3} = -2$.

Chapter

3 *POLYNOMIAL AND RATIONAL FUNCTIONS*

In this chapter, we study functions defined by polynomial expressions and rational functions. The following ideas will be discussed:

- *Straight Lines*
- *Quadratic Functions*
- *Polynomial Functions*
- *Zeros of Polynomial Functions*
- *Division of Polynomials: Long and Synthetic Division*
- *The Fundamental Theorem of Algebra*
- *Rational Functions*

STRAIGHT LINES

Definition. *(Slope of a Line)*

The slope of the line m passing through the two points (x_1, y_1) and (x_2, y_2) is given by the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Vertical Shift (rise)}}{\text{Horizontal Shift (run)}}$$

Note that if a line is

1. rising as x moves from left to right, then it has positive slope. /
2. falling as x moves from left to right, then it has negative slope. \
3. horizontal, then it has zero slope. —
4. vertical, then the slope is not defined. |

Definition. *(Linear Function)* The function

$$f(x) = ax + b$$

is called a *linear function*, where a, b are real numbers.
The graph of the *linear function* is a line.

Equation of a Line.

1. *(Slope-Intercept Form)* The equation of the line with slope m and y-intercept b is
$$y = mx + b$$
2. *(Point-Slope Form)* The equation of the line with slope m and passing through the point (x_1, y_1) is given by
$$y - y_1 = m(x - x_1)$$

Theorem. *(Vertical and Horizontal Lines)*

1. The equation of a vertical line passing through the point $(a, 0)$ is $x = a$.
2. The equation of a horizontal line passing through the point $(0, b)$ is $y = b$.

Theorem. *(Parallel and Perpendicular Lines)* Given two non-vertical lines l_1 and l_2 with slopes m_1 and m_2, respectively, then

1. $l_1 \parallel l_2$ if and only if $m_1 = m_2$
2. $l_1 \perp l_2$ if and only if $m_1 m_2 = -1$

Example 1. Find the slope of the line that passes through each pair of points.

(a) $(-3, 0)$ and $(2, 3)$ (b) $(5, 4)$ and $(-3, 4)$

(c) $(6, 5)$ and $(6, 7)$ (d) $(-1, 5)$ and $(-2, -5)$

Solution.

(a) $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3 - 0}{2 - (-3)} = \dfrac{3}{5}$

(b) $m = \dfrac{4 - 4}{-3 - 5} = \dfrac{0}{-8} = 0.$

This means that the line passing through the two points is horizontal.

(c) $m = \dfrac{7 - 5}{6 - 6} = \dfrac{2}{0},$

which is undefined, so no slope. This implies that the line is vertical.

(d) $m = \dfrac{-5 - 5}{-2 - (-1)} = \dfrac{-10}{-1} = 10$

Example 2. Determine the slope and y-intercept of the line

$$5x + 3y - 6 = 0$$

Solution.

We rewrite the equation in standard form.

$$3y = -5x + 6$$

$$y = -\frac{5}{3}x + 2$$

Thus, the slope is the coefficient of x, i.e. $m = -\dfrac{5}{3}$, and the y-intercept is 2.

Example 3. Find the equation of the line with the given properties.

(a) Passes through $(-4, 5)$ and $(-2, 13)$.

(b) Passes through $(2, -3)$ and with a slope of -7.

(c) Passes through $(0,5)$ and is parallel to the line $2x-3y=-9$.

(d) Passes through $(4,-1)$ and is perpendicular to the line $2x+5y=20$.

Solution.

(a) First, we find the slope.

$$m=\frac{13-5}{-2-(-4)}=\frac{8}{2}=4$$

We can use either one of the given points to find the equation, let's choose the point $(-2,13)$.

$$\begin{aligned} y-y_1&=m(x-x_1)\\ y-13&=4\,(x-(-2))\\ y-13&=4x+8\\ y&=4x+21 \end{aligned}$$

(b) The equation is

$$\begin{aligned} y-(-3)&=-7(x-2)\\ y+3&=-7x+14\\ y&=-7x+11 \end{aligned}$$

(c) To find the slope of the given line, rewrite the equation in standard form.

$$\begin{aligned} 2x-3y&=-9\\ -3y&=-2x-9\\ y&=\frac{2}{3}x+3 \end{aligned}$$

Therefore, the given line has slope $m=\frac{2}{3}$. Because the second line is parallel to the first one, it has the same slope, i.e. $\frac{2}{3}$. Thus, the equation of the required line is

$$\begin{aligned} y-5&=\frac{2}{3}(x-0)\\ y&=\frac{2}{3}x+5 \end{aligned}$$

(d) Rewrite the equation of the given line in standard form.

$$\begin{aligned} 2x+5y&=20\\ 5y&=-2x+20\\ y&=-\frac{2}{5}x+4 \end{aligned}$$

Hence, the given line has slope $m=-\frac{2}{5}$. It follows that any line perpendicular to the given line should have a slope of $m=\frac{5}{2}$ (since $\frac{5}{2}$ is the negative reciprocal of $-\frac{2}{5}$). Consequently, the equation of the line through $(4,-1)$ is

$$y-(-1)=\frac{5}{2}(x-4)$$

$$y + 1 = \frac{5}{2}x - 10$$

$$y = \frac{5}{2}x - 11$$

QUADRATIC FUNCTIONS

Definition. *(Quadratic Function)* The function

$$f(x) = ax^2 + bx + c$$

is called a *quadratic function*, where a, b and c are real numbers with $a \neq 0$.

Properties of Graphs of Quadratic Functions.

1. The graph of the *quadratic function* is a U-shaped curve called *parabola.*
2. If $a > 0$, the parabola opens upward. If $a < 0$, the parabola opens downward.
3. As the value of $|a|$ increases, the parabola becomes narrower. If $|a|$ decreases, the parabola becomes wider.
4. The lowest point of a parabola (when $a > 0$) or the highest point (when $a < 0$) is called the *vertex.*
5. The graph of a quadratic function is symmetric with respect to a vertical line passing through the vertex. This line is called the *axis of symmetry.* If (h, k) is the vertex of a vertical parabola, then the equation of the axis of symmetry is $x = h$.

Theorem. The graph of the function

$$f(x) = ax^2 + bx + c$$

is a *parabola* such that

1. If $a > 0$, then the parabola is concave upward with vertex (the minimum point) at $x = -\dfrac{b}{2a}$.
2. If $a < 0$, then the parabola is concave downward with vertex (the maximum point) at $x = -\dfrac{b}{2a}$.

Theorem. The quadratic function

$$f(x) = a(x - h)^2 + k$$

is said to be in *standard form.* The graph of f a parabola such that if $a > 0$, then the parabola is concave upward and is concave downward if $a < 0$. The axis of symmetry of the parabola is the line $x = h$, and the vertex is at the point (h, k).

Vertex and Intercepts.

To find the vertex of the graph of $f(x) = ax^2 + bx + c$, we can either:

1. Use the method of completing the square to rewrite the function in the form
$$f(x) = a(x-h)^2 + k$$
Then, the vertex is at (h, k).

2. Use the formula $x = -\dfrac{b}{2a}$ to get the x-coordinate of the vertex. The y-coordinate of the vertex can be determined by evaluating $f\left(-\dfrac{b}{2a}\right)$.

To find the y-intercept of the graph of $f(x) = ax^2 + bx + c$, we evaluate $f(0)$. As for the x-intercepts, we solve the quadratic equation $ax^2 + bx + c = 0$.

Example 1. Describe the graph of $f(x) = 2x^2 + 8x + 7$.

Solution. We rewrite the function in standard form using the method of completing the square.
$$\begin{aligned} f(x) &= 2x^2 + 8x + 7 \\ &= 2(x^2 + 4x) + 7 \end{aligned}$$
Add and subtract 4 within the parentheses. We get the 4 by dividing the coefficient of x, namely 4, by 2 and then square the answer.
$$\begin{aligned} &= 2(x^2 + 4x + 4 - 4) + 7 \\ &= 2(x^2 + 4x + 4) - 2(4) + 7 \\ &= 2(x+2)^2 - 1 \end{aligned}$$
Hence, the graph of f is a parabola, which is concave upward (since $2 > 0$) and has vertex at $(-2, -1)$. Equivalently, we can find the vertex using the fact that it occurs at the point $x = -\dfrac{b}{2a} = -\dfrac{8}{2(2)} = -2$. To find the y coordinate of the vertex, we substitute $x = -2$ in the function to obtain $y = -1$. Thus, the vertex occurs at the point $(-2, -1)$.

Example 2. An object is thrown vertically upward with an initial velocity of 32 ft/sec. The height h of the object after t seconds is given by
$$h(t) = -16t^2 + 32t + 48$$

(a) What is the initial height of the object?

(b) When does the object reaches its maximum height? Find the maximum height.

(c) When does the object hit the ground?

Solution.

(a) To find the initial height we substitute $t = 0$ and obtain $h = 48$. Therefore, the object was thrown initially from a height of 48 ft.

(b) Since the graph of h is a parabola that opens down (since $a = -16 < 0$), the maximum height occurs at the vertex. The x-coordinate of the vertex is $x = -\dfrac{b}{2a} = -\dfrac{32}{2(-16)} = 1$. If we substitute $x = 1$, we obtain $h = -16+32+48 = 64$. So the object reaches its maximum height after 1 sec and the maximum height is 64 ft.

(c) To find the time when the object hits the ground we substitute $h = 0$ and solve the resulting equation.

$$0 = -16t^2 + 32t + 48$$
$$0 = -16(t^2 - 2t - 3)$$
$$0 = -16(t - 3)(t + 1)$$

so $t = 3$ or $t = -1$. The second answer makes no sense (time t cannot be negative). Thus, the object hits the ground 3 sec after it is thrown.

POLYNOMIAL FUNCTIONS

Definition. A *polynomial function of x with degree n* is a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

where $a_0, a_1, ..., a_n$ are called the *coefficients* with $a_n \neq 0$.

End Behavior of Polynomials. The shape of the graph of a polynomial is related to the shape of the graph of the leading term, i.e. the monomial $Q(x) = a_n x^n$. They both have the same end behavior as summarized next.
Let $y = P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0, \qquad a_n \neq 0.$

1. Let $a_n > 0$ and n is even. Then,

   $$y \to \infty \quad \text{as} \quad x \to \infty$$

   and

   $$y \to \infty \quad \text{as} \quad x \to -\infty$$

2. Let $a_n < 0$ and n is even. Then,

   $$y \to -\infty \quad \text{as} \quad x \to \infty$$

   and

   $$y \to -\infty \quad \text{as} \quad x \to -\infty$$

3. Let $a_n > 0$ and n is odd. Then,

   $$y \to \infty \quad \text{as} \quad x \to \infty$$

   and

   $$y \to -\infty \quad \text{as} \quad x \to -\infty$$

4. Let $a_n < 0$ and n is odd. Then,

   $$y \to -\infty \quad \text{as} \quad x \to \infty$$

   and

   $$y \to \infty \quad \text{as} \quad x \to -\infty$$

Properties of the graph of a polynomial function.

If P is an nth degree polynomial with real coefficients, then

1. The domain of P is all real numbers.
2. The graph of P is continuous everywhere.

3. The graph of P has at most n x-intercepts.
4. The graph of P has at most $n-1$ turning points.

ZEROS OF POLYNOMIAL FUNCTIONS

Let P be a polynomial and a is a real number. The following statements are equivalent.

1. $x = a$ is *zero* of P.
2. $x = a$ is a *solution* or *root* to the equation $P(x) = 0$.
3. $(x - a)$ is a *factor* of P.
4. The graph of P crosses the x-axis at $x = a$ (the x-intercept).

DIVISION OF POLYNOMIALS: Long and Synthetic Division

The Division Algorithm.

If $P(x)$ and $D(x)$ are polynomials such that $D(x) \neq 0$, and the degree of $D(x)$ is less or equal to the degree of $P(x)$, then there exist unique polynomials $Q(x)$ and $R(x)$ such that

$$P(x) = D(x).Q(x) + R(x)$$

where $R(x)$ is either identically 0 or has a degree less than the degree of $D(x)$. The polynomials $D(x)$ and $P(x)$ are called the *divisor* and *dividend*, respectively, $Q(x)$ is the *quotient*, and $R(x)$ is the *remainder*.

To divide two polynomials P and Q, i.e. to find the quotient $\dfrac{P(x)}{Q(x)}$, where degree of P is greater or equal to the degree of Q, we use:

1. ***Long Division.***
2. ***Synthetic Division***: It works if Q is of degree 1, i.e. if $Q(x) = x - c$.

It is worth mentioning that *Synthetic Division* is a shortcut for long division of polynomials by divisors of the form $(x - c)$.

Theorem. *(The Factor Theorem)*

A polynomial $P(x)$ has a factor $(x - c)$ if and only if $x = c$ is a zero of P, i.e $P(c) = 0$.

Theorem. *(Remainder Theorem)*

If the polynomial $P(x)$ is divided by $(x - c)$, then the remainder is $r = P(c)$.

Theorem. *(Rational Zeros Test)*

If the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ has integer coefficients, then every rational zero of P has the from

$$Rational\ Zero\ = \frac{p}{q}$$

where p is a factor of the constant term a_0.
and q is a factor of the leading term a_n.

THE FUNDAMENTAL THEOREM OF ALGEBRA

Theorem *(Fundamental Theorem of Algebra)*

Every polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0, \qquad n \geq 1,\ a_n \neq 0$$

with complex coefficients, has at least one complex zero.

Theorem *(Complete Factorization Theorem)*

If P is a polynomial of degree $n > 0$, then P has exactly n linear factors

$$P(x) = a\,(x - c_1)\,(x - c_2)\,...\,(x - c_n)$$

where $c_1, c_2, ..., c_n$ are complex numbers and a is the leading coefficient of P.

Theorem *(Factorization of Polynomials)*

If P is a polynomial of degree $n > 0$ and with real coefficients, then P can be written as the product of linear and irreducible quadratic factors with real coefficients.

Theorem. Assume P is a polynomial with real coefficients. If $a + ib$, where $b \neq 0$, is a complex zero of P, then the conjugate $a - ib$ is also a zero of P.

Example 1. Use the *Leading Coefficient Test* to determine the end behavior of each polynomial function.

(a) $f(x) = -2x^3 + x^2 - 5x + 3$ (b) $f(x) = x^4 - 7x^2 + 8$

(c) $f(x) = x^7 - x^2 + 1$

Solution.

(a) Since the degree is odd and the coefficient of the leading term is negative, the graph rises to the left (i.e. $f \to \infty$ as $x \to -\infty$) and falls to the right (i.e. $f \to -\infty$ as $x \to \infty$).

(b) Since the degree is even and the coefficient of the leading term is positive, the graph rises to the left and right (i.e. $f \to \infty$ as $x \to \pm\infty$).

(c) Since the degree is odd and the coefficient of the leading term is positive, the graph falls to the left and rises to the right.

Example 2. Use long division to divide $x^3 - 1$ by $x - 1$.

Solution.

Using long division we obtain:

$$\begin{array}{r|l}
 & x^2 + x + 1 \\ \hline
x - 1 & x^3 + 0x^2 + 0x - 1 \\
 & x^3 - x^2 \\ \cline{2-2}
 & \quad\ x^2 + 0x - 1 \\
 & \quad\ x^2 - x \\ \cline{2-2}
 & \qquad\quad x - 1 \\
 & \qquad\quad x - 1 \\ \cline{2-2}
 & \qquad\qquad\ 0
\end{array}$$

Hence $(x-1)$ divides evenly into (x^3-1) and we have

$$\frac{x^3-1}{x-1} = x^2+x+1$$

Example 3. Divide $4x^4+3x^3+2x+1$ by x^2+x+2.

Solution.

Long division yields

$$
\begin{array}{r@{\;}l}
 & 4x^2 - x - 7 \\ \cline{2-2}
x^2+x+2 \,\big) & 4x^4+3x^3+0x^2+2x+1 \\
 & 4x^4+4x^3+8x^2 \\ \cline{2-2}
 & \quad -x^3-8x^2+2x+1 \\
 & \quad -x^3-x^2-2x \\ \cline{2-2}
 & \qquad -7x^2+4x+1 \\
 & \qquad -7x^2-7x-14 \\ \cline{2-2}
 & \qquad\qquad 11x+15
\end{array}
$$

Thus, the quotient is $4x^2-x-7$ and the remainder is $11x+15$, i.e.

$$\frac{4x^4+3x^3+2x+1}{x^2+x+2} = 4x^2-x-7+\frac{11x+15}{x^2+x+2}$$

Example 4. Use synthetic division to divide $2x^4+7x^3+2x^2-4x-3$ by $x+3$.

Solution.

The synthetic division process works as follows:

Step 1:
In the first line we put the zero of the divisor on the left side of the vertical line , i.e. the zero of $x+3$ which is -3. This is followed on the first line (on the right side of the vertical line) by the coefficients of the dividend, written in descending powers of x. It is important to remark that missing powers of x must be represented by a 0 in that place. For our case, the coefficients of the dividend $2x^4+7x^3+2x^2-4x-3$ are $2, 7, 2, -4$ and -3.

$$
\begin{array}{r|rrrrr}
\cline{2-6}
-3 & 2 & 7 & 2 & -4 & -3
\end{array}
$$

Step 2:
In this step, we bring down the first coefficient of the dividend, which is 2:

$$
\begin{array}{r|rrrrr}
\cline{2-6}
-3 & 2 & 7 & 2 & -4 & -3 \\
 & & & & & \\ \cline{2-6}
\multicolumn{1}{r}{} & 2 & & & &
\end{array}
$$

Step 3:

In this step, we multiply the number 2, which we dropped down, by -3 to get -6. We write the result in the middle row as shown next and then add the two numbers in that column, namely -6 and 7, to get 1.

$$\begin{array}{r|rrrrr} -3 & 2 & 7 & 2 & -4 & -3 \\ & & -6 & & & \\ \hline & 2 & 1 & & & \end{array}$$

Step 4:

This procedure of multiplying by -3 and adding the two resulting numbers in the same column is repeated until the table is complete.

$$\begin{array}{r|rrrrr} -3 & 2 & 7 & 2 & -4 & -3 \\ & & -6 & -3 & 3 & 3 \\ \hline & 2 & 1 & -1 & -1 & 0 \end{array}$$

It is important to note that the last number in the last row, which is 0, is the remainder. The first numbers $2, 1, -1$ and -1 in the third line are the coefficients of the quotient polynomial, which is $2x^3 + x^2 - x - 1$. Thus,

$$\frac{2x^4 + 7x^3 + 2x^2 - 4x - 3}{x + 3} = 2x^3 + x^2 - x - 1$$

Example 5. Divide $2x^3 - 7x^2 + 5$ by $x - 3$.

Solution.

The coefficients of the dividend $2x^3 - 7x^2 + 5$ are $2, -7, 0$ and 5. Note that the term involving x is missing in the dividend, so it must be represented by a 0 in that place. Synthetic division gives:

$$\begin{array}{r|rrrr} 3 & 2 & -7 & 0 & 5 \\ & & 6 & -3 & -9 \\ \hline & 2 & -1 & -3 & -4 \end{array}$$

So, the quotient is $2x^2 - x - 3$ and the remainder is -4, i.e.

$$\frac{2x^3 - 7x^2 + 5}{x - 3} = 2x^2 - x - 3 - \frac{4}{x - 3}$$

Example 6. Find a polynomial with the following zeros.

(a) $-3, 1, 1, 5$ (b) $-\frac{1}{2}, 0, 3$

Solution.

Note that there are many correct polynomials with the given zeros.

(a) For each zero we form a corresponding factor, for example $x = -3$ corresponds to the factor $(x + 3)$. Therefore, one possible polynomial is

$$P(x) = (x + 3)(x - 1)(x - 1)(x - 5) = x^4 - 4x^3 - 10x^2 + 28x - 15$$

(b) The zero $-\frac{1}{2}$ corresponds to the factor $\left(x + \frac{1}{2}\right)$ or to the factor $(2x + 1)$. The root 0 corresponds to the factor x. Therefore, one possible polynomial is

$$P(x) = (2x + 1)x(x - 3) = 2x^3 - 5x^2 - 3x$$

Example 7. Find all the real zeros of

$$f(x) = x^3 - x^2 - 4x + 4$$

Solution.

By using the *Rational Zero Test*, the possible real zeros of f are the factors of 4 (since the coefficient of x^3 is 1). The possible zeros are:

$$\pm 1,\ \pm 2,\ \pm 4$$

Now, we check which of these possible zeros are actually zeros for f. By trial and error, $x = 1$ is a zero. Since $x = 1$ is a zero, it follows that $(x - 1)$ is a factor of $f(x)$. Using synthetic division or long division, we find that

$$\frac{x^3 - x^2 - 4x + 4}{x - 1} = x^2 - 4$$

Therefore,

$$x^3 - x^2 - 4x + 4 = (x^2 - 4)(x - 1) = (x - 2)(x + 2)(x - 1)$$

This factorization implies that f has the three real zeros $1, 2$ and -2.

Example 8. Find all the real zeros of

$$f(x) = 6x^3 - 4x^2 + 3x - 2$$

Solution.

The possible real zeros of f are:

$$\frac{\text{Factors of } 2}{\text{Factors of } 6} = \frac{\pm 1,\ \pm 2}{\pm 1,\ \pm 2,\ \pm 3,\ \pm 6}$$

$$= \pm 1,\ \pm\frac{1}{2},\ \pm\frac{1}{3},\ \pm\frac{1}{6},\ \pm\frac{2}{3}, \pm 2$$

We need to check which of these possible zeros are actually zeros for f. By trial and error, we find that $x = \frac{2}{3}$ is a zero. Hence $\left(x - \frac{2}{3}\right)$ is a factor of $f(x)$. Using synthetic division or long division, it follows that

$$\frac{6x^3 - 4x^2 + 3x - 2}{\left(x - \frac{2}{3}\right)} = 6x^2 + 3$$

Therefore,

$$6x^3 - 4x^2 + 3x - 2 = \left(x - \frac{2}{3}\right)(6x^2 + 3)$$

Note that $6x^2 + 3$ cannot be factored into real linear terms. The quantity $6x^2 + 3 = 0$ when $x = \pm\sqrt{-1/2} = \pm\sqrt{1/2}\, i$, where $i = \sqrt{-1}$. This factorization implies that f has one real zero $x = \frac{2}{3}$ and two complex zeros $x = \pm\sqrt{1/2}\, i$.

Example 9. Find all the real zeros of

$$f(x) = 2x^3 - 7x^2 + 4x + 3$$

Solution.

The possible real zeros of f are:

$$\frac{\text{Factors of } 3}{\text{Factors of } 2} = \frac{\pm 1,\ \pm 3}{\pm 1,\ \pm 2}$$

$$= \pm 1,\ \pm\frac{1}{2},\ \pm\frac{3}{2},\ \pm 3$$

By trial and error, we find that $x = \dfrac{2}{3}$ is a zero. Thus, using synthetic division, we have

$$2x^3 - 7x^2 + 4x + 3 = \left(x - \frac{2}{3}\right)(2x^2 - 4x - 2)$$

We can use the quadratic formula to find the zeros of the quadratic term $Q(x) = 2x^2 - 4x - 2$.

$$x = \frac{4 \pm \sqrt{16 - 4(2)(-2)}}{2(2)} = \frac{4 \pm \sqrt{32}}{4} = \frac{4 \pm 4\sqrt{2}}{4} = 1 \pm \sqrt{2}$$

The exact zeros of f are

$$\frac{2}{3}, \qquad \text{and} \qquad 1 \pm \sqrt{2}$$

RATIONAL FUNCTIONS

Definition. A *rational function*, $R(x)$, is a function that has the form

$$R(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials.

The domain of a rational function is the set of all real numbers except the values of x for which the denominator is zero.

Definition. *(Horizontal and Vertical Asymptotes)*

1. The line $x = a$ is a *vertical asymptote* of the function $y = f(x)$ if

$$f(x) \to \infty \qquad \text{or} \qquad f(x) \to -\infty$$

as $x \to a$, either from the right or the left.

2. The line $y = b$ is a *horizontal asymptote* of the function $y = f(x)$ if

$$f(x) \to b \qquad \text{as} \qquad x \to \infty \quad \text{or} \quad x \to -\infty$$

Asymptotes of Rational Functions.

Let R be the rational function

$$R(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + ... + b_1 x + b_0}$$

where P and Q have *no* common factors.

1. The graph of R has vertical asymptotes at the zeros of $Q(x)$, i.e. at $x = a$ where a is the zero of the denominator.
2. As for the horizontal asymptotes it is as follows:
   a. If $n < m$, then R has horizontal asymptote $y = 0$ (i.e. the x-axis).

   b. If $n = m$, then R has horizontal asymptote $y = \dfrac{a_n}{a_m}$.

   c. If $n > m$, then R has no horizontal asymptote.

Example 1. For each rational function find the asymptotes, if any.

(a) $f(x) = \dfrac{3x^2 - 15}{x^2 - 2x - 3}$ (b) $f(x) = \dfrac{x^3 - 2x^2 + 3}{x^2 - 4}$

(c) $f(x) = \dfrac{x - 2}{x^2 + 1}$ (d) $f(x) = \dfrac{x^2 - 25}{x - 5}$

Solution.

(a) The denominator $x^2 - 2x - 3 = (x+1)(x-3)$ is zero at -1 and 3. Hence the graph of f has vertical asymptotes at $x = -1$ and $x = 2$. Since the numerator and denominator have the same degree, the line

$$y = \frac{3}{1} = 3$$

is a horizontal asymptote. The graph of f is shown in Figure 1. Clearly, $f(x) \to 3$ as $x \to \pm\infty$, hence $y = 3$ is a horizontal asymptote. Note that near the vertical asymptotes $x = -1$ and $x = 3$, the function $f(x)$ tends to ∞ or $-\infty$.

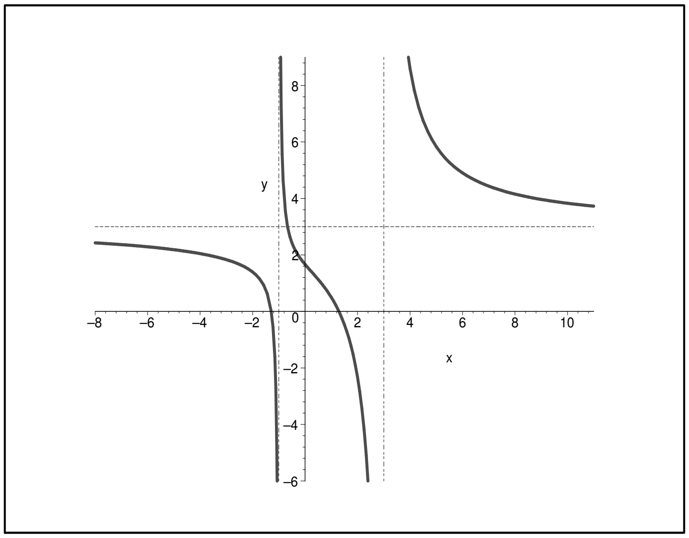


Figure 1: Graph of $f(x) = \dfrac{3x^2 - 15}{x^2 - 2x - 3}$

(b) The denominator $x^2 - 4 = (x - 2)(x + 2)$ is zero at 2 and -2. Hence the graph of f has vertical asymptotes at $x = 2$ and $x = -2$. Since the numerator has a degree greater than that of the denominator, hence f has no horizontal asymptote.

(c) The denominator $x^2 + 1$ is nonzero. Hence the graph of f has no vertical asymptotes. Since the numerator has a degree less than that of the denominator, hence $y = 0$ is a horizontal asymptote.

(d) The numerator has a degree greater than that of the denominator, therefore f has no horizontal asymptote. Note that the numerator and denominator have a common factor which is $(x - 5)$. So, for $x \neq 5$ we have

$$f(x) = \frac{x^2 - 25}{x - 5} = \frac{(x - 5)(x + 5)}{x - 5} = x + 5$$

Thus f has no vertical asymptote since the denominator was cancelled. The graph of f is the same as that of the line $y = x + 5$ with a hole at the point $(5, 10)$.

Chapter

4 *EXPONENTIAL AND LOGARITHMIC FUNCTIONS*

In this chapter, we will study exponential and logarithmic functions and their application to real-life problems. In particular, we will study:

- *Exponential Functions*
- *Applications of Exponential Functions*
- *Logarithmic Functions*
- *Applications of Logarithmic Functions*

EXPONENTIAL FUNCTIONS

Definition. *(Exponential Function)*

The function

$$f(x) = a^x \qquad\qquad a > 0, \quad a \neq 1$$

defines an *exponential function* for each value of the constant a, called the *base*. The exponent x may assume any real value. The domain of f is the set of all real numbers, while its range is the set of all positive numbers.

Definition. *(Exponential Function with Base e)*

For any real x the function

$$f(x) = e^x$$

defines an *exponential function with base e*. To 3 decimal places, the value of the constant e is 2.718.

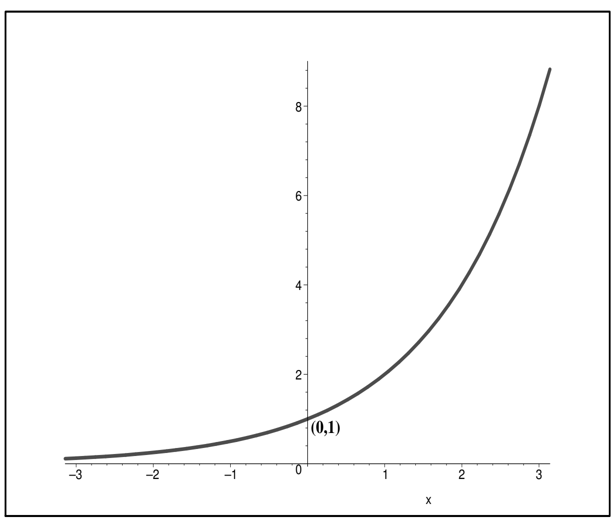


Figure 1: Graph of $y = a^x$ for $a > 1$

Properties of the Graph of the Exponential Function $f(x) = a^x$.

1. The graph is continuous, with no breaks, holes, or jumps.

2. Since $a^0 = 1$, hence the graph passes through the point $(0, 1)$.

3. The x-axis is a horizontal asymptote.

4. The graph of f is increasing if $a > 1$.

5. The graph of f is decreasing if $0 < a < 1$.

6. The function f is one-to-one.

The graph of the exponential function for various values of a is shown in Figures 1 and 2.

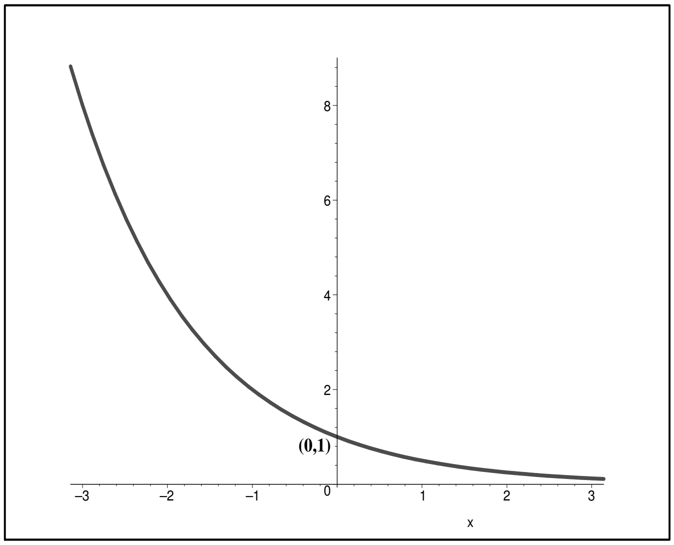


Figure 2: Graph of $y = a^x$ for $0 < a < 1$

Properties of Exponential Functions.

For any positive numbers a and b, $a \neq 1$, $b \neq 1$, and real numbers x and y:

1. $a^x a^y = a^{x+y}$

2. $(a^x)^y = a^{xy}$

3. $(ab)^x = a^x b^x$

4. $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

5. $\frac{a^x}{a^y} = a^{x-y}$

6. $a^x = a^y$ if and only if $x = y$

7. If $x \neq 0$, then $a^x = b^x$ if and only if $a = b$

APPLICATIONS OF EXPONENTIAL FUNCTIONS

Exponential functions arise in problems of:

1. *Exponential growth in biology and economics.*

2. *Radioactive decay in physics and chemistry.*

One of the most familiar examples of exponential growth is *compound interest.*

Formulas for Compound Interest.

A principal P is invested at an annual interest rate r. The amount A in the balance after t years is given by the following formulas.

1. For n compoundings per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$
2. For continuous compounding: $A = Pe^{rt}$

LOGARITHMIC FUNCTIONS

Definition. *(Logarithmic Function)*

Assume $a > 0$ and $a \neq 1$. The *logarithmic function with base* a, denoted by $\log_a$, is defined by

$$y = \log_a x \qquad \text{if and only if} \qquad a^y = x$$

This means that $\log_a x$ is the exponent to which the base a must be raised to yield x.

The domain of a *logarithmic function* is the set of all positive real numbers, while its range is the set of all real numbers. The graph of the logarithmic function for various values of a is shown in Figures 3 and 4.

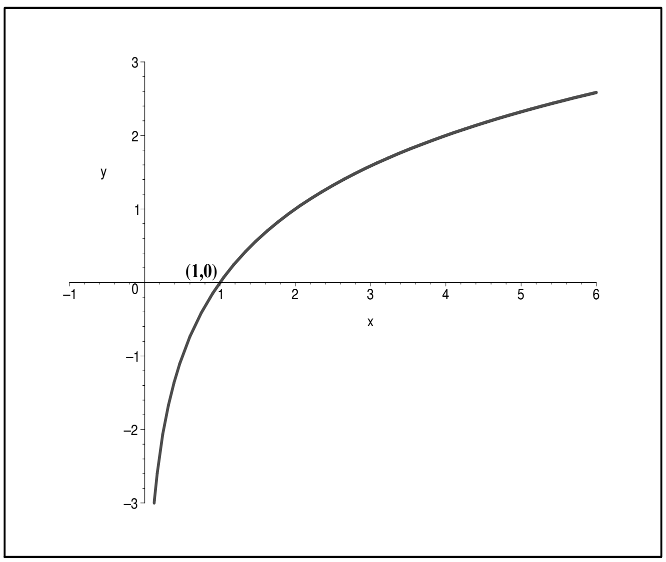


Figure 3: Graph of $y = \log_a x$ for $a > 1$

Properties Logarithmic Functions.

Let a, b, M, and N are positive real numbers, $a \neq 1$, and x and r are real numbers.

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a a^x = x$
4. $a^{\log_a x} = x, \quad x > 0$

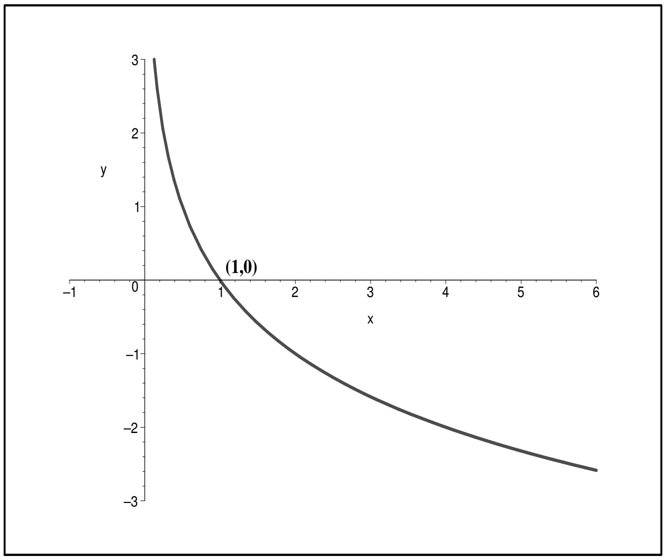


Figure 4: Graph of $y = \log_a x$ for $0 < a < 1$

5. $\log_a(MN) = \log_a M + \log_a N$
6. $\log_a\left(\dfrac{M}{N}\right) = \log_a M - \log_a N$
7. $\log_a M^r = r \log_a M$
8. $\log_a M = \log_a N$ if and only if $M = N$
9. *Change of Base:* $\log_a x = \dfrac{\log_b x}{\log_b a}$

Logarithmic Notation.

1. *Common Logarithm:* $\log x = \log_{10} x$
2. *Natural Logarithm:* $\ln x = \log_e x$

Therefore, we have the following logarithmic-exponential relationships:

$$\log x = y \quad \text{if and only if} \quad x = 10^y$$

$$\ln x = y \quad \text{if and only if} \quad x = e^y$$

APPLICATIONS OF LOGARITHMIC FUNCTIONS

Since logarithmic functions are inverses of exponential functions, hence they arise in the same application problems as exponential functions, and many other phenomena. In particular, they arise in problems dealing with

1. *Exponential growth.*
2. *Radioactive decay.*
3. *Loudness of Sounds.*
2. *Intensity of Earthquakes.*

Example 1. Draw the graph of each function.

(a) $f(x) = \left(\dfrac{1}{5}\right)^x$ (b) $f(x) = e^x$

(c) $f(x) = \pi^x$ (d) $f(x) = e^{-x}$

Solution.

(a) Since $\dfrac{1}{5} < 1$, so the graph of f is the same as that shown in Figure 2.

(b) The graph of f is the same as in Figure 1 since $e > 1$.

(c) Since $\pi > 1$, so the graph of f is the same as in Figure 1.

(d) Since $f(x) = e^{-x} = \left(\dfrac{1}{e}\right)^x$, and $\dfrac{1}{e} < 1$ so the graph of f is the same as in Figure 2.

Example 2. Draw the graph of each function.

(a) $f(x) = \ln x$ (b) $f(x) = \log x$

(c) $f(x) = \log_{1/2} x$

Solution.

(a) Since $\ln x = \log_e x$ and $e > 1$, so the graph of f is as that in Figure 3.

(b) Since $\log x = \log_{10} x$ and $10 > 1$, so the graph of f is as in Figure 3.

(c) Since $\ln x = \log_{1/2} x$ and $1/2 < 1$, so the graph of f is as that shown in Figure 4.

Example 3. Explain how you can obtain the graph of $g(x) = 1 - 2^{x-3}$ using the graph of $f(x) = 2^x$.

Solution.

Shift f three units to the right, reflect the resulting graph through the x-axis, and then shift the graph upward by one unit.

Example 4. Explain how you can obtain the graph of $g(x) = 3 + 2\log(-x)$ using the graph of $f(x) = \log x$.

Solution.

Reflect f through the y-axis, stretch the curve vertically by a factor of 2, and then shift the graph upward by three units.

Example 5. *(Population Growth)* The population, $P(t)$, of a town after t years is given by the model

$$P(t) = 4000e^{0.034t}$$

Assume $t = 0$ corresponds to the year 1980.

(a) What is the initial population?

(b) What will the population be in the year 2010?

(c) When will the population double, i.e reach 8000?

Solution.

(a) To find the initial population we substitute $t = 0$ in the formula. We obtain

$$P(0) = 4000e^0 = 4000$$

since $e^0 = 1$. The initial population is 4000.

(b) Since $t = 0$ corresponds to the year 1980, hence the year 2010 corresponds to $t = 30$. Substituting $t = 30$ results

$$P = 4000e^{0.034(30)} = 11093$$

(c) To find when the population reach 8000, we replace P by 8000 and then solve the resulting equation for t.

$$8000 = 4000e^{0.034t}$$

Divide both sides by 4000.

$$2 = e^{0.034t}$$

Take the natural logarithm of both sides.

$$\ln 2 = \ln\left(e^{0.034t}\right) = 0.034t \ln e = 0.034t$$

Hence

$$t = \frac{\ln 2}{0.034} \approx 23$$

Therefore, the population doubles after 23 years, i.e. during the year 2003.

Example 6. *(Radioactive Decay)* Let A represent the mass of a quantity of the radioactive radium-226. The amount of radium present after t years is given by

$$A = 30\left(\frac{1}{2}\right)^{t/1620}$$

Note that the *half-life* of a radioactive substance is the time required for half the mass to decay.

(a) What is the initial quantity of radium?

(b) What is the quantity present after 500 years?

(c) Determine the half-life of radium-226.

Solution.

(a) To find the initial amount we substitute $t = 0$. We obtain $A = 30\left(\frac{1}{2}\right)^0 = 30$.

(b) To find the amount present after 500 years, we replace $t = 500$ in A.

$$A = 30\left(\frac{1}{2}\right)^{500/1620} \approx 24.2$$

(c) Since from part (a) the initial amount is 30, so to find the half-life we replace A by 15.

$$15 = 30\left(\frac{1}{2}\right)^{t/1620}$$

Divide both sides by 30.

$$\frac{1}{2} = \left(\frac{1}{2}\right)^{t/1620}$$

Take the natural logarithm of both sides.

$$\ln(0.5) = \frac{t}{1620}\ln(0.5)$$

$$1 = \frac{t}{1620}$$

So the half-life of radium-226 is $t = 1620$ years.

Example 7. *(Compound Interest)* An amount of \$16,000 is deposited in an account paying an annual interest rate of 9%. Find the balance after 6 years if it is compounded

(a) quarterly (b) continuously

Solution.

(a) Since the amount is compounded quarterly, hence we have $n = 4$. Therefore, the balance after 6 years is

$$A = P\left(1+\frac{r}{n}\right)^{nt} = 16,000\left(1+\frac{0.09}{4}\right)^{4(6)} \approx \$27292.3$$

(b) For continuous compounding the balance is

$$A = Pe^{rt} = 16,000e^{0.09(6)} \approx \$27456.1$$

Example 8. Change each logarithmic form to the equivalent exponential form.

(a) $\log_5 125 = 3$ (b) $\log_{49} 7 = \frac{1}{2}$ (c) $\log_2\left(\frac{1}{8}\right) = -3$

Solution.

(a) $\log_5 125 = 3$ is equivalent to $5^3 = 125$

(b) $\log_{49} 7 = \frac{1}{2}$ is equivalent to $(49)^{1/2} = \sqrt{49} = 7$

(c) $\log_2\left(\frac{1}{8}\right) = -3$ is equivalent to $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

Example 9. Change each exponential form to the equivalent logarithmic form.

(a) $4^3 = 64$ (b) $\sqrt{81} = 9$ (c) $\frac{1}{2} = 8^{-1/3}$

Solution.

(a) $4^3 = 64$ is equivalent to $\log_4 64 = 3$

(b) $\sqrt{81} = 9$ is equivalent to $\log_{81} 9 = \frac{1}{2}$

(c) $\frac{1}{2} = 8^{-1/3}$ is equivalent to $\log_8\left(\frac{1}{2}\right) = -\frac{1}{3}$

Example 10. Use logarithmic properties to simplify each of the following.

(a) $\ln 1$ (b) $\log_2(32)$ (c) $\log_9 3$

(d) $5^{\log_5 3}$ (e) $e^{\ln x}$ (f) $\log 0.001$

Solution.

(a) $\ln 1 = \log_e 1 = 0$ since $e^0 = 1$.

(b) $\log_2(32) = 5$ since $2^5 = 32$.

(c) $\log_9 3 = \dfrac{1}{2}$ since $9^{1/2} = \sqrt{9} = 3$.

(d) $5^{\log_5 3} = 3$.

(e) $e^{\ln x} = x$ since $e^{\ln x} = e^{\log_e x}$.

(f) $\log 0.001 = -3$ since $\log 0.001 = \log_{10} 0.001$ and $0.001 = \dfrac{1}{1000} = \dfrac{1}{10^3} = 10^{-3}$.

Example 11. Use logarithmic properties to write each expression in terms of simpler logarithmic forms.

(a) $\log_a\left(\dfrac{xy}{5z}\right)$ (b) $\log_a\left(\sqrt[3]{\dfrac{x^2}{3y}}\right)$ (c) $\log_a\left(\dfrac{x^3y^2}{\sqrt{z}}\right)$

Solution.

(a)

$$\begin{aligned}\log_a\left(\frac{xy}{5z}\right) &= \log_a(xy) - \log_a(5z) = \log_a x + \log_a y - (\log_a 5 + \log_a z)\\ &= \log_a x + \log_a y - \log_a 5 - \log_a z\end{aligned}$$

(b)

$$\begin{aligned}\log_a\left(\sqrt[3]{\frac{x^2}{3y}}\right) &= \log_a\left(\frac{x^2}{3y}\right)^{1/3} = \frac{1}{3}\log_a\left(\frac{x^2}{3y}\right) = \frac{1}{3}\left(\log_a x^2 - \log_a(3y)\right)\\ &= \frac{1}{3}\left(2\log_a x - \log_a 3 - \log_a y\right)\end{aligned}$$

(c)

$$\begin{aligned}\log_a\left(\frac{x^3y^2}{\sqrt{z}}\right) &= \log_a\left(x^3y^2\right) - \log_a\sqrt{z}\\ &= \log_a x^3 + \log_a y^2 - \log_a z^{1/2}\\ &= 3\log_a x + 2\log_a y - \frac{1}{2}\log_a z\end{aligned}$$

Example 12. Evaluate each expression.

(a) $\log_4 2 + \log_4 32$ (b) $\log_2 96 - \log_2 6$

(c) $e^{3\ln 5}$ (d) $\log\sqrt{0.1}$

Solution.

(a)

$$\log_4 2 + \log_4 32 = \log_4(2)(32) = \log_4 64 = 3$$

since $4^4 = 64$.

(b)

$$\log_2 96 - \log_2 6 = \log_2 \left(\frac{96}{6}\right) = \log_2 16 = 4$$

since $2^4 = 16$.

(c)

$$e^{3\ln 5} = e^{\ln 5^3} = e^{\ln 125} = e^{\log_e 125} = 125$$

(d)

$$\begin{aligned} \log\sqrt{0.1} = \log{(0.1)}^{1/2} = \frac{1}{2}\log 0.1 = \frac{1}{2}\log_{10}\frac{1}{10} = \frac{1}{2}\log_{10} 10^{-1} \\ = -\frac{1}{2}\log_{10} 10 = -\frac{1}{2} \end{aligned}$$

since $\log_{10} 10 = 1$

Example 13. Express the following expression as a single expression.

$$5\ln x + \frac{1}{2}\ln x - 3\ln(x+1)$$

Solution.

$$\begin{aligned} 5\ln x + \frac{1}{2}\ln x - 3\ln(x+1) = \ln x^5 + \ln x^{1/2} - \ln(x+1)^3 \\ = \ln\left(\frac{x^5 x^{1/2}}{(x+1)^3}\right) = \ln\left(\frac{x^5\sqrt{x}}{(x+1)^3}\right) \end{aligned}$$

Example 14. Find the domain of each logarithmic function.

(a) $f(x) = \log_4\left(9 - x^2\right)$ (b) $f(x) = \log(8 - 3x)$

(c) $f(x) = \ln x + \ln(3 - x)$

Solution.

(a) Note that any logarithmic function, $\log_a x$, is defined when $x > 0$. Thus, to find the domain of f we require that

$$\begin{aligned} 9 - x^2 > 0 \\ 9 > x^2 \\ 3 > |x| \end{aligned}$$

so the domain is $(-3, 3)$.

(b) We need to solve

$$\begin{aligned} 8 - 3x > 0 \\ 8 > 3x \\ \frac{8}{3} > x \end{aligned}$$

So the domain is $\left(-\infty, \frac{8}{3}\right)$.

(c) For $\ln x$ we need $x > 0$. As for $\ln(3 - x)$, we require $3 - x > 0$ or $x < 3$. Hence, the domain is $x > 0$ and $x < 3$, i.e. $(0, 3)$.

Example 15. *(Richter Scale)*

On the *Richter Scale* the magnitude R of an earthquake of intensity I is given by

$$R = \log \frac{I}{S}$$

where S is the intensity of a standard earthquake whose amplitude is 1 micron $= 10^{-4}$ cm. This means that the magnitude of a standard earthquake is

$$R = \log \frac{R}{R} = \log 1 = 0$$

(a) Find the magnitude, R, of an earthquake of intensity $I = 81,000,000$ (let $I_0 = 1$).

(b) The 1923 earthquake in Tokyo and Yokohoma, Japan had an estimated magnitude of 8.3 on the Richter scale. Find the intensity of the earthquake (let $I_0 = 1$).

Solution.

(a) Since $I = 81,000,000$ and $I_0 = 1$, we obtain

$$R = \log I = \log(10,000,000) = 7$$

(b) Since $R = 8.3$ and $I_0 = 1$, we have $8.3 = \log I$ and so $I = 10^{8.3}$.

Chapter

5 *TRIGONOMETRIC FUNCTIONS*

In this chapter, we introduce the trigonometric functions and ways to evaluate them. A summary of some basic trigonometric identities is also included. In particular, we will discuss:

- *Angle Measure*
- *Trigonometry of Right Triangles*
- *Basic Trigonometric Identities*
- *The Law of Sines and the Law of Cosines*

ANGLE MEASURE

There are two commonly used units for measuring angles:

1. *Degrees:* A circle is divided into 360 equal degrees, so that a right angle is 90.
2. *Radians:* Consider the unit circle (a circle of radius 1) with the vertex of an angle at its center. Then, the angle subtends an arc of the circle, and the length of that arc is the radian measure of the angle.

Converting between Degrees and Radians.

$$360^0 = 2\pi \text{ rad}, \qquad 1^0 = \frac{\pi}{180} \text{ rad}, \qquad 1 \text{ rad} = \left(\frac{180}{\pi}\right)^0$$

1. *To convert from radians to degrees:* Multiply by $\frac{180}{\pi}$
2. *To convert from degrees to radians:* Multiply by $\frac{\pi}{180}$

Length and Area of a Circular Arc.

Given a circle of radius r.

1. The *length* s of an arc that subtends a central angle of θ radians is

$$s = r\theta$$

2. The *area* A of a sector with central angle θ radians is

$$A = \frac{1}{2}r^2\theta$$

TRIGONOMETRY OF RIGHT TRIANGLES

In this section, we give the definition of the basic six trigonometric ratios, and include a summary of their exact values for special angle values.

The Trigonometric Ratios.

Consider the right triangle shown in Figure 1 with θ as one of its angles. Assume the lengths of the adjacent (adj), opposite (opp) , and hypotenuse (hyp) sides are given by a, b, and c, respectively. Then, the six trigonometric ratios are defined as follows:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{b}{c} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{a}{c} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{b}{a}$$

$$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{a}{b} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{c}{a} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{c}{b}$$

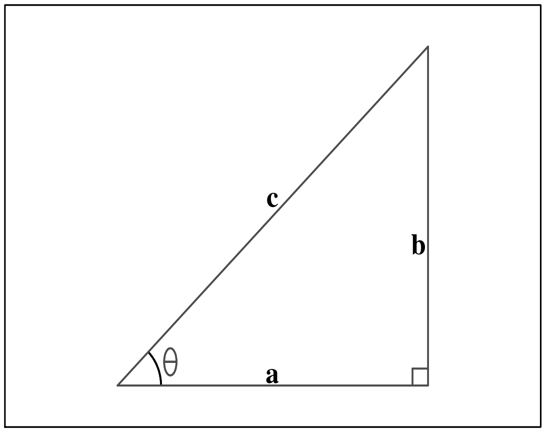


Figure 1:

Special Triangles.

Next, we consider two special triangles where the trigonometric ratios can be computed easily by using the Pythagorean Theorem.

1. *A* 45^o *right triangle:* Two angles are 45^o so the opposite and adjacent sides are equal, assume their length is 1. By the Pythagorean Theorem the hypotenuse has length $\sqrt{1^2+1^2} = \sqrt{2}$. See Figure 2.

2. *A* 30^o-60^o *right triangle:* Note that the side opposite to a 30^o angle is one-half the hypotenuse, i.e. if the opposite side has length 1, then the hypotenuse will have length 2, as is shown in Figure 3. The adjacent side can be calculated by the Pythagorean Theorem, which gives $\sqrt{2^2-1^2} = \sqrt{3}$.

These triangles can be easily used to compute the trigonometric ratios for angles with measures 30^o, 45^o, and 60^o. These values are listed in Table 1.

$\theta(degrees)$	$\theta(radians)$	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
30^0	$\pi/6$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2
45^0	$\pi/4$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60^0	$\pi/3$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$

Table 3: Trigonometric Ratios for 30^o, 45^o, and 60^o angles.

The trigonometric ratios for other important angles are given in Table 2. It

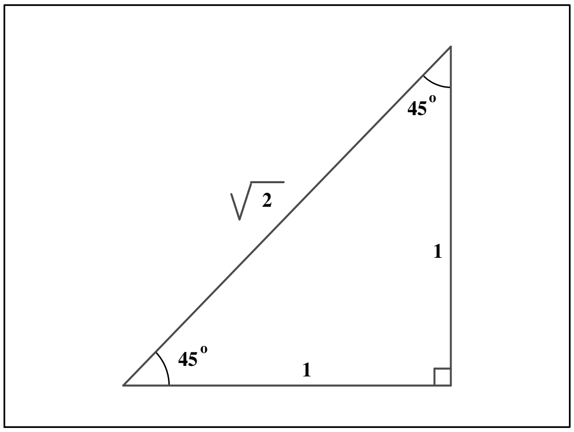


Figure 2:

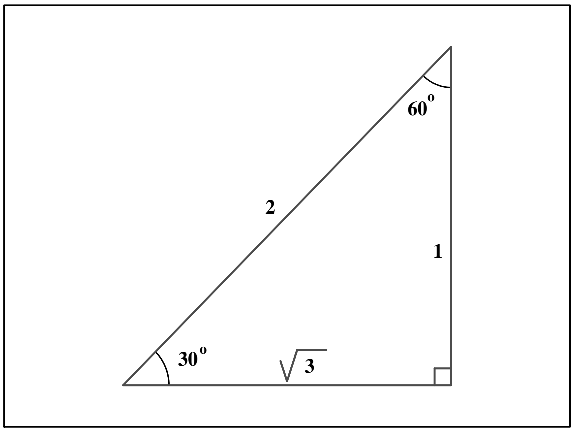


Figure 3:

includes a summary of the exact values for angles with measures 0^o, 90^o, 180^o, and 270^o. It is helpful to know these values since they occur quite often.

$\theta(deg)$	$\theta(rad)$	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
0^0	0	0	1	0	undefined	1	undefined
90^0	$\pi/2$	1	0	undefined	0	undefined	1
180^0	π	0	-1	0	undefined	-1	undefined
270^0	$3\pi/2$	-1	0	undefined	0	undefined	0

Table 4: Trigonometric Ratios for 0^o, 90^o, 180^o, and 270^o angles.

Evaluating Trigonometric Functions at any angle.

To evaluate the trigonometric functions for any angle θ, we follow the following step:

1. Find the *reference angle* $\overline{\theta}$ (acute angle formed by the terminal side of θ and the x-axis) that corresponds to the angle θ.

2. Determine the sign of the trigonometric function of θ.

3. The value of the trigonometric function at θ is the same as at $\overline{\theta}$, except a

possible difference in sign, i.e. $f(\theta) = \pm f(\bar{\theta})$ where f is any trigonometric function.

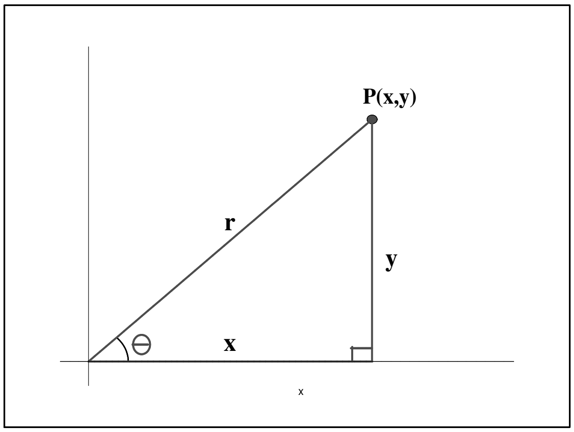


Figure 4:

Draw a right triangle in the plane such that the angle θ is placed in standard position (See Figure 4). Then, the adjacent side corresponds to the x-coordinate while the opposite side corresponds to the y-coordinate. It follows from the definition that the values of the trigonometric functions are all positive if the angle θ has its terminal side in quadrant I. The reason is that x and y are positive in this quadrant. Since $x < 0$ and $y > 0$ in quadrant II, hence $\sin\theta$ is positive while $\cos\theta$ is positive in that quadrant. The signs of the trigonometric functions in the various quadrants is summarized in Table 3.

Quadrant	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
I	+	+	+	+	+	+
II	+	−	−	−	−	+
III	−	−	+	+	−	−
IV	−	+	−	−	+	−

Table 5: Signs of the Trigonometric Functions.

Periodic Properties.

The functions *sine*, *cosine*, *secant*, and *cosecant* have period 2π:

$$\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x$$

$$\sec(x + 2\pi) = \sec x \qquad \csc(x + 2\pi) = \csc x$$

The functions *tangent* and *cotangent* have period π:

$$\tan(x + \pi) = \tan x \qquad \cot(x + \pi) = \cot x$$

Sine and Cosine Curves.

For the sine and cosine curves:

$$y = A\sin Bx \qquad \text{and} \qquad y = A\cos Bx$$

$$Amplitude \ = |A|; \qquad \text{and} \qquad Period = \frac{2\pi}{B}$$

where the *amplitude* is the largest value these functions attain. One complete period can be graphed on the interval $\left[0, \dfrac{2\pi}{B}\right]$.

More generally, for the sine and cosine curves:

$$y = A\sin(Bx + C) \qquad \text{and} \qquad y = A\cos(Bx + C)$$

We have

$$Amplitude \ = |A|; \qquad Period = \frac{2\pi}{B}; \qquad \text{and} \qquad Phase\ Shift = -\frac{C}{B}$$

The *phase shift* is the amount of horizontal shift of the original curves $y = A\sin Bx$ and $y = A\cos Bx$ that will yield the curves of $y = A\sin(Bx + C)$ and $y = A\cos(Bx + C)$. One complete period can be graphed on the interval $\left[-\dfrac{C}{B}, -\dfrac{C}{B} + \dfrac{2\pi}{B}\right]$.

BASIC TRIGONOMETRIC IDENTITIES

In this section, we give a summary of some of the basic trigonometric identities.

I. Reciprocal Identities.

$$\cot x = \frac{1}{\tan x} \qquad \csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x}$$

II. Quotient Identities.

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

III. Identities for Negatives.

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x$$

IV. Pythagorean Identities.

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad \cot^2 x + 1 = \csc^2 x$$

V. Sum Identities.

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

VI. Double-Angle Formulas.

$$\sin 2x = 2\sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

VII. Formulas For Lowering Powers.

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

LAW OF SINES AND LAW OF COSINES

The Law of Sines. In the triangle ABC, shown in Figure 5, we have

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The Law of Cosines. In the triangle ABC, shown in Figure 5, we have

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

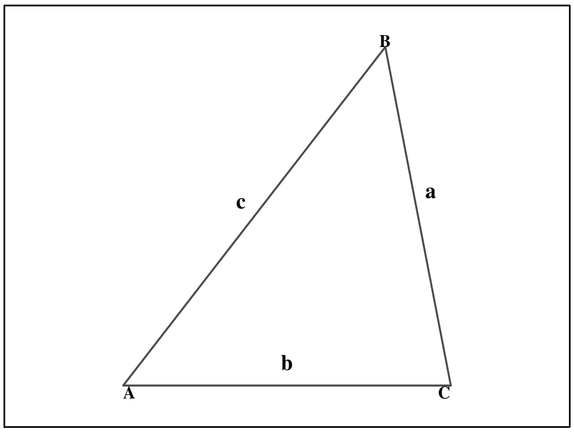


Figure 5:

Example 1. Express the following angles in radians.

(a) 240^0 (b) 15^0

Solution.

(a) $240^0 = 240\left(\frac{\pi}{180}\right)$ rad $= \frac{4}{3}\pi$ rad

(b) $15^0 = 15\left(\frac{\pi}{180}\right)$ rad $= \frac{\pi}{12}$ rad

Example 2. Express the following angles in degrees.

(a) $\frac{\pi}{3}$ rad (b) $\frac{5\pi}{6}$ rad

Solution.

(a) $\frac{\pi}{3}$ rad $= \left(\frac{\pi}{3}\right)\left(\frac{180}{\pi}\right) = 60^0$

(b) $\frac{5\pi}{6}$ rad $= \left(\frac{5\pi}{6}\right)\left(\frac{180}{\pi}\right) = 150^0$

Example 3. *(Coterminal Angles)*

(a) Find angles that are coterminal with the angle 60^0.

(b) Find angles that are coterminal with the angle $\frac{\pi}{6}$.

Solution.

(a) To find coterminal angles we add or subtract multiples of 360^0. Possible angles are:

$$60^0 + 360^0 = 420^0 \qquad \text{and} \qquad 60^0 + 2(360^0) = 780^0$$

$$60^0 - 360^0 = -300^0 \qquad \text{and} \qquad 60^0 - 2(360^0) = -660^0$$

(b) To find coterminal angles we add or subtract any multiple of 2π. Thus, we have the following coterminal angles:

$$\frac{\pi}{6} + 2\pi = \frac{13\pi}{6} \qquad \text{and} \qquad \frac{\pi}{6} + 2(2\pi) = \frac{25\pi}{6}$$

$$\frac{\pi}{6} - 2\pi = -\frac{11\pi}{6} \qquad \text{and} \qquad \frac{\pi}{6} - 2(2\pi) = -\frac{23\pi}{6}$$

Example 4. Given a circle of radius 10 m.

1. Find the length of the arc that subtends a central angle of $\theta = 45^0$.
2. Find the area of the sector with central angle $\theta = 60^0$.

Solution.

(a) Since $45^0 = 45\left(\frac{\pi}{180}\right)$ rad $= \frac{\pi}{4}$ rad, the length of the arc is

$$s = r\theta = 10\left(\frac{\pi}{4}\right) = \frac{5}{2}\pi \text{ m}$$

(b) We have $60^0 = \frac{\pi}{3}$ rad, so the area of the sector is

$$A = \frac{1}{2}r^2\theta = \frac{1}{2}(10)^2\left(\frac{\pi}{3}\right) = \frac{50}{3}\pi \text{ m}^2$$

Example 5. Find

(a) $\cos 240^0$ $\qquad$ (b) $\csc(-630^0)$ $\qquad$ (c) $\cot(-210^0)$

(d) $\sin\left(-\frac{7\pi}{3}\right)$ $\qquad$ (e) $\sec\left(\frac{5\pi}{4}\right)$ $\qquad$ (f) $\tan\left(\frac{11\pi}{6}\right)$

Solution.

(a) The terminal side of this angle is in quadrant III. The reference angle is $240^0 - 180^0 = 60^0$. Therefore, $\cos 240^0 = \pm\cos 60^0$. Since the cos is negative in the third quadrant, hence $\cos 240^0 = -\cos 60^0 = -\frac{1}{2}$.

(b) Note that $-630^0 = -360^0 - 270^0$, so the terminal side of this angle lies on the y-axis. Therefore, $\csc(-630^0) = \csc 90^0 = 1$.

(c) The terminal side of this angle is in quadrant II. The reference angle is $210^0 - 180^0 = 30^0$. Therefore, $\cot(-210^0) = \pm\cot 30^0$. Since the cot is negative in the second quadrant, hence $\cot(-210^0) = -\cot 30^0 = -\sqrt{3}$.

(d) Note that $-\frac{7\pi}{3} = -2\pi - \frac{\pi}{3}$. Thus, the angle is coterminal with $-\frac{\pi}{3}$ and lies in quadrant IV. Since the sin is negative in the fourth quadrant, so

$\sin\left(-\frac{7\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}.$

(e) The terminal side of this angle is in quadrant III. The reference angle is $\frac{5\pi}{4} - \pi = \frac{\pi}{4}$. Thus, $\sec\left(\frac{5\pi}{4}\right) = \pm\sec\left(\frac{\pi}{4}\right)$. Since the sec is negative in the third quadrant, hence $\sec\left(\frac{5\pi}{4}\right) = -\sqrt{2}$.

(f) The terminal side of this angle is in quadrant IV. The reference angle is $2\pi - \frac{11\pi}{6} = \frac{\pi}{6}$. Since the tan is negative in the fourth quadrant, so

$$\tan\left(\frac{11\pi}{6}\right) = -\tan\left(\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$$

Example 6. Verify the identity:

$$\tan\theta\,\sin\theta + \cos\theta = \sec\theta$$

Solution.

$$\begin{aligned}
\tan\theta\,\sin\theta + \cos\theta &= \frac{\sin\theta}{\cos\theta}\,\sin\theta + \cos\theta \\
&= \frac{\sin^2\theta}{\cos\theta} + \frac{\cos\theta}{1} = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta} \\
&= \frac{1}{\cos\theta} = \sec\theta
\end{aligned}$$

where we used the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$.

Example 7. Verify the identity:

$$\cos\theta\,(\sec\theta - \cos\theta) = \sin^2\theta$$

Solution.

$$\begin{aligned}
\cos\theta\,(\sec\theta - \cos\theta) &= \cos\theta\left(\frac{1}{\cos\theta} - \cos\theta\right) \\
&= \cos\theta\left(\frac{1}{\cos\theta}\right) - \cos^2\theta \\
&= 1 - \cos^2\theta = \sin^2\theta
\end{aligned}$$

Example 8. Verify the identity:

$$\frac{1}{1-\sin\theta} = \sec^2\theta + \tan\theta\,\sec\theta$$

Solution.

Multiply both sides of the left-hand side by the conjugate of the denominator.

$$\begin{aligned}
\frac{1}{1-\sin\theta} &= \frac{1}{1-\sin\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta} \\
&= \frac{1+\sin\theta}{1-\sin^2\theta} = \frac{1+\sin\theta}{\cos^2\theta} \\
&= \frac{1}{\cos^2\theta} + \frac{\sin\theta}{\cos^2\theta}
\end{aligned}$$

$$= \left(\frac{1}{\cos\theta}\right)^2 + \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta} = \sec^2\theta + \tan\theta\sec\theta$$

Example 9. Verify the identity:

$$\csc 2\theta - \cot 2\theta = \tan\theta$$

Solution.

Starting with the left-hand side we have

$$\csc 2\theta - \cot 2\theta = \frac{1}{\sin 2\theta} - \frac{\cos 2\theta}{\sin 2\theta} = \frac{1-\cos 2\theta}{\sin 2\theta}$$

Using the double-angle formulas we obtain

$$= \frac{1-\left(1-2\sin^2\theta\right)}{2\sin\theta\,\cos\theta}$$

$$= \frac{2\sin^2\theta}{2\sin\theta\,\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

Example 10. Verify the identity:

$$\frac{\sec^4\theta - 1}{\tan^2\theta} = 2+\tan\theta$$

Solution.

Factoring the left-hand side we get

$$\frac{\sec^4\theta - 1}{\tan^2\theta} = \frac{\left(\sec^2\theta-1\right)\left(\sec^2\theta+1\right)}{\tan^2\theta}$$

Using the identity $\sec^2\theta - 1 = \tan^2\theta$ yields

$$= \frac{\tan^2\theta\left(\sec^2\theta+1\right)}{\tan^2\theta}$$

$$= \sec^2\theta + 1 = \left(\tan^2\theta+1\right)+1 = \tan^2\theta + 2$$

Example 11. Express $4\sin^2\theta\cos^2\theta$ in terms of the first power of cosine.

Solution.

Using the formulas for lowering powers we get

$$4\sin^2\theta\cos^2\theta = 4\left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cos 2\theta}{2}\right)$$

$$= 1-\cos^2 2\theta = 1-\left(\frac{1+\cos 4\theta}{2}\right)$$

$$= \frac{1}{2} - \frac{\cos 4\theta}{2} = \frac{1}{2}\left(1-\cos 4\theta\right)$$

Example 12. Solve the triangle shown in Figure 6.

Solution.

Clearly, the third angle must be equal to 60^o. To find a, we use the equation

$$\sin 30^o = \frac{a}{10}$$

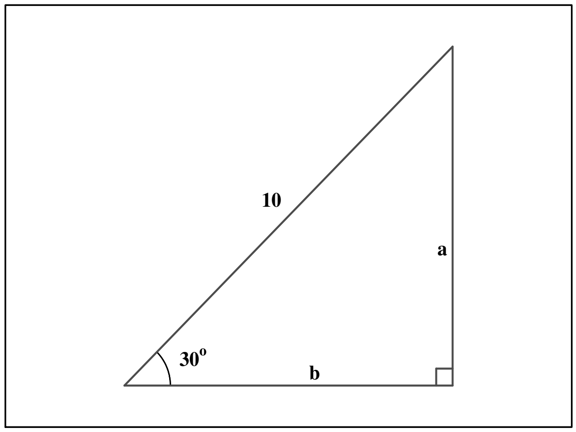


Figure 6:

So

$$a = 10\sin 30^o = 10\left(\frac{1}{2}\right) = 5$$

In a similar fashion, we have $\cos 30^o = \frac{b}{10}$. Thus

$$b = 10\cos 30^o = 10\left(\frac{\sqrt{3}}{2}\right) = 5\sqrt{3}$$

Example 13. If $\csc\theta = 2$ and θ is in quadrant IV, find $\tan\theta$.

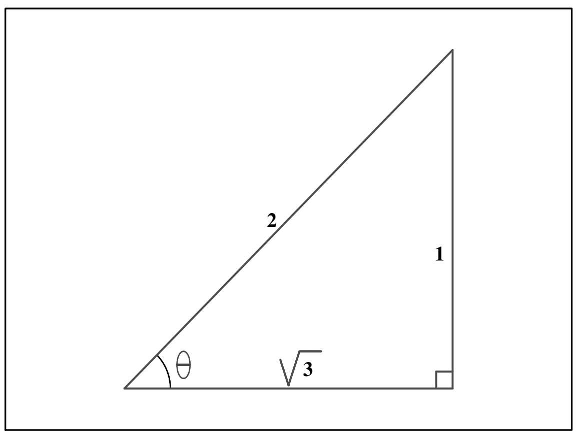


Figure 7:

Solution.

Using the identity $\csc^2\theta = \cot^2\theta + 1$, we can write $\cot\theta$ in terms of $\csc\theta$ as follows:

$$\cot\theta = \pm\sqrt{\csc^2\theta - 1}$$

Since $\cot\theta$ is negative in quadrant IV, so

$$\cot\theta = -\sqrt{\csc^2\theta - 1}$$

Thus

$$\tan\theta = \frac{1}{\cot\theta} = \frac{1}{-\sqrt{2^2 - 1}} = -\frac{1}{\sqrt{3}}$$

Alternatively, the problem can be solved by noting that, except for sign, the values of the trigonometric functions of any angle are the same as those of the *reference angle* $\overline{\theta}$. Thus, as shown in Figure 7, we sketch a triangle with the acute angle $\overline{\theta}$ satisfying $\csc\theta = 2 = \frac{2}{1}$. By the Pythagorean theorem the third side of the triangle is $\sqrt{2^2 - 1^2} = \sqrt{3}$. From the triangle it follows that $\tan\overline{\theta} = \frac{1}{\sqrt{3}}$. Since θ is in quadrant IV, $\tan\theta$ is negative and thus $\tan\overline{\theta} = -\frac{1}{\sqrt{3}}$.

Example 14. If $\tan\theta = \frac{2}{3}$ and θ is in quadrant III, find the other five trigonometric functions.

Solution.

Similar to example 13, we sketch a triangle (see Figure 8) such that $\tan\overline{\theta} = \frac{2}{3}$, where $\overline{\theta}$ is the *reference angle*. Taking into consideration that θ is in quadrant III, we obtain

$$\sin\theta = -\frac{2}{\sqrt{13}} \qquad \cos\theta = -\frac{3}{\sqrt{13}} \qquad \tan\theta = \frac{2}{3}$$

$$\cot\theta = \frac{3}{2} \qquad \sec\theta = -\frac{\sqrt{13}}{3} \qquad \csc\theta = -\frac{\sqrt{13}}{2}$$

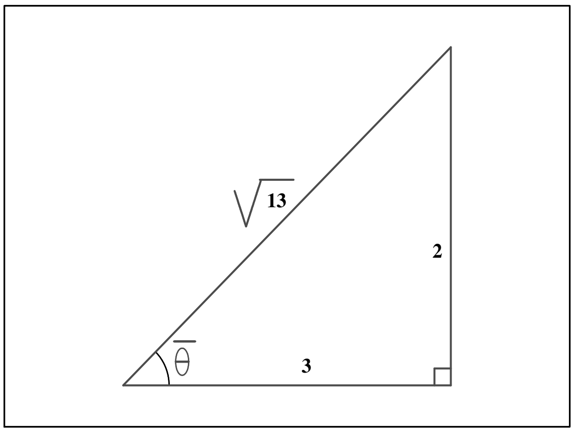


Figure 8:

Example 15. For each function find the amplitude, period, and phase shift.

(a) $y = -5\sin 3\pi x$ (b) $y = \frac{1}{2}\cos\frac{1}{3}x$

(c) $y = -7\cos\left(3x - \frac{\pi}{6}\right)$ (d) $y = -\sin 2\left(x - \frac{\pi}{4}\right)$

Solution.

(a) Amplitude $= |A| = |-5| = 5$ and the period $= \frac{2\pi}{3\pi} = \frac{2}{3}$.

(b) Amplitude $= |A| = \frac{1}{2}$ and the period $\frac{2\pi}{\frac{1}{3}} = 6\pi$.

(c) Amplitude $= |A| = |-7| = 7$ and the period $\frac{2\pi}{3}$. The phase shift is $= -\frac{C}{B} = -\frac{\left(-\frac{\pi}{6}\right)}{3} = \frac{\pi}{18}$.

(d) Amplitude $= |A| = |-1| = 1$, period $= \frac{2\pi}{2} = \pi$, and the phase shift is $= \frac{\pi}{4}$.

Example 16. Two sides of a triangle have lengths $a = 10$ and $b = 20$, and the angle between them is 60^o. Solve the triangle.

Solution.

Using the *Law of Cosines* we have

$$c^2 = a^2 + b^2 - 2ab\cos C = 10^2 + 20^2 - 2(10)(20)\cos 60^o$$

$$= 100 + 400 - 400\left(\frac{1}{2}\right) = 500 - 200 = 300$$

Therefore, $c = \sqrt{300}$.

Applying the *Law of Sines* we obtain

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

or

$$\frac{\sin 60^o}{\sqrt{300}} = \frac{\sin A}{10}$$

which yields

$$\sin A = 10\frac{\sin 60^o}{\sqrt{300}} = \frac{\sin 60^o}{\sqrt{3}} = \frac{\sqrt{3}/2}{\sqrt{3}} = \frac{1}{2}$$

But $\sin A = \frac{1}{2}$ implies that $A = 30^o$. As for the third angle:

$$B = 180^o - 30^o - 60^o = 90^o$$

This means that the given triangle is a right-angle triangle.

Chapter
6 *REVIEW EXERCISES*

1. Find numbers x and y such that $\sqrt{x^2 - y^2} \neq x - y$.

2. Simplify $\sqrt{32x^4y^8z^2}$.

3. Solve for x: $3x^2 = 2 - 5x$.

4. Simplify $\dfrac{9x^4y^6 - 6x^2y^3}{3x^2y^3}$

5. Let $f(x) = 3x - 7$ and $g(x) = \dfrac{x+2}{2-x}$. Find $(gof)(-1)$

6. Evaluate $\dfrac{27^{4/3}\,16^{1/4}}{81^{1/2}}$.

7. Find the coordinates (x, y) of the point of intersection of the graphs of $2y + 4x = -6$ and $y - 3x = 7$.

8. Find the equation of the line that passes through the point $(-3, 2)$ and is perpendicular to the line $\dfrac{4}{3}x + 2y = -12$.

9. Find the vertex of the parabola $y = x^2 - x - 2$, then sketch it.

10. Graph: $|3 - x|$.

11. Let $f(x) = |x - 1| + x^{-3} + 2^{-x}$. Find $f(-1)$.

12. Solve for x $\log_3(2x - 5) = 2$.

13. Graph $y = 1 + 2^{-x}$.

14. Find the distance and the midpoint between the following points of the plane: $P = (-3, 5)$ and $Q = (-7, -9)$.

15. Let $f(x) = \dfrac{1}{x}$. Find the value of $\dfrac{f(x+h) - f(x)}{h}$.

16. Solve the following equation for y in terms of x: $x = e^{5y-2}$.

17. Find the domain of $f(x) = \sqrt{4x - x^2}$.

18. Evaluate $\tan(\pi/4) + \sec(0)$.

19. Sketch the graph of $2x + 5y = 20$, and find the x and y intercepts.

20. Simplify $\sin^2 x + \dfrac{1}{1 + \tan^2 x}$.

21. Solve for x $|5 - 2x| < 4$.

22. Let $\dfrac{x}{x-1}$. Find $f(f(x))$.

23. Given a right-angle triangle where the opposite side of the angle C has length 1.5. Assume $\cos(C) = 4/5$. Find the length of the side adjacent to C.

24. Find the surface area of an open-top box with a height z, length x and width y.

25. Find the domain of $f(x) = \dfrac{\sqrt{x+2}}{x}$.

26. Simplify $\sin^4 x - \cos^4 x$.

27. Let $f(x) = \log(x-2) + |1-x|$. Evaluate $f(3)$.

28. Solve the equation: $\sqrt{3x+4} + 2 = x$.

29. Find a logarithmic function which has a zero at 3, and a vertical asymptote at 2.

30. Find the vertical and horizontal asymptotes of the rational function $f(x) = \dfrac{x^2}{x^2-9}$.

31. Find the inverse of $f(x) = 2x - 5$.

32. Solve the inequality $x(x-1) \leq 6$

33. Compute $\sec\left(\dfrac{13\pi}{6}\right)$.

34. Solve for q: $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

35. If $\sin\theta = \dfrac{4}{5}$ and $\dfrac{\pi}{2} < \theta < \pi$. Find $\tan\theta$.

36. Solve for x: $9^x = \left(\dfrac{1}{3}\right)^{x-3}$.

37. Find the center and radius of the circle having equation $x^2 + (y+3)^2 = 36$.

38. If $\dfrac{(2x-3)(x+1)}{x-1} = 0$. Then $x =$?

39. Solve the inequality $\dfrac{3}{x} > 2$.

40. Solve the equation $\dfrac{1}{3} - \dfrac{x}{6} = 1 + \dfrac{x}{5}$.

41. If $7^x = 3$, find x.

42. If $y = e^{x+2}$, find x.

43. Find the solution set of $-3 < 5x + 2 \leq 12$

44. Solve for x, where $x \in [0, 2\pi)$: $2\cos^2 x + \cos x - 1$

45. Find the minimum value of $x^2 - 2x + 3$.

46. Find the value of $\ln\left(\dfrac{e^3}{e^4}\right)$

47. Solve the equation $|2 - x| = 5$

48. Solve $2|2x-3| + 5 > 15$.

49. Solve for x: $\log x + \log(x-3) = 1$

50. Solve for x: $\log_3(x^2-3)\log_3(3x-2) = 0$

51. Given that $\sin\theta = -\dfrac{2}{5}$, where θ is in quadrant III. Find the value of $\cot\theta$.

52. Solve for x: $\dfrac{1}{x} + \dfrac{1}{x+1} = \dfrac{5}{6}$

53. Solve the equation $x^2 - \dfrac{3}{2} = 2x$.

54. Solve for x: $1 - \dfrac{x-1}{3-x} = \dfrac{x+2}{x+3} - \dfrac{x^2}{x^2-9}$

55. Solve the equation $2x = 1 - \sqrt{2-x}$.

56. Solve the equation $5 + 2\ln x = 6$.

57. Solve the equation $e^{2x} - 3e^x + 2 = 0$.

58. Solve the equation $\log_4 x - \log_4 (x-1) = \dfrac{1}{2}$.

59. Solve for x, where $x \in [0, 2\pi)$: $2\sin^2 x = \sin 2x$.

60. Solve for x, where $x \in [0, 2\pi)$: $\cos x = \sin x - 1$.

61. Solve for x, where $x \in [0, 2\pi)$: $\tan x - 3\cot x = 0$.

62. Solve the equation $\tan^5 x - 9\tan x = 0$.

63. Find the minimum or maximum value of the quadratic function $f(x) = 4x^2 + 4x + 5$.

64. Divide by synthetic division: $\left(x^4 - 3x^3 - 7x^2 - 11x - 20\right) \div (x-5)$

65. Determine the right-hand and left-hand behavior of the graph of the polynomial function $f(x) = x^6 - 5x^2 + 3$.

66. Perform the division: $\left(8x^3 - 3x^2 + 17x - 6\right) \div (x^2 + 2)$

67. Determine the right-hand and left-hand behavior of the graph of the polynomial function $f(x) = -3x^5 - x^4 + 2x^2 - 11$.

68. Divide by synthetic division: $\left(-2x^3 + x + 7\right) \div (x+1)$

69. Perform the division: $\left(x^4 + x^3 - x^2 + 2x\right) \div (x^2 + 2x)$

70. Find all the zeros (roots) of the function $f(x) = x^3 + x^2 - 5x + 3$.

71. Find all the zeros (roots) of the function $f(x) = 3x^3 - 6x^2 + x - 2$.

72. Use the laws of logarithms to expand the expression $\ln \sqrt[3]{\dfrac{x}{y^3 z^2}}$.

73. Determine the horizontal and vertical asymptotes, if any, of the rational function $f(x) = \dfrac{2x+7}{x^2 - 4x + 3}$.

74. Find the equation of the line that passes through the point $(2, -1)$ and is parallel to the line $2x - 3y = 5$.

75. Find the equation of the line passing through the points $(1, 0.6)$ and $(-2, -0.6)$.

76. Find the slope and y-intercept of the equation of the line $7x + 6y - 30 = 0$.

77. Evaluate the expressions: $\log 1000$; $\log_4 \dfrac{1}{16}$; $\log_{25} 5$; $\log_7 1$.

78. Evaluate the expressions: $\ln e^5$; $3^{\log_3 5}$; $\log_{27} 9$; $\log_8 \dfrac{1}{2}$.

79. Find the domain of the function $f(x) = \log_5(10 - 2x)$.

80. Express $5\log_2 x + \dfrac{1}{2}\log_2 y - \log_2 z^3$.

81. Determine the horizontal and vertical asymptotes, if any, of the rational function $f(x) = \dfrac{x^2 - 9}{x+3}$.

82. Find the domain and range of the function $f(x) = 4 - 2e^{x-4}$.

83. Find the domain of the function $f(x) = \dfrac{x}{x^2 + x - 6}$.

84. Find the domain of the function $f(x) = \dfrac{\sqrt[3]{x-1}}{x^2+1}$.

85. Determine whether the graph of the following equations is a function of x: $x^2 + y^2 = 9$; $x = 2y^2$; $y = x^2 - 2x + 3$.

86. Determine whether the following functions are one-to-one: $f(x) = x^2$; $f(x) = 2x - 5$; $f(x) = x^3$.

87. z is directly proportional to x and inversely proportional to the square of y. If $x = 2$ and $y = 3$, then $z = 10$. Find the constant of proportionality.

88. s is jointly proportional to the squares of r and q, and inversely proportional to the square root of w. If $r = 2$, $q = 5$, and $w = \dfrac{1}{4}$, then $s = 60$. Find the constant of proportionality.

89. Find the inverse of the function $f(x) = \dfrac{1}{2}x^3 - 5$.

90. Find the inverse of the function $f(x) = \dfrac{2x-3}{5x+1}$.

91. Find the x- and y- intercepts of the graph of $x^2 - 2x + 2y^2 = 8$.

92. If $\sec\theta = 5$ and $\tan\theta = 2\sqrt{6}$, find $\sin\theta$.

93. The graph of $y = f(x)$ is shifted left 3 units, stretched vertically by a factor of 2, reflected through the x-axis, and then shifted upward by 6 units. Find the equation of the resulting graph.

94. The graph of $y = \sqrt{x}$ is shifted right 2 units, and then reflected through the y-axis. Find the equation of the resulting graph.

95. The graph of $y = x^2$ is shifted right 1 units, and then shifted downward by 5 units. Find the equation of the resulting graph.

96. Given $f(x) = x + 2$ and $g(x) = 4 - x^2$, find $(fog)(x)$ and $(gof)(x)$.

97. Given $f(x) = 4 + x^2$ and $g(x) = \sqrt{x-1}$, find $(fog)(5)$, $(gog)(17)$, and $(gof)(-1)$.

98. Evaluate: $\cot(-45^o)$; $\csc(-30^o)$; $\tan(495^o)$; $\sin(240^o)$.

99. Evaluate: $\sin\dfrac{7\pi}{6}$; $\tan\dfrac{7\pi}{4}$; $\cot\left(\dfrac{-2\pi}{3}\right)$; $\csc\dfrac{3\pi}{2}$.

100. Evaluate: $\tan\left(-\dfrac{\pi}{2}\right)$; $\sec\left(-\dfrac{2\pi}{3}\right)$; $\cos\left(\dfrac{11\pi}{3}\right)$.

101. Assume that $\tan\theta = \dfrac{2}{3}$ and θ is in quadrant III, find $\cos\theta$.

102. Assume that $\sec\theta = 2$ and θ is in quadrant IV, find $\sec\theta$, $\csc\theta$ and $\cot\theta$.

103. Assume that $\cos\theta = -\dfrac{2}{7}$ and $\tan\theta < 0$, find $\sec\theta$, $\csc\theta$ and $\cot\theta$.

104. Simplify $\cot\theta\cos\theta + \sin\theta$.

105. Simplify $\dfrac{1+\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{1+\sin\theta}$.

106. Simplify $\dfrac{1-\tan^2\theta}{1+\tan^2\theta}$.

107. Simplify $\dfrac{2\tan\theta}{1+\tan^2\theta}$.

108. Simplify $(\sin\theta + \cos\theta)^2 - 1$.

109. Simplify $8\sin^2\theta\cos^2\theta$.

110. Find the amplitude and period of the function $y = 5\cos 3\theta$.

111. Find the amplitude and period of the function $y = -3\sin\frac{1}{2}\theta$.

112. Find the amplitude, period, and phase shift of the function $y = -\frac{2}{5}\sin\left(2\theta + \frac{2\pi}{3}\right)$.

ANSWERS

1. $x = 5, y = 4$	2. $4\sqrt{2}x^2y^4z$	3. $\frac{1}{3}, -2$	4. $3x^2y^3 - 1$
5. $-\frac{2}{3}$	6. 18	7. $(-2, 1)$	8. $y = \frac{1}{2}(3x + 13)$
9. $\left(\frac{1}{2}, -\frac{9}{4}\right)$	11. 3	12. 7	14. $\sqrt{212}$; $(-5, -2)$
15. $\frac{-1}{x^2 + xh}$	16. $\frac{1}{5}(\ln x + 2)$	17. $[0, 4]$	18. 2
19. $(0, 4)$; $(10, 0)$	20. 1	21. $\left(\frac{3}{2}, \frac{7}{2}\right)$	22. $\frac{x}{2x + 1}$
23. 2	24. $xy + 2xz + 2yz$	25. $[-2, \infty)/\{0\}$	26. $-\cos 2x$
27. 2	28. 7	29. $\log(x - 2)$	30. $x = \pm 3$, $y = 1$
31. $f^{-1}(x) = \frac{x + 5}{2}$	32. $[-2, 3]$	33. $\frac{2}{\sqrt{3}}$	34. $\frac{p}{p - 1}$
35. $-\frac{4}{3}$	36. 1	37. $(0, -3)$; 6	38. $\frac{3}{2}$; -1
39. $\left(0, \frac{3}{2}\right)$	40. $-\frac{20}{11}$	41. $\frac{\log 3}{\log 7}$	42. $\ln y - 2$
43. $(-1, 2]$	44. $\pi, \frac{\pi}{3}, \frac{5\pi}{3}$	45. $(1, 2)$	46. -1
47. $-3, 7$	48. $(-\infty, -1) \cup (4, \infty)$	49. 5	50. $1, 2$
51. $\frac{\sqrt{21}}{2}$	52. $-\frac{3}{5}, 2$	53. $\frac{2 \pm \sqrt{10}}{2}$	54. $\frac{-3 \pm \sqrt{57}}{4}$
55. $-\frac{1}{4}$	56. $\sqrt{e}$	57. 0, $\ln 2$	58. 2
59. $0, \pi, \frac{\pi}{4}, \frac{5\pi}{4}$	60. $\frac{\pi}{2}, \pi$	61. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$	62. $n\frac{\pi}{3}$
63. min: $(-1/2, 4)$	64. $x^3 + 2x^2 + 3x + 4$	65. $\infty; \infty$	66. $8x - 3 + \frac{x}{x^2 + 2}$
67. $(\infty; -\infty)$	68. $-2x^2 + 2x - 1 + \frac{8}{x + 1}$	69. $x^2 - x + 1$	70. $1, -3$
71. 2	72. $\frac{1}{3}\ln x - \ln y - \frac{2}{3}\ln z$	73. $y = 0$; $x = 1, 3$	74. $y = \frac{1}{3}(2x - 7)$
75. $2x - 5y + 1 = 0$	76. $-7/6$; 5	77. 3; -2; $1/2$; 0	78. 5; 5; $2/3$; $-1/3$
79. $(-\infty, 5)$	80. $\log_2\left(\frac{x^5\sqrt{y}}{z^3}\right)$	81. none	82. $(-\infty, \infty)$; $(-\infty, 4)$
83. $(-\infty, \infty)/\{-3, 2\}$	84. $(-\infty, \infty)$	85. no; no; yes	86. no; yes; yes
87. 45	88. 0.3	85. no; no; yes	89. $f^{-1}(x) = \sqrt[3]{2x + 10}$
90. $f^{-1}(x) = \frac{3 + x}{2 - 5x}$	91. -2 & 4; ± 2	92. $\frac{2\sqrt{6}}{5}$	93. $y = -2f(x + 3) + 6$
94. $y = \sqrt{-2 - x}$	95. $y = (x - 1)^2 - 5$	96. $6 - x^2$; $-x^2 - 4x$	97. 8; $\sqrt{3}$; 2
98. -1; -2; -1; $-\frac{\sqrt{3}}{2}$	99. $-\frac{1}{2}$; -1; $\frac{1}{\sqrt{3}}$; -1	100. undefined; -2; $\frac{1}{2}$	101. $-\frac{3}{\sqrt{13}}$
102. 2; $-\frac{2}{\sqrt{3}}$; $-\frac{1}{\sqrt{3}}$	103. $-\frac{7}{2}$; $\frac{7\sqrt{5}}{15}$; $-\frac{2\sqrt{5}}{15}$	104. $\csc\theta$	105. $2\sec\theta$
106. $\cos 2\theta$	107. $\sin 2\theta$	108. $\sin 2\theta$	109. $1 - \cos 4\theta$
110. 5; $\frac{2\pi}{3}$	111. 3; 4π	112. $\frac{2}{5}$; π; $-\frac{\pi}{3}$	

Chapter

7 *PRACTICE EXAM*

1. Which of the following equations represents the equation of a line parallel to $2x + 20y = 5$?

   (A) $30y + 3x = 1$ (B) $10x + y = 20$ (C) $y = 10x + 2$
   (D) $x + 20y = 10$ (E) None of these

2. Which of the following is equivalent to $x^{-3/5}$?

   (A) $x^{-5/3}$ (B) $\dfrac{1}{\sqrt[3]{x^5}}$ (C) $\dfrac{1}{\sqrt[5]{x^3}}$
   (D) $x^{5/3}$ (E) None of these

3. What is the vertex of $y = 2 - 5x^2$?

   (A) $(2, 0)$ (B) $(2, -5)$ (C) $(-2, -5)$
   (D) $(0, -5)$ (E) None of these

4. Which of the following is always *true*.

   (A) $\log(x + y) = \log x + \log y$ (B) $\log x \log y = \log xy$
   (C) $\log 1 = 1$ (D) $\log 0 = 0$
   (E) None of these

5. Which of the following is equivalent to

$$\frac{1}{2x} + \frac{1}{3x}$$

   (A) $\dfrac{5}{6x}$ (B) $\dfrac{2}{5x}$ (C) $\dfrac{1}{5x}$
   (D) $\dfrac{5}{6x^2}$ (E) None of these

6. Which of the following points is not on the graph of $y = (x - 3)^4$?

   (A) $(0, 81)$ (B) $(3, 0)$ (C) $(4, 1)$
   (D) $(2, -1)$ (E) None of these

7. Factor completely: $x^3 + 4x^2 - 100x - 400$

   (A) $(x + 4)(x - 10)^2$ (B) $4x(x + 10)(x - 10)$
   (C) $4(x + 10)(x - 10)$ (D) $(x + 4)(x + 10)(x - 10)$
   (E) None of these

8. Solve $|8 - 3x| < 4$

   (A) $\left(\dfrac{4}{3}, 4\right)$ (B) $\left[\dfrac{4}{3}, 4\right]$ (C) $\left(-\infty, \dfrac{4}{3}\right)$
   (D) $\left(-\infty, \dfrac{4}{3}\right) \cup (4, \infty)$ (E) $(-\infty, 4)$

9. Solve the equation $xy = zu - 2xv$ for x.

   (A) $x = \dfrac{zu - 2xv}{y}$ (B) $x = \dfrac{zu}{y + 2v}$ (C) $x = \dfrac{y + 2v}{zu}$
   (D) $x = \dfrac{zu - xy}{2v}$ (E) None of these

10. Solve the inequality $\dfrac{4}{x} - 1 > 0$.

$(A)\ x > 4$ $(B)\ x < 4$ $(C)\ x > \dfrac{1}{4}$
$(D)\ 0 < x < 4$ (E) None of these

11. Solve $(5x-7)^2 = 22$.

$(A)\ \dfrac{7+\sqrt{22}}{5}$ $(B)\ \dfrac{7\pm\sqrt{22}}{5}$ $(C)\ \dfrac{-5+\sqrt{22}}{7}$
$(D)\ \pm 3$ (E) None of these

12. Find the equation of the line that passes through $(4,-7)$ and is perpendicular to the line $2x - 5y = 3$.

$(A)\ 2y + 5x = 6$ $(B)\ 5y = 2x - 43$ $(C)\ y = \dfrac{5}{2}x - 17$
$(D)\ y = -\dfrac{5}{2}x - 7$ (E) None of these

13. How can you get the graph of $y = -f(x+2) - 5$ from the graph of $y = f(x)$?

(A) Shift $y = f(x)$ left 2 units, down 5 units, then reflect through the x-axis.

(B) Shift $y = f(x)$ right 2 units, down 5 units, then reflect through x-axis.

(C) Shift $y = f(x)$ left 2 units, reflect through x-axis, then down 5 units.

(D) Shift $y = f(x)$ right 5 units, reflect through the x-axis, then vertical stretch by 2 units.

(E) None of these

14. The domain of $f(x) = \dfrac{3\sqrt{x}}{1-x}$ is:

$(A)\ [1,\infty)$ $(B)\ (0,1)\cup(1,\infty)$ $(C)\ [0,1)\cup(1,\infty)$
$(D)\ (0,\infty)$ (E) none of these

15. The end behavior of the polynomial $y = -3x^6 + x^5 - x + 6$ is:

(A) Rises to the left and the right

(B) Falls to the left and the right

(C) Rises to the right and falls to the right

(D) Rises to the left and falls to the right

(E) None of these

16. Use the properties of logarithms to expand $\log_9\left(\dfrac{3y^6}{z^3\sqrt[5]{x}}\right)$.

(A) $2 + 6\log_9 y - 3\log_9 z - \dfrac{1}{5}\log_9 x$

(B) $\dfrac{1}{2} + 6\log_9 y - 3\log_9 z + \dfrac{1}{5}\log_9 x$

(C) $2 + 6\log_9 y - 3\log_9 z - 5\log_9 x$

(D) $\dfrac{1}{2} + 6\log_9 y - 3\log_9 z - \dfrac{1}{5}\log_9 x$

(E) None of these

17. Let $f(x) = \log(x-2)$. Which of the following statements is *true* about the function f?

(A) x-intercept at $(1, 0)$ and has a vertical asymptote at $x = 2$.

(B) x-intercept at $(3, 0)$ and has a vertical asymptote at $x = 0$.

(C) domain $[2, \infty)$ and has no horizontal asymptote.

(D) domain $(2, \infty)$ and no vertical asymptote.

(E) None of these

18. Let $f(x) = 3^{x+2} - 3$. Which of the following statements is *true* about the function f?

(A) The domain of f is $(-\infty, \infty)$ and f has a horizontal asymptote at $y = 0$.

(B) The domain of f is $(-2, \infty)$ and f has a horizontal asymptote at $y = -3$.

(C) The range of f is $(-3, \infty)$ and f has no vertical asymptote.

(D) The function f has a vertical asymptote at $x = -2$ and intersects the x-axis at $x = -2$.

(E) None of these

19. $\log_6 4 + \log_6 9 =$

(A) 2 $\quad(B)$ $\log_6 \frac{4}{9}$ $\quad(C)$ $\log_6 13$

(D) 6 $\quad(E)$ None of these

20. $\sin\left(-\frac{5\pi}{4}\right) =$

(A) $-\frac{1}{\sqrt{2}}$ $\quad(B)$ $-\frac{\sqrt{3}}{2}$ $\quad(C)$ $-\frac{1}{2}$

(D) $\frac{1}{2}$ $\quad(E)$ None of these

21. Assume $\frac{\pi}{2} < \theta < \frac{3\pi}{4}$, which of the following could be a *possible* value of $\tan\theta$?

(A) -20 $\quad(B)$ 0 $\quad(C)$ $\frac{1}{2}$

(D) -0.6 $\quad(E)$ None of these

22. For all real numbers x, $\cos^2 x - \sin^2 x =$

(A) 1 $\quad(B)$ $\cos 2x$ $\quad(C)$ $\sin 2x$

(D) -1 $\quad(E)$ None of these

23. The *Period* of $\sin(\pi x)$ is

(A) 2π $\quad(B)$ π $\quad(C)$ $\frac{1}{2}$

(D) 2 $\quad(E)$ None of these

24. Which of the following is equivalent to $\frac{1-\cos^2\theta}{\cos^2\theta}$.

(A) 1 $\quad(B)$ $\tan^2\theta$ $\quad(C)$ $\frac{1}{\cos^2\theta}$

(D) $\sec^2\theta - \cos^2\theta$ $\quad(E)$ None of these

25. If $f(x) = \log_3 x$, then $f(1/3) =$

(A) $\sqrt[3]{3}$ (B) 1 (C) -1
(D) -3 (E) None of these

26. If $\log x = \dfrac{3}{2}$ and $\log y = 5$, then $\log\left(\dfrac{x^2}{y}\right) =$

(A) $\dfrac{9}{20}$ (B) $-\dfrac{11}{4}$ (C) -2
(D) 8 (E) None of these

27. For all real numbers x, $2 + \tan^2\theta - \sec^2\theta =$

(A) $1 + 3\tan^2\theta$ (B) $\sec^2\theta$ (C) $3\tan^2\theta$
(D) 1 (E) None of these

28. If $f(x) = \dfrac{4x}{1-x}$ and $g(x) = \dfrac{2}{x}$, then $f(g(x)) =$

(A) $\dfrac{8}{x-2}$ (B) $\dfrac{8}{1-x}$ (C) $\dfrac{1-x}{2x}$
(D) $\dfrac{8(x-2)}{x^2}$ (E) None of these

29. $\dfrac{9a^2 + 3a}{3a} =$

(A) $9a^2$ (B) $3a + 1$ (C) $4a$
(D) $9a^2 + 1$ (E) None of these

30. If $7^{x-1} = 3$, then $x =$

(A) $\dfrac{10}{7}$ (B) $1 + \log\dfrac{3}{7}$ (C) $1 + \log_3 7$
(D) $1 + \log_7 3$ (E) None of these

31. If $f(x) = x^2 + x - 9$, then $f(a+2) =$

(A) $a^2 + 2a - 5$ (B) $a^2 + a - 3$ (C) $a^2 + a - 7$
(D) $a^2 + a - 5$ (E) None of these

32. Solve for x: $\log_4 x = -3$

(A) -12 (B) -16 (C) $\dfrac{1}{64}$
(D) -64 (E) None of these

33. The width of a rectangular garden is 10 ft. less than its length x. Write an expression for the perimeter of the garden in terms of its length x.

(A) $2x + 2(10 - x)$ (B) $2x(x - 10)$ (C) $4x - 20$
(D) $2(x + 10) + 2x$ (E) None of these

34. If $\dfrac{1}{x-3} + 4 = \dfrac{x}{x-3}$, then which of the following best approximates x?

(A) 3.67 (B) 0.67 (C) 0.40
(D) -0.67 (E) None of these

35. The graphs of the equations $\begin{cases} x + 3y = 1 \\ 2x - 6y = 2 \end{cases}$ consist of

(A) one line (B) two distinct parallel lines
(C) two lines intersecting at $x = 1$ (D) two lines intersecting at $x = 2$
(E) None of these

36. Let $f(x) = \dfrac{1}{(x-1)^2}$. As x approaches 1, $f(x)$ approaches

(A) ∞ (B) $-\infty$ (C) 0
(D) 1 (E) None of these

37. Solve for x: $|5x - 6| + 3 = 10$

(A) $-\dfrac{1}{5}$ (B) $\dfrac{13}{5}$ (C) $-\dfrac{7}{5}$ and $\dfrac{19}{5}$
(D) $-\dfrac{7}{5}$ (E) None of these

38. $\cos\theta \cot\theta \sec^2\theta =$

(A) $\cos\theta$ (B) $\tan\theta$ (C) $\csc\theta$
(D) $\sec\theta$ (E) None of these

39. The graph of the equation $y - 7x = -5$ is

(A) A horizontal line (B) Not a line (C) A line rising to the right
(D) A line falling to the right (E) None of these

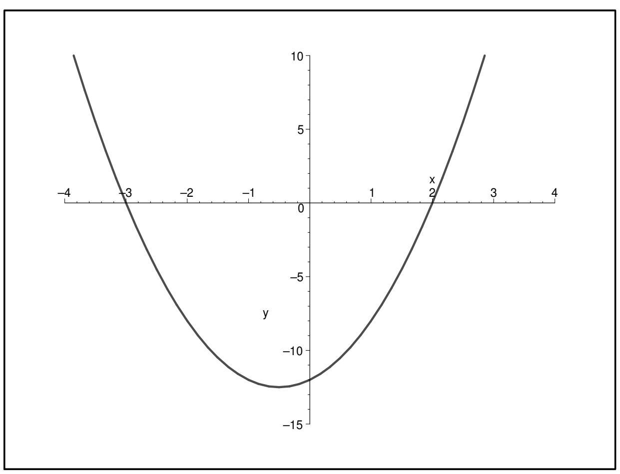
40. Solve for x: $\dfrac{3}{2}(x+5) - \dfrac{1}{4}x = 6$

(A) $-\dfrac{6}{5}$ (B) $\dfrac{1}{7}$ (C) $\dfrac{6}{5}$
(D) -1 (E) None of these

41. The interval solution of the inequality $x^2 - 14x > 15$ is

(A) $(3, 5)$ (B) $(-\infty, 3) \cup (5, \infty)$ (C) $(-1, 15)$
(D) $(-\infty, -1) \cup (15, \infty)$ (E) None of these

42. If f is a function whose graph is shown in the Figure below, then $f(x) < 0$ if



(A) $x < 0$ (B) $x > 2$ or $x < -3$ (C) $-3 < x < 2$
(D) $x > -3$ (E) None of these

43. The radical equation $\sqrt{16x + 36} - 4 = x$ has two solutions. The *sum* of the two solutions is

(A) 10 (B) 12 (C) -12
(D) 8 (E) None of these

44. Use long division or synthetic division to find the quotient and remainder when $2x^3 - x^2 + 4x - 6$ is divided by $x - 1$.

(A) $x^2 + 2x + 1$; remainder -5 (B) $2x^2 + x + 5$; remainder -1
(C) $2x^2 + 3x - 1$; remainder -5 (D) $2x^2 + 3x - 1$; remainder 3
(E) None of these

45. Solve the inequality $\frac{1}{4}x - \frac{3}{2}(x + 1) \geq \frac{1}{3}(x - 2) + 1$.

(A) $\left[-\frac{22}{19}, \infty\right)$ (B) $\left(-\infty, -\frac{22}{10}\right]$ (C) $\left(-\infty, \frac{11}{19}\right]$
(D) $\left(\frac{11}{19}, \infty\right)$ (E) None of these

46. If $f(x) = 2\sqrt[3]{5 - x}$ find the inverse function $f^{-1}(x)$.

(A) $\frac{1}{2\sqrt[3]{5 - x}}$ (B) $\frac{1}{2}\sqrt[3]{5 + x}$ (C) $5 - \frac{x^3}{8}$
(D) $x^3 - \frac{5}{8}$ (E) None of these

47. Let $f(x) = 3x^2 + 12x - 7$. Which of the following statements is *false* about f?

(A) The graph of f is a function of x.

(B) f has an absolute minimum at $x = -2$.

(C) f is increasing on the interval $(-2, \infty)$.

(D) f is one-to-one.

(E) None of these

48. The graph of $2x - 6y + 15 = 0$ crosses the y-axis at $y =$

(A) 7.5 (B) -7.5 (C) 2.5
(D) -2.5 (E) 15

49. Which of the following best approximates the smaller solution of $x^2 - 6x = 5$?

(A) 1 (B) -5 (C) -0.74
(D) 2.24 (E) 6.24

50. The price of a car is inversely proportional to its age. If the price of the car after two years is 10,000, then the price of the car after 5 years is

(A) \$5000 (B) \$4000 (C) \$2000
(D) \$15000 (E) \$2500

ANSWERS

1. *A*	2. *C*	3. *E*	4. *E*	5. *A*	6. *D*	7. *D*
8. *A*	9. *B*	10. *D*	11. *B*	12. *A*	13. *C*	14. *C*
15. *B*	16. *D*	17. *E*	18. *C*	19. *A*	20. *E*	21. *A*
22. *B*	23. *D*	24. *B*	25. *C*	26. *C*	27. *D*	28. *A*
29. *B*	30. *D*	31. *E*	32. *C*	33. *C*	34. *A*	35. *C*
36. *A*	37. *E*	38. *C*	39. *C*	40. *A*	41. *D*	42. *C*
43. *D*	44. *B*	45. *B*	46. *C*	47. *D*	48. *C*	49. *C*
50. *B*						