Rn Contains a Division Ring if and only if $R$ Does

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$R_n$ Contains a Division Ring iff $R$ Does

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INTRODUCTION. Let $R$ be a ring with 1, and let $R_n$ denote the complete matrix ring of all $n \times n$ matrices over $R$ under the usual matrix addition and multiplication. Recall $A, B \in R_n$ are similar if and only if there exists $P \in R_n$ such that $A = PBP^{-1}$. If $A \in R_n$ is similar over $R$ to a diagonal matrix, then $A$ is called [1] diagonalizable over $R$. For $B \in R_n$, $b_{ij}$ denotes the entry of $B$ in the $i$th row and $j$th column.

In this note, we give an alternative proof of [1, Theorem 1] which is quite shorter than that in [1]. We would like to point out that our proof begins exactly like the original.

Theorem ([1, Theorem 1]). Let $R$ be a ring with 1 for which each idempotent matrix in $R_n$ is diagonalizable over $R$. Then $R$ contains a division ring if and only if $R_n$ contains a division ring.

Proof: If $R$ contains a division ring, then clearly $R_n$ contains a division ring. Assume $R_n$ contains a division ring $K$. The division ring $K$ has an identity—call it $J$—and by the hypothesis $PJP^{-1} = I$ a diagonal matrix for some invertible matrix $P \in R_n$. Since the conjugation of $R_n$ by $P$ induces a ring automorphism of $R_n$, $M = PKP^{-1}$ is a division ring of $R_n$ and has $I$ as the identity. Hence $I$ is a nonzero idempotent of $R_n$. Let $S = \{A \in M: A$ is diagonal$\}$. Since $I \in S$, $S$ is not empty. We leave it to the reader to verify that $S$ is a division subring of $M$. Since $I \neq 0$, there exists $1 \leq j \leq n$ such that $i_{jj}$ is a nonzero idempotent of $R$. Let $D = \{a_{ii}; A \in S\}$. Then $D$ is a division ring of $R$ with $i_{jj}$ as the identity.

We end this note with some examples that satisfy the hypothesis of the Theorem and with one example where the hypothesis fails. Let $R$ be a commutative ring with 1. Then $R$ is called $ID$ (basal) as in [7] ([2]) iff for every $n \geq 1$ the idempotents of $R_n$ are diagonalizable. Foster [2] has shown that if $R$ is a principal ideal domain, then $R$ is $ID$. Seshadri [6] has shown that if $R$ is a principal ideal domain, then $R[x]$ is $ID$. In particular if $F$ is a field, then $F[x, y]$ is $ID$. Steger [7] has shown that if $R$ is an elementary division ring (i.e., for every $n \geq 1$ and $A \in R_n$ there exist invertible matrices $P, Q$ in $R_n$ such that $PAQ$ is diagonal) then $R$ is $ID$. Also; Steger has shown that if $R$ is $\pi$-regular ring (i.e., for every $x$ in $R$ there exists $n \geq 1$ and $y$ in $R$ ($n$ and $y$ depending on $x$) such that $x^n y x^n = x^n$) then $R$ is $ID$. In particular for every $m \geq 1 Z_m$ (i.e., $Z/mZ$) is $ID$ (Foster has shown independently that $Z_m$ is $ID$).

Finally, Theorem 3 in [7] states that if $R$ is $ID$, then every invertible ideal of $R$ is principal. Thus if $R$ is a Dedekind domain which is not principal, then $R$ is not $ID$. In particular, let $R = Z[\sqrt{-5}]$ ($Z$ is the set of all integers). Then $R$ is a Dedekind domain, see [4, EX. 37, P. 70]. But $R$ is not a unique factorization domain, for example 21 does not have unique factorization in $R$. Thus $R$ is not principal and therefore it is not $ID$. 

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Dedicated to Prof. Nick Vaughan on his retirement.

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A Further Simplification of Dixon’s Proof of Cauchy’s Integral Theorem

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The modification in [1] of Dixon’s proof of the Cauchy Integral Theorem and Formula is based on the proposition stated below. In this note we give a proof of that proposition which is more suitable for undergraduate students. In what follows, $G$ will be an open set in the complex plane $C$, and $\gamma$ will be a closed rectifiable curve. We write $f \in H(G)$ if $f$ is holomorphic, i.e. analytic, in $G$, and we use the notation $D(z, r)$ for the disk $\{w \in C : |w - z| < r\}$. The trace of $\gamma$ in $C$ is denoted by $\{\gamma\}$; we say the curve $\gamma$ is in $G$ when $\{\gamma\} \subset G$.

Proposition. If $\gamma$ is a curve in $G$, then for any $z \in \{\gamma\}$ there is a closed curve $\sigma$ in $G$ with $z \notin \{\sigma\}$ such that $\int_{\gamma} f = \int_{\sigma} f$ for all $f \in H(G)$.

Proof: We assume that there is a point $\zeta \neq z$ with $\zeta \in \{\gamma\}$; otherwise the result is trivial. Pick $r > 0$ so that $D(z, r) \subset G$ and $\zeta \notin D(z, r)$. We will assume that $\gamma$ is given by $\gamma(t)$ for $t \in [0, 1]$ and $\gamma(0) = \gamma(1) = \zeta$. By the uniform continuity of the mapping $\gamma$, there is a natural number $n$ such that if $s, t \in [0, 1]$ and $|t - s| < 1/n$, then $|\gamma(t) - \gamma(s)| < r$. Partition the interval $[0, 1]$ using the points $0 < 1/n < \cdots < (n - 1)/n < 1$. Let $0 = x_0 < x_1 < x_2 < \cdots < x_m = 1$ be the set of partition points $k/n$ such that $\gamma(k/n) \neq z$. If between adjacent points $x_i$ and $x_{i+1}$ there is a point of the form $k/n$ or any other point $t_0$ with $\gamma(t_0) = z$, then the path $\gamma(t)$, $x_i \leq t \leq x_{i+1}$, is in the disk $D(z, r)$. In this case, we may replace the