A CHARACTERIZATION OF VALUATION DOMAINS VIA m-CANONICAL IDEALS

Ayman Badawi
Department of Mathematics & Statistics
American University of Sharjah
P.O. Box 26666
Sharjah, United Arab Emirates

E-mail: abadawi@aus.ac.ae

Abstract. A nonzero ideal I of an integral domain R is said to be an m-canonical ideal of R if $(I:(I:J)) = J$ for every nonzero ideal J of R. In this paper, we show that if a quasi-local integral domain (R, M) admits a proper m-canonical ideal I of R, then the following statements are equivalent:

(1) R is a valuation domain;
(2) I is a divided m-canonical ideal of R;
(3) $cM = I$ for some nonzero $c \in R$;
(4) $(I : M)$ is a principal ideal of R;
(5) $(I : M)$ is an invertible ideal of R;
(6) R is an integrally closed domain and $(I : M)$ is a finitely generated of R;
(7) $(M : M) = R$ and $(I : M)$ is a finitely generated of R;
(8) If $J = (I : M)$, then J is a finitely generated of R and $(J : J) = R$.

Among the many results in this paper, we show that an integral domain R is a valuation domain if and only if R admits a divided proper m-canonical ideal, iff R is a root closed domain which admits a strongly primary proper m-canonical ideal, also we show that an integral domain R is a one-dimensional valuation domain if and only if R is a completely integrally closed domain which admits a powerful proper m-canonical ideal of R.

1. Intruduction

Throughout this paper, R denotes a commutative integral domain with identity $1 \neq 0$ having quotient field K and (R, M) denotes a quasi-local domain with maximal ideal M. If J and L are fractional ideals of R, then $(J : L) = \{x \in K \mid xL \subset J\}$ and $J^{-1} = (R : J)$. Recall that an ideal I of R is called *divisorial* if $(R : (R : I)) = I$. We recall from [13] that a nonzero ideal I of R is said to be *m-canonical* if $(I : (I : J)) = J$ for every nonzero ideal J of R. This type of ideals has been studied extensively in [13] and [5]. Other related studies can be found in [6], [7], [12], [14], [15], [16], [17], [18], and [19]. We say that an ideal I of R is *proper* if $I \neq \{0\}$ and $I \neq R$. In this paper, we show (Corollary 2.15) that if a quasi-local integral domain (R, M) admits a proper m-canonical ideal I of R, then the following statements are equivalent:

(1) R is a valuation domain;
(2) I is a divided m-canonical ideal of R;
(3) $cM = I$ for some nonzero $c \in R$;

(4) $(I : M)$ is a principal ideal of R;
(5) $(I : M)$ is an invertible ideal of R;
(6) R is an integrally closed domain and $(I : M)$ is a finitely generated of R;
(7) $(M : M) = R$ and $(I : M)$ is a finitely generated of R;
(8) If $J = (I : M)$, then J is a finitely generated of R and $(J : J) = R$.

Recall that a proper ideal I of an integral domain R is said to be *divided* in the sense of Dobbs [8] and Badawi [3] if $I \subset (c)$ for every $c \in R \setminus I$. We show (Corollary 2.5 and Theorem 3.3) that an integral domain R is a valuation domain if and only if R admits a divided proper m-canonical ideal of R. We recall from [4] that an ideal I of R is said to be *strongly primary* if, whenever $xy \in I$ with $x, y \in K$, we have $x \in I$ or $y^n \in I$ for some $n \geq 1$. We show (Corollary 3.4) that an integral domain R is a valuation domain if and only if R is a root closed domain which admits a strongly primary proper m-canonical ideal; also recall from [4] that an ideal I of R is called *powerful* if, whenever $xy \in I$ with $x, y \in K$, we have $x \in R$ or $y \in R$. We show (Corollary 3.5) that an integral domain R is a one-dimensional valuation domain if and only if R is a completely integrally closed domain which admits a powerful proper m-canonical ideal of R. We recall that R is called an *h-local domain* if each nonzero ideal of R is contained in only finitely many maximal ideals of R and each nonzero prime ideal of R is contained in a unique maximal ideal of R. Suppose that an integrally closed domain R admits a proper m-canonical I such that $(I : M)$ is a finitely generated ideal of R for every maximal ideal M of R containing I. Then we show (Theorem 3.6) that R is an h-local Prüfer domain with only finitely many maximal ideals of R that are not finitely generated. We show (Proposition 3.8) that if an integrally closed domain R admits a proper m-canonical ideal I, then the following statements are equivalent:

(1) R is an h-local Prüfer domain with only finitely many maximal ideals of R that are not finitely generated;
(2) For every maximal ideal M of R, we have either $I_M = R_M$ or $(I_M : M_M)$ is a finitely generated ideal of R_M;
(3) For every maximal ideal M of R, we have either $I_M = R_M$ or I_M is a divided proper ideal of R_M
(4) For every maximal ideal M of R, we have either $I_M = R_M$ or $(I_M : M_M)$ is a principal ideal of R;
(5) For every maximal ideal M of R, we have either $I_M = R_M$ or $(I_M : M_M)$ is an invertible ideal of R_M.

Remark 1.1. *Suppose that R is an m-canonical ideal of R. Then dR is an m-canonical ideal of R for every nonzero nonunit d of R by* [13, Lemma 2.2(c)]. *Hence an integral domain admits a nonzero m-canonical ideal if and only if it admits a proper m-canonical ideal.*

2. On quasi-local domains that admit m-canonical ideals

Observe that if I is an m-canonical ideal of R, then $(I : (I : R)) = R$ and hence $(I : I) = R$. In the following proposition, we show that a nonzero ideal I of R is an m-canonical ideal if and only if $(I : (I : J)) = J$ for every nonzero proper ideal J of R.

Proposition 2.1. *Let I be a nonzero ideal of R. Then the following statements are equivalent:*

(1) *I is an m-canonical ideal of R;*
(2) *$(I:(I:J)) = J$ for every nonzero proper ideal J of R.*

Proof. If $I = R$, then there is nothing to prove. Hence we may assume that I is a proper ideal of R. **(1) $\Rightarrow$ (2)**. No comments. **(2) $\Rightarrow$ (1)**. First, we show that $(I:I) = R$. Let $x = a/b \in (I:I)$ for some $a \in R$ and nonzero $b \in R$. Since I is an m-canonical ideal of R, we have $(I:(I:(b))) = (b)$. Since $(I:(b)) = \{i/b \mid i \in I\}$ and $x = a/b \in (I:I)$, we conclude that $a \in (I:(I:(b))) = (b)$. Thus $b \mid a$ (in R), and thus $(I:I) = R$. Hence $(I:(I:R)) = (I:I) = R$ and therefore I is an m-canonical ideal of R. □

We have the following important observation.

Proposition 2.2. *Suppose that R admits a nonzero proper m-canonical ideal I. Then for each maximal ideal M of R containing I, there is a $c \in R \setminus I$ such that $(I:M) = I + (c)$. In particular, if R is a quasi-local domain with maximal ideal M and $I \neq M$, then there is a $c \in M \setminus I$ such that $(I:M) = I + (c)$.*

Proof. Let M be a maximal ideal of R containing I. Since $(I:I) = R$, it is clear that $(I:M)$ is an ideal of R. Since $(I:(I:M)) = M$ and $(I:I) = R$, we conclude that there is a $c \in (I:M) \setminus I$. It is clear that $M \subset (I:I+(c))$. Once again, Since $(I:I) = R$, $(I:I+(c))$ is an ideal of R. Since M is a maximal ideal of R and $M \subset (I:I+(c)) \subset R$, the only possibilities are that either $(I:I+(c)) = M$ or $(I:I+(c)) = R$. Since I is m-canonical and $c \notin I$, the latter is ruled out since $(I:R) = I$ and $(I:(I:I+(c))) = I + (c) \neq I$. Thus $(I:I+(c)) = M$ and $(I:M) = (I:(I:I+(c))) = I + (c)$. The "in particular" statement is clear. □

It is shown in [13, Lemma 2.2(i)] that a prime m-canonical ideal of an integral domain R is a maximal ideal of R. In the following result, we show that a proper radical m-canonical ideal of a quasi-local domain (R, M) is a maximal ideal of R.

Proposition 2.3. *Suppose that (R, M) admits a proper radical m-canonical ideal I. Then I is a maximal ideal of R.*

Proof. Suppose that $I \neq M$. Then $(I:M) = I + (c)$ for some $c \in M \setminus I$ by Proposition 2.2. Hence $c^2 \in I$. Thus $c \in I$, a contradiction. Since (R, M) has exactly one maximal ideal, I is "the" maximal ideal of R. Hence $I = M$ is a maximal ideal of R. □

We give the following characterization of valuation domains in terms of m-canonical ideals.

Theorem 2.4. *Suppose that (R, M) is a quasi-local domain. Then R is a valuation domain if and only if R admits a proper m-canonical ideal I such that $(I:M)$ is a principal ideal of R.*

Proof. Suppose that (R, M) is a valuation domain. Then M is an m-canonical ideal of R by [5, Proposition 4.1] and hence $(M:M) = R$ is a principal ideal of R. Conversely, suppose that R admits a nonzero proper m-canonical ideal I such that $(I:M)$ is a principal ideal of R. Then $(I:M) = (d)$ for some $d \in R \setminus I$. It is clear that $R \subset (I:dM)$. Now let $x \in (I:dM)$. Then $xdM \subset I$. Since

$(I : M) = (d)$ and $xdM \subset I$, we conclude that $xd \subset (d)$. Thus $x \in R$. Hence $(I : dM) = R$, and thus $dM = (I : (I : dM)) = (I : R) = I$. Let J be a nonzero ideal of R. Since $(M : (M : J)) = (dM : (dM : J))$ by [13, Lemma 2.1], we have $(M : (M : J)) = (dM : (dM : J)) = (I : (I : J)) = J$. Thus M is an m-canonical ideal of R, and therefore R is a valuation domain by [5, Proposition 4.1]. □

Recall that a proper ideal I of an integral domain R is said to be *divided* in the sense of Dobbs [8] and Badawi [3] if $I \subset (c)$ for every $c \in R \setminus I$. It is clear that every proper ideal of a valuation domain is divided. We have the following result.

Corollary 2.5. *Let (R, M) be a quasi-local domain. Then R is a valuation domain if and only if R admits a divided proper m-canonical ideal.*

Proof. Suppose that R is a valuation domain. Then M is an m-canonical ideal of R by [5, Proposition 4.1] and hence it is clear that M is a divided ideal of R. Conversely, suppose that I is a divided proper m-canonical ideal of R. Then $(I : M) = I + (c)$ for some $c \in R \setminus I$ by Proposition 2.2. Since I is divided, $(I : M) = I + (c) = (c)$. Since $(I : M)$ is a principal ideal of R, we conclude that R is a valuation domain by Theorem 2.4. □

Corollary 2.6. *Suppose that a quasi-local domain (R, M) admits a proper m-canonical ideal I. Then R is a valuation domain if and only if $(I : M)$ is a principal ideal of R.*

Proof. Suppose that R is a valuation domain. Then I is divided. Hence as in the proof of Corollary 2.5 we have $(I : M)$ is a principal ideal of R. The converse is clear by Theorem 2.4. □

Corollary 2.7. *Let (R, M) be a qausi-local domain. Then R is a valuation domain if and only if cM is an m-canonical ideal of R for some nonzero $c \in R$.*

Proof. Suppose that R is a valuation domain. Then M is an m-canonical ideal of R by [5, Proposition 4.1]. Conversely, suppose that cM is an m-canonical ideal of R for some nonzero $c \in R$. Since $(M : (M : J)) = (cM : (cM : J))$ by [13, Lemma 2.1], for every nonzero ideal J of R we have $(M : (M : J)) = (cM : (cM : J)) = J$. □

Corollary 2.8. *Let (R, M) be a quasi-local domain. Then R is a valuation domain if and only if $\{cM \mid c \text{ is a nonzero element of } R\}$ is the set of all proper m-canonical ideals of R.*

Proof. Suppose that (R, M) is a valuation domain. Let I be a proper m-canonical ideal of R. Then $(I : M) = (c)$ for some nonzero element $c \in R$ by Corollary 2.6. Hence by an argument similar to the one just given in the proof of Theorem 2.4, we conclude that $I = cM$. □

The following lemma is needed.

Lemma 2.9. *Let $J(R)$ be the Jacobson radical of R and suppose that $J(R) \neq \{0\}$. Suppose that I is a proper ideal of R, $c_1, c_2, \cdots, c_m \in R \setminus I$ such that $J = I + (c_1, c_2, \cdots, c_m)$ is a finitely generated ideal of R. If L is a nonzero ideal of R which is contained in $J(R)$ and $JL = I$, then $(c_1, c_2, \cdots, c_m)L = I$.*

Proof. Let L be a nonzero ideal of R which is contained in $J(R)$ and suppose that $JL = I$. Since J is a finitely generated ideal of R, we may choose $i_1, i_2, \cdots, i_n \in I$ such that $J = (i_1, i_2, \cdots, i_n, c_1, c_2, \cdots, c_m)$. Since $JL = I$, there are $d_1, d_2, \cdots, d_{n+m} \in L$ such that $d_1 i_1 + \cdots + d_n i_n + d_{n+1} c_1 + \cdots + d_{n+m} c_m = i_1$. Hence $i(1 - d_1) \in (i_2, \cdots, i_n, c_1, \cdots, c_m)L$. Since $d_1 \in J(R)$, $1 - d_1$ is a unit of R. Thus $i_1 \in (i_2, \cdots, i_n, c_1, \cdots, c_m)L$. A similar argument will show that $i_2 \in (i_3, \cdots, i_n, c_1, \cdots, c_m)L$, $i_3 \in (i_4, \cdots, i_n, c_1, \cdots, c_m)L$, $\cdots$, $i_n \in (c_1, \cdots, c_m)L$. Thus $(c_1, \cdots, c_m)L = I$. □

We have the following result.

Corollary 2.10. *Let (R, M) be a quasi-local domain. Then R is a valuation domain if and only if $(M : M) = R$ and R admits a proper m-canonical ideal I of R such that $(I : M)$ is a finitely generated ideal of R.*

Proof. Suppose that R is a valuation domain. Then it is clear that $(M : M) = R$ and M is an m-canonical ideal of R by [5, Proposition 4.1]. Conversely, suppose that $(M : M) = R$ and R admits a proper m-canonical ideal I of R such that $(I : M)$ is a finitely generated ideal of R. Then $J = (I : M) = I + (c)$ for some $c \in R \setminus I$ by Proposition 2.2. It is clear that $R \subset (I : MJ)$. Let $x \in (I : MJ)$. Since $xMJ \subset I$, $(M : M) = R$, and $(I : J) = M$, we conclude that $xM \subset M$. Hence $x \in R$ since $(M : M) = R$. Thus $R = (I : MJ)$, and hence $MJ = (I : (I : MJ)) = (I : R) = I$. Since $J = I + (c)$ and $MJ = I$, we conclude that $cM = I$ by Lemma 2.9. Hence $(I : M) = I + (c) = cM + (c) = (c)$ is a principal ideal of R. Thus R is a valuation domain by Theorem 2.4. □

Corollary 2.11. *Let (R, M) be a quasi-local domain. Then R is a valuation domain if and only if R admits a proper m-canonical ideal I of R such that $J = (I : M)$ is a finitely generated ideal of R and $(J : J) = R$.*

Proof. Suppose that R is a valuation domain. Then M is an m-canonical ideal of R by [5, Proposition 4.1] and hence the claim is clear. Conversely, suppose that R admits a proper m-canonical ideal I of R such that $J = (I : M)$ is a finitely generated ideal of R and $(J : J) = R$. Then $J = (I : M) = I + (c)$ for some $c \in R \setminus I$ by Proposition 2.2. It is clear that $R \subset (I : JM)$. Let $x \in (I : JM)$. Since $xJM \subset I$, $(J : J) = R$, and $(I : M) = J$, we conclude that $xJ \subset J$. Hence $x \in R$ since $(J : J) = R$. Thus $R = (I : JM)$, and hence $JM = (I : (I : JM)) = (I : R) = I$. Since $J = I + (c)$ and $JM = I$, we conclude that $cM = I$ by Lemma 2.9. Hence $(I : M) = I + (c) = cM + (c) = (c)$ is a principal ideal of R. Thus R is a valuation domain by Theorem 2.4.

□

Corollary 2.12. *Let (R, M) be a quasi-local domain. Then R is a valuation domain if and only if R is an integrally closed domain which admits a proper m-canonical ideal I of R such that $J = (I : M)$ is a finitely generated ideal of R.*

Proof. For the converse, just observe that $(J : J) = R$ since R is integrally closed and J is a finitely generated ideal of R. Hence we are done by Corollary 2.11 □

The following is an example of a quasi-local domain (R, M) which admits a proper m-canonical ideal I such that $(I : M)$ is a finitely generated ideal of R but (R, M) is not a valuation domain.

Example 2.13. *Let* $V = GF(4)[[X]] = GF(4) + XGF(4)[[X]]$ *is a valuation domain, and let* $R = Z_2 + XGF(4)[[X]]$. *Then* R *is a pseudo-valuation domain(pseudo-valuation domains have been defined and studied in* [11]*) with maximal ideal* $M = XGF(4)[[X]]$ *which is not a valuation domain. Since* $GF(4)$ *is a finite algebraic extension of* Z_2, R *has a finitely generated m-canonical ideal* I *by* [5, Theorem 2.16 and Theorem 3.1]. *Since* $(I : M) = I + (c)$ *for some* $c \in R \setminus I$ *by Proposition 2.2 and* I *is a finitely generated ideal of* R, *we conclude that* $(I : M)$ *is a finitely generated of* R.

Remark 2.14. *Observe that since* (R, M) *is quasi-local, the condition* $(I : M)$ *is a principal ideal of* R *in Theorem 2.4 can be replaced by* $(I : M)$ *is an invertible ideal of* R.

Suppose that (R, M) is a valuation domain. Since every proper m-canonical ideal I of R is divided, we have $(I : M) = I + (c) = (c)$ for some $c \in R \setminus I$ by Proposition 2.2. Hence in light of the different characterizations of valuation domains above, the reader can now easily prove the following corollary.

Corollary 2.15. *Suppose that a quasi-local domain* (R, M) *admits a proper m-canonical ideal* I. *Then the following statements are equivalent:*

(1) R *is a valuation domain;*
(2) I *is a divided m-canonical ideal of* R*;*
(3) $cM = I$ *for some nonzero* $c \in R$*;*
(4) $(I : M)$ *is a principal ideal of* R*;*
(5) $(I : M)$ *is an invertible ideal of* R*;*
(6) R *is an integrally closed domain and* $(I : M)$ *is a finitely generated of* R*;*
(7) $(M : M) = R$ *and* $(I : M)$ *is a finitely generated of* R*;*
(8) *If* $J = (I : M)$, *then* J *is a finitely generated of* R *and* $(J : J) = R$.

Suppose that a quasi-local domain (R, M) admits a proper finitely generated m-canonical ideal I. Then $(I : M) = I + (c)$ by Proposition 2.2, and thus $(I : M)$ is a finitely generated ideal of R. Hence we have the following corollary.

Corollary 2.16. *Suppose that a quasi-local domain* (R, M) *admits a proper finitely generated m-canonical ideal* I. *Then the following statements are equivalent:*

(1) R *is a valuation domain;*
(2) I *is a divided m-canonical ideal of* R*;*
(3) $cM = I$ *for some nonzero* $c \in R$*;*
(4) $(I : M)$ *is a principal ideal of* R*;*
(5) $(I : M)$ *is an invertible ideal of* R*;*
(6) R *is an integrally closed domain;*
(7) $(M : M) = R$*;*
(8) *If* $J = (I : M)$, *then* $(J : J) = R$.

3. On integral domains that admit specific m-canonical ideals

In this section, we investigate the behavior of integral domains that admit specific m-canonical ideals. Recall from [11] that a prime ideal P of R is said to be a strongly prime ideal if, whenever $xy \in P$ with $x, y \in K$, we have $x \in P$ or $y \in P$, also recall from [4] that an ideal I of R is called *powerful* if, whenever

$xy \in I$ with $x, y \in K$, we have $x \in R$ or $y \in R$. We recall that R is called an *h-local domain* if each nonzero ideal of R is contained in only finitely many maximal ideals of R and each nonzero prime ideal of R is contained in a unique maximal ideal of R. If I is a proper ideal of R, then $Rad(I)$ denotes the radical ideal of I in R. We start this section with the following "useful" lemma.

Lemma 3.1. *Let M be a maximal ideal of R and let P be a prime ideal contained in M such that $PR_M = P$. If R_M is a valuation domain, then P is a divided prime ideal of R.*

Proof. Let $r \in R \setminus P$. If R_M is a valuation domain, then $rR_M \supset P_M = P$ with $r \notin P_M$. Moreover, $rP = rP_M = P_M = P$. Thus P is a divided prime ideal of R. □

We have the following result.

Theorem 3.2. *Let I be a proper divided ideal of an integral domain R. Then*

(1) *If R is a Prüfer domain, then $Rad(I)$ is is a divided prime of R.*
(2) *If R is an h-local Prüfer domain, then R is a valuation domain.*

Proof. **(1)**. Suppose that R is a Prüfer domain and let M be a maximal ideal of R. Since I is divided, $I \subset M$. Moreover, $I_M \subset M$ is an ideal of R. Since R_M is a valuation domain, $Rad(I_M)$ (in R_M) is a prime ideal of the form $PR_M = P_M$ for some prime P of R minimal over I. Let $s \in P_M$. Then some power of s is in I_M. As I_M is an ideal of R and R is a a Prüfer domain, s must be an element of R. Thus $P_M \subset R$ and therefore we have $P = P_M$. By Lemma 3.1, P is a divided prime of R. Thus each maximal ideal of R contains P. This implies that P is the unique minimal prime of I and therefore $Rad(I) = P$ is a divided prime of R.
(2). Suppose that R is an h-local Prüfer domain. Since R is a Prüfer domain, by (1), we conclude that $Rad(I)$ is a divided prime ideal of R, and hence $Rad(I)$ is contained in every maximal ideal of R. Since R is an h-local domain, we conclude that $Rad(I)$ is contained in a unique maximal ideal implying that R is quasi-local and therefore a valuation domain. □

We state our main result of this section.

Theorem 3.3. (compare with Corollary 2.5) *Suppose that R admits a divided proper m-canonical ideal I. Then R is a valuation domain.*

Proof. First observe that R must be an h-local domain by [13, Proposition 2.4]. Let M be a maximal ideal of R. Since I is divided, $I \subset M$. Hence I_M is a proper m-canonical ideal of R_M by [13, Proposition 5.5]. It is clear that I_M is a divided ideal of R_M since I is divided. Thus R_M is a valuation domain by Corollary 2.5. Since R_M is a valuation domain for every maximal ideal M of R, we conclude that R is a Prüfer domain and therefore an h-local Prüfer domain. Since I is divided and R is an h-local Prüfer domain, by Theorem 3.2(2), we conclude that R is a valuation domain. □

Recall from [4] that an ideal I of R is said to be *strongly primary* if, whenever $xy \in I$ with $x, y \in K$, we have $x \in I$ or $y^n \in I$ for some $n \geq 1$.

Corollary 3.4. *For an integral domain R, the following statements are equivalent:*

(1) *R is a valuation domain;*
(2) *R is a root closed domain which admits a strongly primary proper m-canonical ideal.*

Proof. **(1) ⇒ (2)**. Suppose that (R, M) is a valuation domain. Then M is a strongly primary (prime) ideal of R and M is a proper m-canonical ideal of R by [5, Proposition 4.1]. Clearly, R is root closed. **(2) ⇒ (1)**. Let I be a strongly primary proper m-canonical ideal of R and suppose that R is root closed. Let $d \in R \setminus I$. Hence $d(i/d) \in I$ for every $i \in I$. Since $d \notin I$, for every $i \in I$ we have $(i/d)^n \in I$ for some $n \geq 1$. Hence $i/d \in I$ for every $i \in I$ since R is root closed. Thus I is a divided ideal of R. Hence R is a valuation domain by Theorem 3.3. □

Corollary 3.5. *For an integral domain R, the following statements are equivalent:*
(1) *R is a one-dimensional valuation domain;*
(2) *R is a completely integrally closed domain which admits a powerful proper m-canonical ideal of R.*

Proof. **(1) ⇒ (2)**. Suppose that (R, M) is a one-dimensional valuation domain. Then it is clear that R is completely integrally closed and M is a powerful ideal of R. Once again, M is an m-canonical ideal of R by [5, Proposition 4.1]. **(2) ⇒ (1)**. Suppose that R is a completely integrally closed domain and admits a powerful proper m-canonical ideal I of R. Let $d \in R \setminus I$ and let $i \in I$. Then $(i/d)^n(d/i)^n i \in I$ for every $n \geq 1$. Since I is powerful, we have either $(i/d)^n \in R$ or $(d/i)^n i \in R$. Suppose that $(d/i)^n i \in R$ for every $n \geq 1$. Then since R is a completely integrally closed domain, we have $d \in (i) \subset I$, which is a contradiction since $d \notin I$. Hence $(i/d)^n \in R$ for some $n \geq 1$, and thus $i \in (d)$ since R is root closed. Hence I is a divided ideal of R, and thus R is a valuation domain by Theorem 3.3. Since R is a completely integrally closed valuation domain which is not a field, R is one-dimensional. □

Theorem 3.6. *Suppose that an integrally closed domain R admits a proper m-canonical I such that $(I : M)$ is a finitely generated ideal of R for every maximal ideal M of R containing I. Then R is an h-local Prüfer domain with only finitely many maximal ideals of R that are not finitely generated.*

Proof. Let M be a maximal ideal of R. Then I_M is an m-canonical ideal of R_M by [13, Proposition 5.5]. Suppose that $I \not\subset M$. Then $I_M = R_M$, and hence R_M is a valuation domain by [12, Theorem 5.1]. Suppose that $I \subset M$. Then $(I_M : M_M)$ is a finitely generated ideal of R_M since $(I : M)$ is finitely generated ideal of R. Since R_M is an integrally closed domain and $(I_M : M_M)$ is a finitely generated ideal of R_M, we conclude that R_M is a valuation domain by Corollary 2.12. Hence R is a Prüfer domain, and thus R is an h-local Prüfer domain with only finitely many maximal ideals of R that are not finitely generated by [13, Theorem 6.7]. □

Heinzer, Huckaba, and Papick in [13] asked the following question which is, to my knowledge, still open : If (R, M) is an integrally closed domain that has an m-canonical ideal, does it follow that R is a valuation domain?

In case (R, M) admits a finitely generated m-canonical ideal, Barucci, Houston, Lucas, and Papick in [5, Theorem 2.1] gave a positive answer to the question above. In view of Theorem 3.6, we now give an alternative proof of [5, Theorem 2.1].

Corollary 3.7. ([5, Theorem 2.1]) *Suppose that an integrally closed domain R admits a finitely generated m-canonical ideal. Then R is an h-local Prüfer domain such that each nonzero ideal of R is divisorial. In particular, if R is quasi-local, then R is a valuation domain.*

Proof. We may assume that R admits a proper finitely generated m-canonical ideal I. Let M be a maximal ideal of R containing I. Then $(I : M) = I + (c)$ for some $c \in R \setminus I$ by Proposition 2.2, and thus $(I : M)$ is a finitely generated ideal of R since I is finitely generated. Hence R is an h-local Prüfer domain by Theorem 3.6. Since I is a finitely generated ideal of R, we conclude that I is an invertible ideal of R, and thus every nonzero ideal of R is divisorial by [13, Proposition 3.6]. The "in particular" statement is clear by Corollary 2.12 □

In light of Corollary 2.15, [13, Proposition 5.5], [12, Theorem 5.1], and [13, Theorem 6.7], one can easily prove the following proposition.

Proposition 3.8. *Suppose that an integrally closed domain R admits a proper m-canonical I. Then the following statements are equivalent:*

1. *R is an h-local Prüfer domain with only finitely many maximal ideals of R that are not finitely generated;*
2. *For every maximal ideal M of R, we have either $I_M = R_M$ or $(I_M : M_M)$ is a finitely generated ideal of R_M;*
3. *For every maximal ideal M of R, we have either $I_M = R_M$ or I_M is a divided proper ideal of R_M*
4. *For every maximal ideal M of R, we have either $I_M = R_M$ or $(I_M : M_M)$ is a principal ideal of R;*
5. *For every maximal ideal M of R, we have either $I_M = R_M$ or $(I_M : M_M)$ is an invertible ideal of R_M.*

We end up this paper with the following two related results. It is shown in [11] that if P is a nonmaximal strongly prime ideal, then R_P is a valuation domain, also it is shown in [2, Corollary 5] that if P is a nonmaximal strongly prime ideal, then $(P : P) = R_P$ is a valuation domain. We now give an alternative proof of [2, Corollary 5].

Proposition 3.9. ([2, Corollary 5]) *Let P be a nonzero nonmaximal strongly prime ideal of R. Then $(P : P) = R_P$ is a valuation domain.*

Proof. It is well-known by [11] that R_P is a valuation domain with maximal ideal P. Hence P is an m-canonical ideal of R_P by [5, Proposition 4.1]. Hence $(P : P) = R_P$ by [13, Lemma 2.2]. □

Proposition 3.10. *Let P be a nonzero prime ideal of R. Then the following statements are equivalent:*

1. *P is a strongly prime ideal;*
2. *P is an m-canonical prime ideal of some quasi-local overring of R.*

Proof. **(1) ⇒(2)**. Suppose that P is a strongly prime ideal of R. If P is a maximal ideal of R, then $(P : P)$ is a valuation domain with maximal ideal P by [1], and thus P is an m-canonical prime ideal of $(P : P)$ by [5, Proposition 4.1]. Suppose that P is a nonmaximal strongly prime ideal of R. Then R_P is a valuation domain with maximal ideal P by [11], and hence, once again, P is an m-canonical prime ideal of R_P by [5, Proposition 4.1]. **(2)** $\Rightarrow$ **(1)**. Suppose that P is an m-canonical prime ideal of a quasi-local overring B of R. Then P is a maximal ideal of B by [13, Lemma 2.2(i)]. Thus B is a valuation domain by [5, Proposition 4.1], and hence P is a strongly prime ideal of R. □

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