Problem: Prove :

$$\forall a, b \in \mathbb{Z}^+, \quad 49|(a^2+b^2) \Rightarrow 7|a \wedge 7|b$$

Proof: I will prove an equivalent statement which is the contrapositive:

$$\forall a, b \in \mathbb{Z}^+, \quad 7 \not| a \vee 7 \not| b \Rightarrow 49 \not| (a^2+b^2)$$

I will discuss the cases: $7 \not| a \vee 7 \not| b \equiv [\underbrace{(7|a \wedge 7 \not| b) \oplus (7 \not| a \wedge 7|b)}_{\text{case 1,2}} \oplus \underbrace{(7 \not| a \wedge 7 \not| b)}_{\text{case 3}}]$ [1]

Case 1,2: Suppose without loss of generality that $7|a \wedge 7 \not| b$, then $49 \not| (a^2+b^2)$ holds always. Because if not, then we must have $7|b$ and this contradicts with $7 \not| b$.

Case 3: This the core of the proof. Suppose $7 \not| a \wedge 7 \not| b$, then we can write

$$a = 7n_1 + r_1, b = 7n_2 + r_2 \quad : \quad 1 \leqslant r_1, r_2 \leqslant 6$$

Then:

$$a^2 + b^2 = \underbrace{49(n_1^2 + n_2^2) + 14(n_1 r_1 + n_2 r_2)}_{\text{divisable by 7}} + (r_1^2 + r_2^2)$$

Therefore, $7|a^2+b^2$ iff $\exists r_1, r_2$ such that $7|r_1^2 + r_2^2$ and $1 \leqslant r_1, r_2 \leqslant 6$.
Since r_1, r_2 have finite set of possible values, then all this values can be tried. By trying them all (tedious to list), we find that no such r_1, r_2 exists. Therefore, $7 \not| (a^2+b^2)$ which implies $49 \not| (a^2+b^2)$. QED

[1] $\oplus$ denotes exclusive OR