On rings with divided nil ideal: a survey

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Abstract. Let $R$ be a commutative ring with $1 \neq 0$ and $\text{Nil}(R)$ be its set of nilpotent elements. Recall that a prime ideal of $R$ is called a divided prime if $P \subset (x)$ for every $x \in R \setminus P$; thus a divided prime ideal is comparable to every ideal of $R$. In many articles, the author investigated the class of rings $\mathcal{H} = \{ R \mid R$ is a commutative ring and $\text{Nil}(R)$ is a divided prime ideal of $R\}$ (Observe that if $R$ is an integral domain, then $R \in \mathcal{H}$.) If $R \in \mathcal{H}$, then $R$ is called a $\phi$-ring. Recently, David Anderson and the author generalized the concept of Prüfer domains, Bezout domains, Dedekind domains, and Krull domains to the context of rings that are in the class $\mathcal{H}$. Also, Lucas and the author generalized the concept of Mori domains to the context of rings that are in the class $\mathcal{H}$. In this paper, we state many of the main results on $\phi$-rings.

Keywords. Prüfer ring, $\phi$-Prüfer ring, Dedekind ring, $\phi$-Dedekind ring, Krull ring, $\phi$-Krull ring, Mori ring, $\phi$-Mori ring, divided ring.

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1 Introduction

Let $R$ be a commutative ring with $1 \neq 0$ and $\text{Nil}(R)$ be its set of nilpotent elements. Recall from [26] and [7] that a prime ideal of $R$ is called a divided prime if $P \subset (x)$ for every $x \in R \setminus P$; thus a divided prime ideal is comparable to every ideal of $R$. In [6], [8], [9], [10], and [11], the author investigated the class of rings $\mathcal{H} = \{ R \mid R$ is a commutative ring and $\text{Nil}(R)$ is a divided prime ideal of $R\}$ (Observe that if $R$ is an integral domain, then $R \in \mathcal{H}$.) If $R \in \mathcal{H}$, then $R$ is called a $\phi$-ring. Recently, David Anderson and the author, [3] and [4], generalized the concept of Prüfer, Bezout domains, Dedekind domains, and Krull domains to the context of rings that are in the class $\mathcal{H}$. Also, Lucas and the author, [17], generalized the concept of Mori domain to the context of rings that are in the class $\mathcal{H}$. Yet, another paper by Dobbs and the author [14] investigated going-down $\phi$-rings. In this paper, we state many of the main results on $\phi$-rings.

We assume throughout that all rings are commutative with $1 \neq 0$. Let $R$ be a ring. Then $T(R)$ denotes the total quotient ring of $R$, and $Z(R)$ denotes the set of zerodivisors of $R$. We start by recalling some background material. A non-zerodivisor of a ring $R$ is called a regular element and an ideal of $R$ is said to be regular if it contains a regular element. An ideal $I$ of a ring $R$ is said to be a nonnil ideal if $I \not\subseteq \text{Nil}(R)$. If $I$ is a nonnil ideal of a ring $R \in \mathcal{H}$, then $\text{Nil}(R) \subset I$. In particular, this holds if $I$ is a regular ideal of a ring $R \in \mathcal{H}$.

Recall from [6] that for a ring $R \in \mathcal{H}$ with total quotient ring $T(R)$, the map $\phi : T(R) \longrightarrow R_{\text{Nil}(R)}$ such that $\phi(a/b) = a/b$ for $a \in R$ and $b \in R \setminus Z(R)$ is
a ring homomorphism from $T(R)$ into $R_{\text{Nil}(R)}$, and $\phi$ restricted to $R$ is also a ring homomorphism from $R$ into $R_{\text{Nil}(R)}$ given by $\phi(x) = x/1$ for every $x \in R$. Observe that if $R \in \mathcal{H}$, then $\phi(R) \in \mathcal{H}$, $\text{Ker}(\phi) \subseteq \text{Nil}(R)$, $\text{Nil}(T(R)) = \text{Nil}(R)$, $\text{Nil}(R_{\text{Nil}(R)}) = \phi(\text{Nil}(R)) = \text{Nil}(\phi(R)) = Z(\phi(R))$, $T(\phi(R)) = R_{\text{Nil}(R)}$ is quasilocal with maximal ideal $\text{Nil}(\phi(R))$, and $R_{\text{Nil}(R)}/\text{Nil}(\phi(R)) = T(\phi(R))/\text{Nil}(\phi(R))$ is the quotient field of $\phi(R)/\text{Nil}(\phi(R))$.

Recall that an ideal $I$ of a ring $R$ is called a divisorial ideal of $R$ if $(I^{-1})^{-1} = I$, where $I^{-1} = \{x \in T(R) \mid xI \subseteq R\}$. If a ring $R$ satisfies the ascending chain condition (a.c.c.) on divisorial regular ideals of $R$, then $R$ is called a Mori ring in the sense of [46]. An integral domain $R$ is called a Dedekind domain if every nonzero ideal of $R$ is invertible, i.e., if $I$ is a nonzero ideal of $R$, then $II^{-1} = R$. If every finitely generated nonzero ideal $I$ of an integral domain $R$ is invertible, then $R$ is said to be a Prüfer domain. If every finitely generated regular ideal of a ring $R$ is invertible, then $R$ is said to be a Prüfer ring. If $R$ is an integral domain and $x^{-1} \in R$ for each $x \in T(R) \setminus R$, then $R$ is called a valuation domain. Also, recall from [29] that an integral domain $R$ is called a Krull domain if $R = \cap V_i$, where each $V_i$ is a discrete valuation overring of $R$, and every nonzero element of $R$ is a unit in all but finitely many $V_i$. Many characterizations and properties of Dedekind and Krull domains are given in [29], [30], and [40]. Recall from [32] that an integral domain $R$ with quotient field $K$ is called a pseudo-valuation domain (PVD) in case each prime ideal of $R$ is strongly prime in the sense that $xy \in P$, $x \in K$, $y \in K$ implies that either $x \in P$ or $y \in P$. Every valuation domain is a pseudo-valuation domain. In [13], Anderson, Dobbs and the author generalized the concept of pseudo-valuation rings to the context of arbitrary rings. Recall from [13] that a prime ideal $P$ of $R$ is said to be strongly prime if either $aP \subseteq bR$ or $bR \subseteq aP$ for all $a, b \in R$. A ring $R$ is said to be a pseudo-valuation ring (PVR) if every prime ideal of $R$ is a strongly prime ideal of $R$.

Throughout the paper, we will use the technique of idealization of a module to construct examples. Recall that for an $R$-module $B$, the idealization of $B$ over $R$ is the ring formed from $R \times B$ by defining addition and multiplication as $(r,a) + (s,b) = (r + s, a + b)$ and $(r,a)(s,b) = (rs, rb + sa)$, respectively. A standard notation for the “idealized ring” is $R(+)B$. See [38] for basic properties of these rings.

### 2. $\phi$-pseudo-valuation rings and $\phi$-chained rings

In [6], the author generalized the concept of pseudo-valuation domains to the context of rings that are in $\mathcal{H}$. Recall from [6] that a ring $R \in \mathcal{H}$ is said to be a $\phi$-pseudovaluation ring ($\phi$-PVR) if every nonnil prime ideal of $R$ is a $\phi$-strongly prime ideal of $\phi(R)$, in the sense that $xy \in \phi(P)$, $x \in R_{\text{Nil}(R)}$, $y \in R_{\text{Nil}(R)}$ (observe that $R_{\text{Nil}(R)} = T(\phi(R))$ implies that either $x \in \phi(P)$ or $y \in \phi(P)$. We state some of the main results on $\phi$-pseudo-valuation rings.

**Theorem 2.1** ([8, Proposition 2.1]). Let $D$ be a PVD and suppose that $P, Q$ are prime ideal of $D$ such that $P$ is properly contained in $Q$. Let $d \geq 1$ and choose $x \in D$ such that $\text{Rad}(xD) = P$. Then $J = x^{d+1}D_Q$ is an ideal of $D$ and hence $D/J$ is a PVR
with the following properties:

(i) \( \text{Nil}(R) = P / J \) and \( x^d \notin J \);

(ii) \( Z(R) = Q / J \).

**Theorem 2.2** ([8, Corollary 2.7]). Let \( d \geq 2, D, P, Q, x, J, \) and \( R \) be as in Theorem 2.1. Set \( B = R_{\text{Nil}(R)} \). Then the idealization ring \( R(+)B \) is a \( \phi \)-PVR that is not a PVR.

**Theorem 2.3** ([10, Proposition 2.9], also see [23, Theorem 3.1]). Let \( R \in \mathcal{H} \). Then \( R \) is a \( \phi \)-PVR if and only if \( R / \text{Nil}(R) \) is a PVD.

Recall from [9] that a ring \( R \in \mathcal{H} \) is said to be a \( \phi \)-chained ring (\( \phi \)-CR) if for each \( x \in R_{\text{Nil}(R)} \setminus \phi(R) \) we have \( x^{-1} \in \phi(R) \). A ring \( A \) is said to be a chained ring if for every \( a, b \in A \), either \( a \mid b \) (in \( A \)) or \( b \mid a \) (in \( A \)).

**Theorem 2.4** ([9, Corollary 2.7]). Let \( d \geq 2, D \) be a valuation domain, \( P, Q, x, J, R \) be as in Theorem 2.1. Then \( R = D / J \) is a chained ring. Furthermore, if \( B = R_{\text{Nil}(R)} \), then the idealization ring \( R(+)B \) is a \( \phi \)-CR that is not a chained ring.

**Theorem 2.5** ([9, Proposition 3.3]). Let \( R \in \mathcal{H} \) be a quasi-local ring with maximal ideal \( M \) such that \( M \) contains a regular element of \( R \). Then \( R \) is a \( \phi \)-PVR if and only if \( (M : M) = \{ x \in T(R) \mid xM \subseteq M \} \) is a \( \phi \)-CR with maximal ideal \( M \).

**Theorem 2.6** ([3, Theorem 2.7]). Let \( R \in \mathcal{H} \). Then \( R \) is a \( \phi \)-CR if and only if \( R / \text{Nil}(R) \) is a valuation domain.

Recall that \( B \) is said to be an overring of a ring \( A \) if \( B \) is a ring between \( A \) and \( T(A) \).

**Theorem 2.7** ([10, Corollary 3.17]). Let \( R \in \mathcal{H} \) be a \( \phi \)-PVR with maximal ideal \( M \). The following statements are equivalent:

(i) Every overring of \( R \) is a \( \phi \)-PVR;

(ii) \( R[u] \) is a \( \phi \)-PVR for each \( u \in (M : M) \setminus R \);

(iii) \( R[u] \) is quasi-local for each \( u \in (M : M) \setminus R \);

(iv) If \( B \) is an overring of \( R \) and \( B \subset (M : M) \), then \( B \) is a \( \phi \)-PVR with maximal ideal \( M \);

(v) If \( B \) is an overring of \( R \) and \( B \subset (M : M) \), then \( B \) is quasi-local;

(vi) Every overring of \( R \) is quasi-local;

(vii) Every \( \phi \)-CR between \( R \) and \( T(R) \) other than \( (M : M) \) is of the form \( R_P \) for some non-maximal prime ideal \( P \) of \( R \);

(viii) \( R' = (M : M) \) (where \( R' \) is the integral closure of \( R \) inside \( T(R) \)).
3 Nonnil Noetherian rings ($\phi$-Noetherian rings)

Recall that an ideal $I$ of a ring $R$ is said to be a nonnil ideal if $I \not\subseteq \text{Nil}(R)$. Let $R \in \mathcal{H}$. Recall from [11] that $R$ is said to be a a nonnil-Noetherian ring or just a $\phi$-Noetherian ring as in [16] if each nonnil ideal of $R$ is finitely generated. We have the following results.

**Theorem 3.1** ([11, Corollary 2.3]). Let $R \in \mathcal{H}$. If every nonnil prime ideal of $R$ is finitely generated, then $R$ is a $\phi$-Noetherian ring.

**Theorem 3.2** ([11, Theorem 2.4]). Let $R \in \mathcal{H}$. The following statements are equivalent:

(i) $R$ is a $\phi$-Noetherian ring;

(ii) $R/\text{Nil}(R)$ is a Noetherian domain;

(iii) $\phi(R)/\text{Nil}(\phi(R))$ is a Noetherian domain;

(iv) $\phi(R)$ is a $\phi$-Noetherian ring.

**Theorem 3.3** ([11, Theorem 2.6]). Let $R \in \mathcal{H}$. Suppose that each nonnil prime ideal of $R$ has a power that is finitely generated. Then $R$ is a $\phi$-Noetherian ring.

**Theorem 3.4** ([11, Theorem 2.7]). Let $R \in \mathcal{H}$. Suppose that $R$ is a $\phi$-Noetherian ring. Then any localization of $R$ is a $\phi$-Noetherian ring, and any localization of $\phi(R)$ is a $\phi$-Noetherian ring.

**Theorem 3.5** ([11, Theorem 2.9]). Let $R \in \mathcal{H}$. Suppose that $R$ satisfies the ascending chain condition on the nonnil finitely generated ideals. Then $R$ is a $\phi$-Noetherian ring.

**Theorem 3.6** ([11, Theorem 3.4]). Let $R$ be a Noetherian domain with quotient field $K$ such that $\text{dim}(R) = 1$ and $R$ has infinitely many maximal ideals. Then $D = R(+)^{K} \in \mathcal{H}$ is a $\phi$-Noetherian ring with Krull dimension one which is not a Noetherian ring. In particular, $\mathbb{Z}(+)\mathbb{Q}$ is a $\phi$-Noetherian ring with Krull dimension one which is not a Noetherian ring (where $\mathbb{Z}$ is the set of all integer numbers with quotient field $\mathbb{Q}$).

**Theorem 3.7** ([11, Theorem 3.5]). Let $R$ be a Noetherian domain with quotient field $K$ and Krull dimension $n \geq 2$. Then $D = R(+)^{K} \in \mathcal{H}$ is a $\phi$-Noetherian ring with Krull dimension $n$ which is not a Noetherian ring. In particular, if $K$ is the quotient field of $R = \mathbb{Z}[x_{1}, \ldots, x_{n-1}]$, then $R(+)^{K}$ is a $\phi$-Noetherian ring with Krull dimension $n$ which is not a Noetherian ring.

In the following result, we show that a $\phi$-Noetherian ring is related to a pullback of a Noetherian domain.
Theorem 3.8 ([16, Theorem 2.2]). Let $R \in \mathcal{H}$. Then $R$ is a $\phi$-Noetherian ring if and only if $\phi(R)$ is ring-isomorphic to a ring $A$ obtained from the following pullback diagram:

$$
\begin{array}{c}
A \quad \longrightarrow \\
\downarrow \quad \;
\end{array}
\begin{array}{c}
S = A/M \\
\downarrow \quad \;
\end{array}
\begin{array}{c}
T \quad \longrightarrow \\
T/M
\end{array}
$$

where $T$ is a zero-dimensional quasilocal ring containing $A$ with maximal ideal $M$, $S = A/M$ is a Noetherian subring of $T/M$, the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

Theorem 3.9 ([16, Proposition 2.4]). Let $R \in \mathcal{H}$ be a $\phi$-Noetherian ring and let $I \neq R$ be an ideal of $R$. If $I \subset \text{Nil}(R)$, then $R/I$ is a $\phi$-Noetherian ring. If $I \not\subset \text{Nil}(R)$, then $\text{Nil}(R) \subset I$ and $R/I$ is a Noetherian ring. Moreover, if $\text{Nil}(R) \subset I$, then $R/I$ is both Noetherian and $\phi$-Noetherian if and only if $I$ is either a prime ideal or a primary ideal whose radical is a maximal ideal.

Theorem 3.10 ([16, Corollary 2.5]). Let $R \in \mathcal{H}$ be a $\phi$-Noetherian ring. Then a homomorphic image of $R$ is either a $\phi$-Noetherian ring or a Noetherian ring.

Our next result shows that a $\phi$-Noetherian ring satisfies the conclusion of the Principal Ideal Theorem (and the Generalized Principal Ideal Theorem).

Theorem 3.11 ([16, Theorem 2.7]). Let $R \in \mathcal{H}$ be a $\phi$-Noetherian ring and let $P$ be a prime ideal. If $P$ is minimal over an ideal generated by $n$ or fewer elements, then the height of $P$ is less than or equal to $n$. In particular, each prime minimal over a nonnil element of $R$ has height one.

Other statements about primes of Noetherian rings that can be easily adapted to statements about primes of $\phi$-Noetherian rings include the following.

Theorem 3.12 ([16, Proposition 2.8] and [40, Theorem 145]). Let $R \in \mathcal{H}$ satisfy the ascending chain condition on radical ideals. If $R$ has an infinite number of prime ideals of height one, then their intersection is $\text{Nil}(R)$.

Theorem 3.13 ([16, Proposition 2.9]). Let $R \in \mathcal{H}$ be a $\phi$-Noetherian ring and $P$ be a nonnil prime ideal of $R$ of height $n$. Then there exist nonnil elements $a_1, \ldots, a_n$ in $R$ such that $P$ is minimal over the ideal $(a_1, \ldots, a_n)$ of $R$, and for any $i$ $(1 \leq i \leq n)$, every (nonnil) prime ideal of $R$ minimal over $(a_1, \ldots, a_i)$ has height $i$.

Theorem 3.14 ([16, Proposition 2.10]). Let $R \in \mathcal{H}$ be a $\phi$-Noetherian ring and let $I$ be an ideal of $R$ generated by $n$ elements with $I \neq R$. If $P$ is a prime ideal containing $I$ with $P/I$ of height $k$, then the height of $P$ is less than or equal to $n + k$.

Theorem 3.15 ([16, Proposition 3.1]). Let $R \in \mathcal{H}$ be a $\phi$-Noetherian ring and let $P$ be a height $n$ prime of $R$. If $Q$ is a prime of $R[x]$ that contracts to $P$ but properly contains $PR[x]$, then $PR[x]$ has height $n$ and $Q$ has height $n + 1$. 

Similar height restrictions exist for the primes of $R[x_1, \ldots, x_m]$.

**Theorem 3.16** ([16, Proposition 3.2]). Let $R \in \mathcal{H}$ be a $\phi$-Noetherian ring and let $P$ be a height $n$ prime of $R$. If $Q$ is a prime of $R[x_1, \ldots, x_m]$ that contracts to $P$ but properly contains $PR[x_1, \ldots, x_m]$, then $PR[x_1, \ldots, x_m]$ has height $n$ and $Q$ has height at most $n + m$. Moreover the prime $PR[x_1, \ldots, x_m] + (x_1, \ldots, x_m)R[x_1, \ldots, x_m]$ has height $n + m$.

**Theorem 3.17** ([16, Corollary 3.3]). If $R$ is a finite dimensional $\phi$-Noetherian ring of dimension $n$, then $\dim(R[x_1, \ldots, x_m]) = n + m$ for each integer $m > 0$.

In our next result, we show that each ideal of $R[x]$ that contracts to a nonnil ideal of $R$ is finitely generated.

**Theorem 3.18** ([16, Proposition 3.4]). Let $R \in \mathcal{H}$ be a $\phi$-Noetherian ring. If $I$ is an ideal of $R[x_1, \ldots, x_n]$ for which $I \cap R$ is not contained in $\text{Nil}(R)$, then $I$ is a finitely generated ideal of $R[x_1, \ldots, x_n]$.

Since three distinct comparable primes of $R[x]$ cannot contract to the same prime of $R$, a consequence of Theorem 3.18 is that the search for primes of $R[x]$ that are not finitely generated can be restricted to those of height one. A similar statement can be made for primes of $R[x_1, \ldots, x_n]$.

**Theorem 3.19** ([16, Corollary 3.5]). Let $R \in \mathcal{H}$ be a $\phi$-Noetherian ring and let $P$ be a prime of $R[x_1, \ldots, x_n]$. If $P$ has height greater than $n$, then $P$ is finitely generated.

The ring in our next example shows that the converse of Theorem 3.18 does not hold even for prime ideals.

**Example 3.20** ([16, Example 3.6]). Let $R = D(+)L$ be the idealization of $L = K((y))/D$ over $D = K[[y]]$. Then $R$ is a quasilocal $\phi$-Noetherian ring with nilradical $\text{Nil}(R)$ isomorphic to $L$. Consider the polynomial $g(x) = 1 - xy$. Since the coefficients of $g$ generate $D$ as an ideal and $g$ is irreducible, $P = gD[x]$ is a height-one principal prime of $D[x]$ with $P \cap D = (0)$. Each nonzero element of $L$ can be written in the form $d/y^n$ where $n$ is a positive integer, $y$ denotes the image of $y$ in $L$ and $d = d_0 + d_1y + \cdots + d_{n-1}y^{n-1}$ with $d_0 \neq 0$. Given such an element, let $f(x) = 1 + yx + \cdots + y^{n-1}x^{n-1} \in L[x]$. Then $g(x)(df(x)/y^n) = d/y^n$ since $dy^n/y^n = 0$ in $L$. It follows that $g(x)R[x]$ is a height-one principal prime of $R[x]$ that contracts to $\text{Nil}(R)$.

4  $\phi$-Prüfer rings and $\phi$-Bezout rings

We say that a nonnil ideal $I$ of $R$ is $\phi$-invertible if $\phi(I)$ is an invertible ideal of $\phi(R)$. Recall from [3] that $R$ is called a $\phi$-Prüfer ring if every finitely generated nonnil ideal of $R$ is $\phi$-invertible.
Theorem 4.1 ([3, Corollary 2.10]). Let $R \in \mathcal{H}$. Then the following statements are equivalent:

(i) $R$ is a $\phi$-Prüfer ring;
(ii) $\phi(R)$ is a Prüfer ring;
(iii) $\phi(R)/\text{Nil}(\phi(R))$ is a Prüfer domain;
(iv) $R_P$ is a $\phi$-CR for each prime ideal $P$ of $R$;
(v) $R_P/\text{Nil}(R_P)$ is a valuation domain for each prime ideal $P$ of $R$;
(vi) $R_M/\text{Nil}(R_M)$ is a valuation domain for each maximal ideal $M$ of $R$;
(vii) $R_M$ is a $\phi$-CR for each maximal ideal $M$ of $R$.

Theorem 4.2 ([3, Theorem 2.11]). Let $R \in \mathcal{H}$ be a $\phi$-Prüfer ring and let $S$ be a $\phi$-chained overring of $R$. Then $S = R_P$ for some prime ideal $P$ of $R$ containing $Z(R)$.

The following is an example of a ring $R \in \mathcal{H}$ such that $R$ is a Prüfer ring, but $R$ is not a $\phi$-Prüfer ring.

Example 4.3 ([3, Example 2.15]). Let $n \geq 1$ and let $D$ be a non-integrally closed domain with quotient field $K$ and Krull dimension $n$. Set $R = D(+)(K/D)$. Then $R \in \mathcal{H}$ and $R$ is a Prüfer ring with Krull dimension $n$ which is not a $\phi$-Prüfer ring.

Theorem 4.4 ([3, Theorem 2.17]). Let $R \in \mathcal{H}$. Then $R$ is a $\phi$-Prüfer ring if and only if every overring of $\phi(R)$ is integrally closed.

Example 4.5 ([3, Example 2.18]). Let $n \geq 1$ and let $D$ be a Prüfer domain with quotient field $K$ and Krull dimension $n$. Set $R = D(+K)$. Then $R \in \mathcal{H}$ is a (non-domain) $\phi$-Prüfer ring with Krull dimension $n$.

Recall from [21] that a ring $R$ is said to be a pre-Prüfer ring if $R/I$ is a Prüfer ring for every nonzero proper ideal $I$ of $R$.

Theorem 4.6 ([3, Theorem 2.19]). Let $R \in \mathcal{H}$ such that $\text{Nil}(R) \neq \{0\}$. Then $R$ is a pre-Prüfer ring if and only if $R$ is a $\phi$-Prüfer ring.

The following example shows that the hypothesis $\text{Nil}(R) \neq \{0\}$ in Theorem 4.6 is crucial.

Example 4.7 ([3, Example 2.20] and [42, Example 2.9]). Let $D$ be a Prüfer domain with quotient field $F$. For indeterminates $X,Y$, let $K = F(Y)$ and let $V$ be the valuation domain $K + XK[[X]]$. Then $V$ is one-dimensional with maximal ideal $M = XK[[X]]$. Set $R = D + M$. Then $\text{Nil}(R) = \{0\}$, and $R$ is a pre-Prüfer ring (domain) which is not a Prüfer ring (domain). Hence $R$ is not a $\phi$-Prüfer ring.
Recall from [3] that a ring $R \in \mathcal{H}$ is said to be a $\phi$-Bezout ring if $\phi(I)$ is a principal ideal of $\phi(R)$ for every finitely generated nonnil ideal $I$ of $R$. A $\phi$-Bezout ring is a $\phi$-Prüfer ring, but of course the converse is not true. A ring $R$ is said to be a Bezout ring if every finitely generated regular ideal of $R$ is principal.

**Theorem 4.8** ([3, Corollary 3.5]). Let $R \in \mathcal{H}$. Then the following statements are equivalent:

1. $R$ is a $\phi$-Bezout ring;
2. $R/\text{Nil}(R)$ is a Bezout domain;
3. $\phi(R)/\text{Nil}(\phi(R))$ is a Bezout domain;
4. $\phi(R)$ is a Bezout ring;
5. Every finitely generated nonnil ideal of $R$ is principal.

**Theorem 4.9** ([3, Theorem 3.9]). Let $R \in \mathcal{H}$ be quasi-local. Then $R$ is a $\phi$-CR if and only if $R$ is a $\phi$-Bezout ring.

**Example 4.10** ([3, Example 3.8]). Let $n \geq 1$ and let $D$ be a Bezout domain with quotient field $K$ and Krull dimension $n$. Set $R = D(+)K$. Then $R \in \mathcal{H}$ is a (non-domain) $\phi$-Bezout ring with Krull dimension $n$.

### 5 $\phi$-Dedekind rings

Let $R \in \mathcal{H}$. We say that a nonnil ideal $I$ of $R$ is $\phi$-invertible if $\phi(I)$ is an invertible ideal of $\phi(R)$. If every nonnil ideal of $R$ is $\phi$-invertible, then we say that $R$ is a $\phi$-Dedekind ring.

**Theorem 5.1** ([4, Theorem 2.6]). Let $R \in \mathcal{H}$. Then $R$ is a $\phi$-Dedekind ring if and only if $\phi(R)$ is ring-isomorphic to a ring $A$ obtained from the following pullback diagram:

$$
\begin{array}{ccc}
A & \rightarrow & A/M \\
\downarrow & & \downarrow \\
T & \rightarrow & T/M
\end{array}
$$

where $T$ is a zero-dimensional quasilocal ring with maximal ideal $M$, $A/M$ is a Dedekind subring of $T/M$, the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

**Example 5.2** ([4, Example 2.7]). Let $D$ be a Dedekind domain with quotient field $K$, and let $L$ be an extension ring of $K$. Set $R = D(+L)$. Then $R \in \mathcal{H}$ and $R$ is a $\phi$-Dedekind ring which is not a Dedekind domain.

We say that a ring $R \in \mathcal{H}$ is $\phi$-(completely) integrally closed if $\phi(R)$ is (completely) integrally closed in $T(\phi(R)) = R_{\text{Nil}(R)}$. The following characterization of $\phi$-Dedekind rings resembles that of Dedekind domains as in [40, Theorem 96].
Theorem 5.3 ([4, Theorem 2.10]). Let $R \in \mathcal{H}$. Then the following statements are equivalent:

(i) $R$ is $\phi$-Dedekind;
(ii) $R$ is nonnil-Noetherian ($\phi$-Noetherian), $\phi$-integrally closed, and of dimension $\leq 1$;
(iii) $R$ is nonnil-Noetherian and $R_M$ is a discrete $\phi$-chained ring for each maximal ideal $M$ of $R$.

A ring $R$ is said to be a Dedekind ring if every nonzero ideal of $R$ is invertible.

Theorem 5.4 ([4, Theorem 2.12]). Let $R \in \mathcal{H}$ be a $\phi$-Dedekind ring. Then $R$ is a Dedekind ring.

The following is an example of a ring $R \in \mathcal{H}$ which is a Dedekind ring but not a $\phi$-Dedekind ring.

Example 5.5 ([4, Example 2.13]). Let $D$ be a non-Dedekind domain with (proper) quotient field $K$. Set $R = D(+)/K$. Then $R \in \mathcal{H}$ and $R = T(R)$. Hence $R$ is a Dedekind ring. Since $R/\operatorname{Nil}(R)$ is ring-isomorphic to $D$, $R$ is not a $\phi$-Dedekind ring by [4, Theorem 2.5].

It is well known that an integral domain $R$ is a Dedekind domain iff every nonzero proper ideal of $R$ is (uniquely) a product of prime ideals of $R$. We have the following result.

Theorem 5.6 ([4, Theorem 2.15]). Let $R \in \mathcal{H}$. Then $R$ is a $\phi$-Dedekind ring if and only if every nonnil proper ideal of $R$ is (uniquely) a product of nonnil prime ideals of $R$.

Theorem 5.7 ([4, Theorem 2.16]). Let $R \in \mathcal{H}$. Then the following statements are equivalent:

(i) $R$ is a $\phi$-Dedekind ring;
(ii) Each nonnil proper principal ideal $aR$ can be written in the form $aR = Q_1 \cdots Q_n$, where each $Q_i$ is a power of a nonnil prime ideal of $R$ and the $Q_i$’s are pairwise comaximal;
(iii) Each nonnil proper ideal $I$ of $R$ can be written in the form $I = Q_1 \cdots Q_n$, where each $Q_i$ is a power of a nonnil prime ideal of $R$ and the $Q_i$’s are pairwise comaximal.

Theorem 5.8 ([4, Theorem 2.20]). Let $R \in \mathcal{H}$. Then the following statements are equivalent:

(i) $R$ is a $\phi$-Dedekind ring;
(ii) Each nonnil prime ideal of $R$ is $\phi$-invertible;
(iii) $R$ is a nonnil-Noetherian ring and each nonnil maximal ideal of $R$ is $\phi$-invertible.
Theorem 5.9 ([4, Theorem 2.23]). Let \( R \in \mathcal{H} \) be a \( \phi \)-Dedekind ring. Then every overring of \( R \) is a \( \phi \)-Dedekind ring.

6 Factoring nonnil ideals into prime and invertible ideals

In this section, we give a generalization of the concept of factorization of ideals of an integral domain into a finite product of invertible and prime ideals which was extensively studied by Olberding [48] to the context of rings that are in the class \( \mathcal{H} \). Observe that if \( R \) is an integral domain, then \( R \in \mathcal{H} \). An ideal \( I \) of a ring \( R \) is said to be a nonnil ideal if \( I \not\subseteq \text{Nil}(R) \). Let \( R \in \mathcal{H} \). Then \( R \) is said to be a \( \phi \)-ZPUI ring if each nonnil ideal \( I \) of \( \phi(R) \) can be written as \( I = JP_1 \cdots P_n \), where \( J \) is an invertible ideal of \( \phi(R) \) and \( P_1, \ldots, P_n \) are prime ideals of \( \phi(R) \). If every nonnil ideal \( I \) of \( R \) can be written as \( I = JP_1 \cdots P_n \), where \( J \) is an invertible ideal of \( R \) and \( P_1, \ldots, P_n \) are prime ideals of \( R \), then \( R \) is said to be a nonnil-ZPUI ring. Commutative \( \phi \)-ZPUI rings that are in \( \mathcal{H} \) are characterized in [12, Theorem 2.9]. Examples of \( \phi \)-ZPUI rings that are not ZPUI rings are constructed in [12, Theorem 2.13]. It is shown in [12, Theorem 2.14] that a \( \phi \)-ZPUI ring is the pullback of a ZPUI domain. It is shown in [12, Theorem 3.1] that a nonnil-ZPUI ring is a \( \phi \)-ZPUI ring. Examples of \( \phi \)-ZPUI rings that are not nonnil-ZPUI rings are constructed in [12, Theorem 3.2]. We call a ring \( R \in \mathcal{H} \) a nonnil-strongly discrete ring if \( R \) has no nonnil prime ideal \( P \) such that \( P^2 = P \). A ring \( R \in \mathcal{H} \) is said to be nonnil-h-local if each nonnil ideal of \( R \) is contained in at most finitely many maximal ideals of \( R \) and each nonnil prime ideal \( P \) of \( R \) is contained in a unique maximal ideal of \( R \).

Since the class of integral domains is a subset of \( \mathcal{H} \), the following result is a generalization of [48, Theorem 2.3].

Theorem 6.1 ([12, Theorem 2.9]). Let \( R \in \mathcal{H} \). Then the following statements are equivalent:

(i) \( R \) is a \( \phi \)-ZPUI ring;

(ii) Every nonnil proper ideal of \( R \) can be written as a product of prime ideals of \( R \) and a finitely generated ideal of \( R \);

(iii) Every nonnil proper ideal of \( \phi(R) \) can be written as a product of prime ideals of \( \phi(R) \) and a finitely generated ideal of \( \phi(R) \);

(iv) \( R \) is a nonnil-strongly discrete nonnil-h-local \( \phi \)-Prüfer ring.

In the following result, we show that a nonnil-ZPUI ring is a \( \phi \)-ZPUI ring.

Theorem 6.2 ([12, Theorem 3.1]). Let \( R \in \mathcal{H} \) be a nonnil-ZPUI ring. Then \( R \) is a \( \phi \)-ZPUI ring, and hence all the following statements hold:

(i) \( R/\text{Nil}(R) \) is a ZPUI domain.

(ii) Every nonnil proper ideal of \( R \) can be written as a product of prime ideals of \( R \) and a finitely generated ideal of \( R \).
(iii) Every nonnil proper ideal of $\phi(R)$ can be written as a product of prime ideals of $\phi(R)$ and a finitely generated ideal of $\phi(R)$.

(iv) $R$ is a nonnil-strongly discrete nonnil-h-local $\phi$-Prüfer ring.

(v) $R$ is a nonnil-strongly discrete nonnil-h-local Prüfer ring.

Examples of $\phi$-ZPUI rings that are not nonnil-ZPUI rings are constructed in the following result.

**Theorem 6.3** ([12, Theorem 3.2]). Let $A$ be a ZPUI domain that is not a Dedekind domain with Krull dimension $n \geq 1$ and quotient field $K$. Then $R = A(+)K/A \in \mathcal{H}$ is a $\phi$-ZPUI ring with Krull dimension $n$ which is not a nonnil-ZPUI ring.

Olberding in [48, Corollary 2.4] showed that for each $n \geq 1$, there exists a ZPUI domain with Krull dimension $n$. A Dedekind domain is a trivial example of a ZPUI domain. We have the following result.

**Theorem 6.4** ([12, Theorem 2.13]). Let $A$ be a ZPUI domain (i.e. $A$ is a strongly discrete h-local Prüfer domain by [48, Theorem 2.3]) with Krull dimension $n \geq 1$ and quotient field $F$, and let $K$ be an extension ring of $F$ (i.e. $K$ is a ring and $F \subseteq K$). Then $R = A(+)K \in \mathcal{H}$ is a $\phi$-ZPUI ring with Krull dimension $n$ that is not a ZPUI ring.

In the following result, we show that a $\phi$-ZPUI ring is the pullback of a ZPUI domain. A good paper for pullbacks is the article by Fontana [27].

**Theorem 6.5** ([12, Theorem 2.14]). Let $R \in \mathcal{H}$. Then $R$ is a $\phi$-ZPUI ring if and only if $\phi(R)$ is ring-isomorphic to a ring $A$ obtained from the following pullback diagram:

\[
\begin{array}{ccc}
A & \longrightarrow & A/M \\
\downarrow & & \downarrow \\
T & \longrightarrow & T/M
\end{array}
\]

where $T$ is a zero-dimensional quasilocal ring with maximal ideal $M$, $A/M$ is a ZPUI ring that is a subring of $T/M$, the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

### 7 $\phi$-Krull rings

We say that a ring $R \in \mathcal{H}$ is a discrete $\phi$-chained ring if $R$ is a $\phi$-chained ring with at most one nonnil prime ideal and every nonnil ideal of $R$ is principal. Recall from [4] that a ring $R \in \mathcal{H}$ is said to be a $\phi$-Krull ring if $\phi(R) = \cap V_i$, where each $V_i$ is a discrete $\phi$-chained overring of $\phi(R)$, and for every nonnilpotent element $x \in R$, $\phi(x)$ is a unit in all but finitely many $V_i$.

**Theorem 7.1** ([4, Theorem 3.1]). Let $R \in \mathcal{H}$. Then $R$ is a $\phi$-Krull ring if and only if $R/\text{Nil}(R)$ is a Krull domain.
We have the following pullback characterization of $\phi$-Krull rings.

**Theorem 7.2** ([4, Theorem 3.2]). Let $R \in \mathcal{H}$. Then $R$ is a $\phi$-Krull ring if and only if $\phi(R)$ is ring-isomorphic to a ring $A$ obtained from the following pullback diagram:

$$
\begin{array}{ccc}
A & \longrightarrow & A/M \\
\downarrow & & \downarrow \\
T & \longrightarrow & T/M
\end{array}
$$

where $T$ is a zero-dimensional quasilocal ring with maximal ideal $M$, $A/M$ is a Krull subring of $T/M$, the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

**Example 7.3** ([4, Example 3.3]). Let $D$ be a Krull domain with quotient field $K$, and let $L$ be a ring extension of $K$. Set $R = D(+)L$. Then $R \in \mathcal{H}$ and $R$ is a $\phi$-Krull ring which is not a Krull domain.

It is well known [29, Theorem 3.6] that an integral domain $R$ is a Krull domain if and only if $R$ is a completely integrally closed Mori domain. We have a similar characterization for $\phi$-Krull rings.

**Theorem 7.4** ([4, Theorem 3.4]). Let $R \in \mathcal{H}$. Then $R$ is a $\phi$-Krull ring if and only if $R$ is a $\phi$-completely integrally closed $\phi$-Mori ring.

**Theorem 7.5** ([4, Theorem 3.5]). Let $R \in \mathcal{H}$ be a $\phi$-Krull ring which is not zero-dimensional. Then the following statements are equivalent:

(i) $R$ is a $\phi$-Prüfer ring;

(ii) $R$ is a $\phi$-Dedekind ring;

(iii) $R$ is one-dimensional.

It is well known that if $R$ is a Noetherian domain, then $R'$ is a Krull domain. In particular, an integrally closed Noetherian domain is a Krull domain. We have the following analogous result for nonnil-Noetherian rings.

**Theorem 7.6** ([4, Theorem 3.6]). Let $R \in \mathcal{H}$ be a nonnil-Noetherian ring. Then $\phi(R)'$ is a $\phi$-Krull ring. In particular, if $R$ is a $\phi$-integrally closed nonnil-Noetherian ring, then $R$ is a $\phi$-Krull ring.

It is known [40, Problem 8, page 83] that if $R$ is a Krull domain in which all prime ideals of height $\geq 2$ are finitely generated, then $R$ is a Noetherian domain. We have the following analogous result for nonnil-Noetherian rings.

**Theorem 7.7** ([4, Theorem 3.7]). Let $R \in \mathcal{H}$ be a $\phi$-Krull ring in which all prime ideals of $R$ with height $\geq 2$ are finitely generated. Then $R$ is a nonnil-Noetherian ring.
For a ring $R \in \mathcal{H}$, let $\phi_R$ denotes the ring-homomorphism $\phi : T(R) \rightarrow R_{\text{Nil}(R)}$. It is well known [29, Proposition 1.9, page 8] that an integral domain $R$ is a Krull domain if and only if $R$ satisfies the following three conditions:

(i) $R_P$ is a discrete valuation domain for every height-one prime ideal $P$ of $R$;
(ii) $R = \bigcap R_P$, the intersection being taken over all height-one prime ideals $P$ of $R$;
(iii) Each nonzero element of $R$ is in only a finite number of height-one prime ideals of $R$, i.e., each nonzero element of $R$ is a unit in all but finitely many $R_P$, where $P$ is a height-one prime ideal of $R$.

The following result is an analog of [29, Proposition 1.9, page 8].

**Theorem 7.8** ([4, Theorem 3.9]). Let $R \in \mathcal{H}$ with $\dim(R) \geq 1$. Then $R$ is a $\phi$-Krull ring if and only if $R$ satisfies the following three conditions:

(i) $R_P$ is a discrete $\phi$-chained ring for every height-one prime ideal $P$ of $R$;
(ii) $\phi_R(R) = \bigcap \phi_{R_P}(R_P)$, the intersection being taken over all height-one prime ideals $P$ of $R$;
(iii) Each nonnilpotent element of $R$ lies in only a finite number of height-one prime ideals of $R$, i.e., each nonnilpotent element of $R$ is a unit in all but finitely many $R_P$, where $P$ is a height-one prime ideal of $R$.

Recall that a ring $R$ is called a Marot ring if each regular ideal of $R$ is generated by its set of regular elements. A Marot ring is called a Krull ring in the sense of [38, page 37] if either $R = T(R)$ or if there exists a family $\{V_i\}$ of discrete rank-one valuation rings such that:

(i) $R$ is the intersection of the valuation rings $\{V_i\}$;
(ii) Each regular element of $T(R)$ is a unit in all but finitely many $V_i$.

The following is an example of a ring $R \in \mathcal{H}$ which is a Krull ring but not a $\phi$-Krull ring.

**Example 7.9** ([4, Example 3.12]). Let $D$ be a non-Krull domain with (proper) quotient field $K$. Set $R = D(+)K/D$. Then $R \in \mathcal{H}$ and $R = T(R)$. Hence $R$ is a Krull ring. Since $R/\text{Nil}(R)$ is ring-isomorphic to $D$, $R$ is not a $\phi$-Krull ring by Theorem 7.1.

**8 $\phi$-Mori rings**

According to [46], a ring $R$ is called a Mori ring if it satisfies a.c.c. on divisorial regular ideals. Let $R \in \mathcal{H}$. A nonnil ideal $I$ of $R$ is $\phi$-divisorial if $\phi(I)$ is a divisorial ideal of $\phi(R)$, and $R$ is a $\phi$-Mori ring if it satisfies a.c.c. on $\phi$-divisorial ideals.

The following is a characterization of $\phi$-Mori rings in terms of Mori rings in the sense of [46].
Theorem 8.1 ([17, Theorem 2.2]). Let $R \in \mathcal{H}$. Then $R$ is a $\phi$-Mori ring if and only if $\phi(R)$ is a Mori ring.

The following is a characterization of $\phi$-Mori rings in terms of Mori domains.

Theorem 8.2 ([17, Theorem 2.5]). Let $R \in \mathcal{H}$. Then $R$ is a $\phi$-Mori ring if and only if $R/\text{Nil}(R)$ is a Mori domain.

Theorem 8.3 ([17, Theorem 2.7]). Let $R \in \mathcal{H}$ be a $\phi$-Mori ring. Then $R$ satisfies a.c.c. on nonnil divisorial ideals of $R$. In particular, $R$ is a Mori ring.

The converse of Theorem 8.3 is not valid as it can be seen by the following example.

Example 8.4 ([17, Example 2.8]). Let $D$ be an integral domain with quotient field $L$ which is not a Mori domain and set $R = D(+)(L/D)$, the idealization of $L/D$ over $D$. Then $R \in \mathcal{H}$ is a Mori ring which is not a $\phi$-Mori ring.

Example 8.18 shows how to construct a nontrivial Mori ring (i.e., where $R \not\cong T(R)$) in $\mathcal{H}$ which is not $\phi$-Mori.

Theorem 8.5 ([17, Theorem 2.10]). Let $R \in \mathcal{H}$ be a $\phi$-Noetherian ring. Then $R$ is both a $\phi$-Mori ring and a Mori ring.

Given a Krull domain of the form $E = L + M$, where $L$ is a field and $M$ a maximal ideal of $E$, any subfield $K$ of $L$ gives rise to a Mori domain $D = K + M$. If $L$ is not a finite algebraic extension of $K$, then $D$ cannot be Noetherian (see [19, Section 4]). We make use of this in our next example to build a $\phi$-Mori ring which is neither an integral domain nor a $\phi$-Noetherian.

Example 8.6 ([17, Example 2.11]). Let $K$ be the quotient field of the ring $D = \mathbb{Q} + X\mathbb{R}[[X]]$ and set $R = D(+)$, the idealization of $K$ over $D$. It is easy to see that $\text{Nil}(R) = \{0\}(+)K$ is a divided prime ideal of $R$. Hence $R \in \mathcal{H}$. Now since $R/\text{Nil}(R)$ is ring-isomorphic to $D$ and $D$ is a Mori domain but not a Noetherian domain, we conclude that $R$ is a $\phi$-Mori ring which is not a $\phi$-Noetherian ring.

In light of Example 8.6, $\phi$-Mori rings can be constructed as in the following example.

Example 8.7 ([17, Example 2.12]). Let $D$ be a Mori domain with quotient field $K$ and let $L$ be an extension ring of $K$. Then $R = D(+)$, the idealization of $L$ over $D$, is in $\mathcal{H}$. Moreover, $R$ is a $\phi$-Mori ring since $R/\text{Nil}(R)$ is ring-isomorphic to $D$ which is a Mori domain.

The following result is a generalization of [54, Theorem 1]. An analogous result holds for Mori rings when the chains under consideration are restricted to regular divisorial ideals whose intersection is regular [46, Theorem 2.22].
Theorem 8.8 ([17, Theorem 2.13]). Let \( R \in \mathcal{H} \). Then \( R \) is a \( \phi \)-Mori ring if and only if whenever \( \{I_m\} \) is a descending chain of nonnil \( \phi \)-divisorial ideals of \( R \) such that \( \cap I_m \neq \Nil(R) \), then \( \{I_m\} \) is a finite set.

Let \( D \) be an integral domain with quotient field \( K \). If \( I \) is an ideal of \( D \), then \( (D : I) = \{x \in K \mid xI \subseteq D\} \). Mori domains can be characterized by the property that for each nonzero ideal \( I \), there is a finitely generated ideal \( J \subseteq I \) such that \( (D : I) = (D : J) \) (equivalently, \( I_v = J_v \)) ([51, Theorem 1]). Our next result generalizes this result to \( \phi \)-Mori rings.

Theorem 8.9 ([17, Theorem 2.14]). Let \( R \in \mathcal{H} \). Then \( R \) is a \( \phi \)-Mori ring if and only if for any nonnil ideal \( I \) of \( R \), there exists a nonnil finitely generated ideal \( J \), \( J \subseteq I \), such that \( \phi^{-1}(J) = \phi^{-1}(I) \), equivalently, \( \phi(J)_v = \phi(I)_v \).

In the following theorem we combine all of the different characterizations of \( \phi \)-Mori rings stated in this section.

Theorem 8.10 ([17, Corollary 2.15]). Let \( R \in \mathcal{H} \). The following statements are equivalent:

(i) \( R \) is a \( \phi \)-Mori ring;

(ii) \( R/\Nil(R) \) is a Mori domain;

(iii) \( \phi(R)/\Nil(\phi(R)) \) is a Mori domain;

(iv) \( \phi(R) \) is a Mori ring.

(v) If \( \{I_m\} \) is a descending chain of nonnil \( \phi \)-divisorial ideals of \( R \) such that \( \cap I_m \neq \Nil(R) \), then \( \{I_m\} \) is a finite set;

(vi) For each nonnil ideal \( I \) of \( R \), there exists a nonnil finitely generated ideal \( J \), \( J \subseteq I \), such that \( \phi^{-1}(J) = \phi^{-1}(I) \);

(vii) For each nonnil ideal \( I \) of \( R \), there exists a nonnil finitely generated ideal \( J \), \( J \subseteq I \), such that \( \phi(J)_v = \phi(I)_v \).

The following result is a generalization of [54, Theorem 5].

Theorem 8.11 ([17, Theorem 3.1]). Let \( R \in \mathcal{H} \) be a \( \phi \)-Mori ring and \( I \) be a nonzero \( \phi \)-divisorial ideal of \( R \). Then \( I \) contains a power of its radical.

We recall a few definitions regarding special types of ideals in integral domains. For a nonzero ideal \( I \) of an integral domain \( D \), \( I \) is said to be strong if \( II^{-1} = I \), strongly divisorial if it is both strong and divisorial, and \( v \)-invertible if \( (II^{-1})_v = D \). We will extend these concepts to the rings in \( \mathcal{H} \).

Let \( I \) be a nonnil ideal of a ring \( R \in \mathcal{H} \). We say that \( I \) is strong if \( II^{-1} = I \), \( \phi \)-strong if \( \phi(I)\phi(I)^{-1} = \phi(I) \), strongly divisorial if it is both strong and divisorial, strongly \( \phi \)-divisorial if it is both \( \phi \)-strong and \( \phi \)-divisorial, \( v \)-invertible if \( (II^{-1})_v = R \).
and \(\phi\)-\(v\)-invertible if \((\phi(I)\phi(I)^{-1})_v = \phi(R)\). Obviously, \(I\) is \(\phi\)-strong, strongly \(\phi\)-divisorial or \(\phi\)-\(v\)-invertible if and only if \(\phi(I)\) is, respectively, strong, strongly divisorial or \(v\)-invertible.

In [51, Proposition 1], J. Querré proved that if \(P\) is a prime ideal of a Mori domain \(D\), then \(P\) is divisorial when it is height one. In the same proposition, he incorrectly asserted that if the height of \(P\) is larger than one and \(P^{-1}\) properly contains \(D\), then \(P\) is strongly divisorial. While it is true that such a prime must be strong, a (Noetherian) counterexample to the full statement can be found in [34]. What one can say is that \(P_v\) will be strongly divisorial (see [5]).

**Theorem 8.12** ([17, Theorem 3.3]). Let \(R \in \mathcal{H}\) be a \(\phi\)-Mori ring and \(P\) be a (nonnil) prime ideal of \(R\). If \(\text{ht}(P) = 1\), then \(P\) is \(\phi\)-divisorial. If \(\text{ht}(P) \geq 2\), then either \(\phi(P)^{-1} = \phi(R)\) or \(\phi(P)_v\) is strongly divisorial.

For a \(\phi\)-Mori ring \(R \in \mathcal{H}\), let \(D_m(R)\) denote the maximal \(\phi\)-divisorial ideals of \(R\); i.e., the set of nonnil ideals of \(R\) maximal with respect to being \(\phi\)-divisorial. The following result generalizes [25, Theorem 2.3] and [19, Proposition 2.1].

**Theorem 8.13** ([17, Theorem 3.4]). Let \(R \in \mathcal{H}\) be a \(\phi\)-Mori ring such that \(\text{Nil}(R)\) is not the maximal ideal of \(R\). Then the following hold:

- (a) The set \(D_m(R)\) is nonempty. Moreover, \(M \in D_m(R)\) if and only if \(M/\text{Nil}(R)\) is a maximal divisorial ideal of \(R/\text{Nil}(R)\).
- (b) Every ideal of \(D_m(R)\) is prime.
- (c) Every nonnilpotent nonunit element of \(R\) is contained in a finite number of maximal \(\phi\)-divisorial ideals.

As with a nonempty subset of \(R\), a nonempty set of ideals \(S\) is multiplicative if (i) the zero ideal is not contained in \(S\), and (ii) for each \(I\) and \(J\) in \(S\), the product \(IJ\) is in \(S\). Such a set \(S\) is referred to as a multiplicative system of ideals and it gives rise to a generalized ring of quotients \(R_S = \{t \in T(R) \mid tI \subset R\text{ for some }I \in S\}\). For each prime ideal \(P\), \(R_{\{P\}} = \{t \in T(R) \mid st \in R\text{ for some }s \in R\setminus P\} = R_S\), where \(S\) is the set of ideals (including \(R\)) that are not contained in \(P\). Note that in general a localization of a Mori ring need not be Mori (see Example 8.18 below). On the other hand, if \(S\) is a multiplicative system of regular ideals, then \(R_S\) is a Mori ring whenever \(R\) is Mori ring ([46, Theorem 2.13]).

**Theorem 8.14** ([17, Theorem 3.5], and [17, Theorem 2.2]). Let \(R\) be a \(\phi\)-Mori ring. Then

- (a) \(R_S\) is a \(\phi\)-Mori ring for each multiplicative set \(S\).
- (b) \(R_P\) is a \(\phi\)-Mori ring for each prime \(P\).
- (c) \(R_S\) is a \(\phi\)-Mori ring for each multiplicative system of ideals \(S\).
- (d) \(R_{\{P\}}\) is a \(\phi\)-Mori ring for each prime ideal \(P\).
One of the well-known characterizations of Mori domains is that an integral domain $D$ is a Mori domain if and only if (i) $D_M$ is a Mori domain for each maximal divisorial ideal $M$, (ii) $D = \cap D_M$ where the $M$ range over the set of maximal divisorial ideals of $D$, and (iii) each nonzero element is contained in at most finitely many maximal divisorial ideals ([52, Théorème 2.1] and [54, Théorème I.2]). A similar statement holds for $\phi$-Mori rings. Note that in condition (ii), if $D$ has no maximal divisorial ideals, the intersection is assumed to be the quotient field of $D$. For the equivalence, that means that $D$ is its own quotient field. The analogous statement is that if $D_m$ is empty, then we have $R = T(R) = R_{\text{Nil}(R)}$ with $\text{Nil}(R)$ the maximal ideal.

**Theorem 8.15** ([17, Theorem 3.6]). Let $R \in \mathcal{H}$. Then the following statements are equivalent:

(i) $R$ is a $\phi$-Mori ring;

(ii) (a) $R_M$ is a $\phi$-Mori ring for each maximal $\phi$-divisorial $M$, (b) $\phi(R) = \cap \phi(R)_{\phi(M)}$ where the $M$ range over the set of maximal $\phi$-divisorial ideals, and (c) each non-nil element (ideal) is contained in at most finitely many maximal $\phi$-divisorial ideals;

(iii) (a) $R(M)$ is a $\phi$-Mori ring for each maximal $\phi$-divisorial $M$, (b) $\phi(R) = \cap \phi(R)_{\phi(M)}$ where the $M$ range over the set of maximal $\phi$-divisorial ideals, and (c) each non-nil element (ideal) is contained in at most finitely many maximal $\phi$-divisorial ideals.

In [19], V. Barucci and S. Gabelli proved that if $P$ is a maximal divisorial ideal of a Mori domain $D$, then the following three conditions are equivalent: (1) $D_P$ is a discrete rank-one valuation domain, (2) $P$ is $\nu$-invertible, and (3) $P$ is not strong [19, Theorem 2.5]. A similar result holds for $\phi$-Mori rings.

**Theorem 8.16** ([17, Theorem 3.9]). Let $R \in \mathcal{H}$ be a $\phi$-Mori ring and $P \in D_m(R)$. Then the following statements are equivalent:

(i) $R_P$ is a discrete rank-one $\phi$-chained ring;

(ii) $P$ is $\phi$-$\nu$-invertible;

(iii) $P$ is not $\phi$-strong.

Recall from [38] that if $f(x) \in R[x]$, then $c(f)$ denotes the ideal of $R$ generated by the coefficients of $f(x)$, and $R(x)$ denotes the quotient ring $R[x]_S$ of the polynomial ring $R[x]$, where $S$ is the set of $f \in R[x]$ such that $c(f) = R$.

**Theorem 8.17** ([17, Theorem 4.5]). Let $R$ be an integrally closed ring for which $\text{Nil}(R) = Z(R) \neq \{0\}$. Then the following statements are equivalent:

(1) $R$ is $\phi$-Mori and the nilradical of $T(R[x])$ is an ideal of $R(x)$;

(2) $R(x)$ is $\phi$-Mori;
(3) \(R(x)\) is \(\phi\)-Noetherian;
(4) \(R\) is \(\phi\)-Noetherian and the nilradical of \(T(R[x])\) is an ideal of \(R(x)\);
(5) Each regular ideal of \(R\) is invertible;
(6) \(R/\text{Nil}(R)\) is a Dedekind domain;
(7) \(R\) is a \(\phi\)-Dedekind ring.

As mentioned above, a Mori ring is said to be nontrivial if it is properly contained in its total quotient ring. Our next example is of a nontrivial Mori ring that is in the set \(\mathcal{H}\) but is not a \(\phi\)-Mori ring.

**Example 8.18** ([17, Example 5.3]). Let \(E\) be a Dedekind domain with a maximal ideal \(M\) such that no power of \(M\) is principal (equivalently, \(M\) generates an infinite cyclic subgroup of the class group) and let \(D = E + xF[x]\), where \(F\) is the quotient field of \(E\). Let \(\mathcal{P} = \{ND \mid N \in \text{Max}(E)\setminus\{M\}\}, B = \sum F/D_{P_{\alpha}}\) where each \(P_{\alpha} \in \mathcal{P}\), and let \(R = D(+B)\). Then the following hold:

(a) If \(J\) is a regular ideal, then \(J = I(+B)B = IR\) for some ideal \(I\) that contains a polynomial in \(D\) whose constant term is a unit of \(E\). Moreover, the ideal \(I\) is principal and factors uniquely as \(P_{1}^{r_1}\cdots P_{n}^{r_n}\), where the \(P_i\) are the height-one maximal ideals of \(D\) that contain \(I\).

(b) \(R \neq T(R)\) since, for example, the element \((1 + x, 0)\) is a regular element of \(R\) that is not a unit.

(c) \(R\) is a nontrivial Mori ring but \(R\) is not \(\phi\)-Mori.

(d) \(MR\) is a maximal \(\phi\)-divisorial ideal of \(R\), but \(R_{MR}\) is not a Mori ring.

**References**


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