SOME FINITENESS CONDITIONS ON THE SET OF OVERRINGS OF A $\phi$-RING

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Abstract. Let $\mathcal{H} = \{R \mid R$ is a commutative ring and $\text{Nil}(R)$ is a divided prime ideal of $R\}$. For a ring $R \in \mathcal{H}$ with total quotient ring $T(R)$, let $\phi$ be the natural ring homomorphism from $T(R)$ into $R_{\text{Nil}(R)}$. An integral domain $R$ is said to be an FC-domain (in the sense of Gilmer) if each chain of distinct overrings of $R$ is finite, and $R$ is called an FO-domain if $R$ has finitely many overrings. A ring $R$ is called an FC-ring if each chain of distinct overrings of $R$ is finite, and $R$ is said to be an FO-ring if $R$ has finitely many overrings. A ring $R \in \mathcal{H}$ is said to be a $\phi$-FC-ring if $\phi(R)$ is an FC-ring, and $R$ is called a $\phi$-FO-ring if $\phi(R)$ is an FO-ring. In this paper, we show that the theory of $\phi$-FC-rings and $\phi$-FO-rings resembles that of FC-domains and FO-domains.

1. Introduction

We assume throughout that all rings are commutative with $1 \neq 0$. Let $R$ be a ring. Then $T(R)$ denotes the total quotient ring of $R$, $R'$ denotes the integral closure of $R$ in $T(R)$, $\text{Nil}(R)$ denotes the set of nilpotent elements of $R$, $Z(R)$ denotes the set of zerodivisors of $R$. Recall from [19] and [9] that a prime ideal of $R$ is called a divided prime if $P \subset (x)$ for every $x \in R \setminus P$; thus a divided prime ideal is comparable (under set inclusion) to every ideal of $R$. Throughout this paper, $\mathcal{H} = \{R \mid R$ is a commutative ring and $\text{Nil}(R)$ is a divided prime ideal of $R\}$, and $\mathcal{H}_0 = \{R \in \mathcal{H} \mid \text{Nil}(R) = Z(R)\}$. In [7], [8], [10], [11], [12], and [13] the first-named author investigated the class of rings $\mathcal{H}$. Observe that if $R$ is an integral domain, then $R \in \mathcal{H}_0 \subset \mathcal{H}$. If $R \in \mathcal{H}$, then $R$ is called a $\phi$-ring. For

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a further study on $\phi$-rings, we recommend the references: [3], [4], [14], [15], and [16].

A non-zerodivisor of a ring $R$ is called a regular element and an ideal of $R$ is said to be regular if it contains a regular element. An ideal $I$ of a ring $R$ is said to be a nonnil ideal if $I \not\subseteq \text{Nil}(R)$. If $I$ is a nonnil ideal of a ring $R \in \mathcal{H}$, then $\text{Nil}(R) \subset I$. In particular, $\text{Nil}(R) \subset I$ for every regular ideal of a ring $R \in \mathcal{H}$. Recall from [8] that for a ring $R \in \mathcal{H}$ with total quotient ring $T(R)$, the map $\phi : T(R) \rightarrow R_{\text{Nil}(R)}$ such that $\phi(a/b) = a/b$ for $a \in R$ and $b \in R \setminus Z(R)$ is a ring homomorphism from $T(R)$ into $R_{\text{Nil}(R)}$, and $\phi$ restricted to $R$ is also a ring homomorphism from $R$ into $R_{\text{Nil}(R)}$ given by $\phi(x) = x/1$ for every $x \in R$. Recall that if every finitely generated regular ideal of a ring $R$ is invertible, then $R$ is said to be a Prüfer ring. Recall from [3] that a nonnil ideal $I$ of $R \in \mathcal{H}$ is a $\phi$-invertible if $\phi(I)$ is an invertible ideal of $\phi(R)$, and a ring $R \in \mathcal{H}$ is said to be a $\phi$-Prüfer ring if every finitely generated nonnil ideal of $R$ is $\phi$-invertible, that is, if $\phi(R)$ is a Prüfer ring. Also recall from [11] that a ring $R \in \mathcal{H}$ is said to be a $\phi$-chained ring ($\phi$-CR) if for each $x \in R_{\text{Nil}(R)} \setminus \phi(R)$, we have $x^{-1} \in \phi(R)$.

In this paper, we generalize the concept of FC-domains and FO-domains as in [22] to the context of rings that are in $\mathcal{H}$. Recall from [22] that an integral domain $R$ is said to be an FC-domain if each chain of distinct overrings of $R$ is finite, and $R$ is called an FO-domain if $R$ has finitely many overrings. Recall that $B$ is said to be an overring of a ring $R$ if $R \subseteq B \subseteq T(R)$, where $T(R)$ is the total quotient ring of $R$. Jaballah (the second-named author) asked in [27, Question 1] for a characterization of FO-domain. Gilmer in [22] gave such characterization. A ring $R$ is called an FC-ring if each chain of distinct overrings of $R$ is finite, and $R$ is said to be an FO-ring if $R$ has finitely many overrings. A ring $R \in \mathcal{H}$ is said to be a $\phi$-FC-ring if $\phi(R)$ is an FC-ring, and $R$ is called a $\phi$-FO-ring if $\phi(R)$ is an FO-ring.

We remind the reader with the following important properties of $\phi$-rings (for (1) through (5) see [8],) Let $R \in \mathcal{H}$. Then

1. $\phi(R) \in \mathcal{H}_0$.
2. $\text{Ker}(\phi) \subseteq \text{Nil}(R)$.
3. $\text{Nil}(T(R)) = \text{Nil}(R)$.
4. $\text{Nil}(R_{\text{Nil}(R)}) = \phi(\text{Nil}(R)) = \text{Nil}(\phi(R)) = Z(\phi(R))$.
5. $T(\phi(R)) = R_{\text{Nil}(R)}$ is quasilocal with maximal ideal $\text{Nil}(\phi(R))$, and $R_{\text{Nil}(R)}/\text{Nil}(\phi(R)) = T(\phi(R))/\text{Nil}(\phi(R))$ is the quotient field of $\phi(R)/\text{Nil}(\phi(R))$. 

(6) If \( R \in \mathcal{H}_0 \) and \( D = R/\text{Nil}(R) \), then \( D' = R'/\text{Nil}(R) \) [2, Lemma 2.8].

The technique of idealization as in [24] is used in this paper to construct examples. Recall that for an \( R \)-module \( M \), the idealization of \( M \) over \( R \) is the ring formed from \( R \times M \) by defining addition and multiplication as \((r,a) + (s,m) = (r + s, a + m)\) and \((r,a)(s,m) = (rs, rm + sa)\), respectively.

2. \( \phi \)-FC-Extensions

Let \( R \subseteq S \) be a ring extension. Then \([R,S]\) denotes the set of all rings that are between \( R \) and \( S \), and \((R : S) = \{ r \in R \mid rS \subseteq R \}\) is the conductor of \( R \) in \( S \). We start with the following (trivial) lemma.

Lemma 2.1. Suppose that \( R \subseteq S \) is a ring extension such that \( \text{Nil}(R) = \text{Nil}(S) \). Then

1. \( R/\text{Nil}(R) = S/\text{Nil}(R) \) if and only if \( R = S \).
2. \( R \subseteq S \) is an FC(FO)-extension if and only if \( R/\text{Nil}(R) \subseteq S/\text{Nil}(R) \) is an FC(FO)-Extension.
3. \([R,S]\) satisfies the d.c.c(a.c.c)-condition if and only if \([R/\text{Nil}(R), S/\text{Nil}(R)]\) satisfies the d.c.c(a.c.c)-condition.
4. \((R/\text{Nil}(R) : S/\text{Nil}(R)) = (R : S)/\text{Nil}(R)\).

The following result is a generalization of [23, Theorem 5].

Theorem 2.2. Let \( R \in \mathcal{H}_0 \). Then each \( \alpha \in T(R) \) is the root of a polynomial in \( R[X] \) with unit coefficient (i.e. one of the coefficients is a unit) if and only if the integral closure of \( R \) (in \( T(R) \)) is a Pr"ufer ring. In particular, an integrally closed ring \( R \in \mathcal{H}_0 \) is a Pr"ufer ring if and only if each \( \alpha \in T(R) \) is the root of a polynomial in \( R[X] \) with unit coefficient.

Proof. Let \( D = R/\text{Nil}(R) \). Suppose that \( R' \) is a Pr"ufer ring. Let \( \alpha \in T(R) \). Since \( D \) is a Pr"ufer domain by [3, Theorem 2.6] and \( T(D) = T(R)/\text{Nil}(R) \), \( \alpha + \text{Nil}(R) \) is the root of a polynomial in \( D[X] \) with unit coefficient. Since an element \( b \in R \) is a unit of \( R \) if and only if \( b + \text{Nil}(R) \) is a unit of \( D \), we conclude that \( \alpha \) is the root of a polynomial in \( R[X] \) with unit coefficient.

Conversely, suppose that each \( \alpha \in T(R) \) is the root of a polynomial in \( R[X] \) with unit coefficient. Then it is clear that each \( \beta \in T(R)/\text{Nil}(R) \) is the root of a polynomial in \( (R/\text{Nil}(R))[X] \) with unit coefficient. Since \( T(R)/\text{Nil}(R) \) is the total quotient field of the integral domain \( R/\text{Nil}(R) \), the integral closure of \( R/\text{Nil}(R) \) (in \( T(R)/\text{Nil}(R) \)) is a Pr"ufer domain by [23, Theorem 5]. Since the integral closure of \( R/\text{Nil}(R) \) is of the form of \( R'/\text{Nil}(R) \) by [2, Lemma 2.8], we conclude that \( R' \) is a Pr"ufer ring by [3, Theorem 2.6]. \( \square \)
The following result is a generalization of [22, Corollary 1.2].

**Corollary 2.3.** Let $R \in \mathcal{H}_0$. If d.c.c is satisfied in $[R, T(R)]$, then $R'$ is a Prüfer ring. In particular, the integral closure of an FC-ring in $\mathcal{H}_0$ is a Prüfer ring.

**Proof.** Since $[R, T(R)]$ satisfies the d.c.c, each $\alpha \in T(R)$ is the root of a polynomial in $R[X]$ with unit coefficient by [22, Proposition 1.1]. Thus the claim is now clear by Theorem 2.2 and by the fact that an FC-ring satisfies the d.c.c condition. \qed

Let $S$ be a ring extension of a ring $R$. Then recall that $S$ is said to be strongly affine over $R$ if every subring $B$ of $S$ such that $R \subseteq B \subseteq S$ is finitely generated as a ring extension of $R$. The following result is a generalization of [22, Proposition 1.3].

**Proposition 2.4.** Let $R \in \mathcal{H}_0$. If $R$ is an FC-ring, then $T(R)$ is strongly affine over $R$; hence the integral closure of $R$ (inside $T(R)$) is a finite $R$-module.

**Proof.** Suppose that $R$ is an FC-ring. Let $D = R/\text{Nil}(R)$. Since $T(D) = T(R)/\text{Nil}(R)$, $D$ is an FC-domain by Lemma 2.1. Thus $T(D)$ is strongly affine over $D$ by [22, Proposition 1.3]. It is easily verified that $T(D)$ is strongly affine over $D$ if and only if $T(R)$ is strongly affine over $R$. Since $D' = R'/\text{Nil}(R)$ and $D'$ is a finite $D$-module by [22, Proposition 1.3], it is easily verified that $R'$ is a finite $R$-module. \qed

The following result is a generalization of [22, Theorem 1.5].

**Theorem 2.5.** Let $R \in \mathcal{H}_0$ be an integrally closed ring. The following conditions are equivalent:

1. $R$ is a Prüfer ring with finitely many prime ideals;
2. $R/\text{Nil}(R)$ is a Prüfer domain with finitely many prime ideals;
3. $R$ is a finite dimensional Prüfer ring with finitely many maximal ideals;
4. $R/\text{Nil}(R)$ is a finite dimensional Prüfer domain with finitely many maximal ideals;
5. $R/\text{Nil}(R)$ is an FC-domain;
6. $R/\text{Nil}(R)$ is an FO-domain;
7. $R$ is an FO-ring;
8. $R$ is an FC-ring.

**Proof.** Let $D = R/\text{Nil}(R)$. Then $D$ is an integral domain with quotient field $T(R)/\text{Nil}(R)$. Since $D' = R'/\text{Nil}(R)$ and $R$ is an integrally closed ring, we conclude that $D$ is an integrally closed domain. We will prove
(2) ⇒ (3) and (8) ⇒ (1). The reader should be able to verify the other implications. (2) ⇒ (3). Since \( D \) is a Prüfer domain with finitely many prime ideals, \( D \) is a finite dimensional Prüfer domain with finitely many maximal ideals by [22, Theorem 1.5]. Thus \( R \) is a finite dimensional Prüfer ring with finitely many maximal ideals by [3, Theorem 2.6]. (8) ⇒ (1). Since \( D \) is an FC-domain, \( D \) is a Prüfer domain with finitely many prime ideals by [22, Theorem 1.5]. Hence \( R \) is a Prüfer ring by [3, Theorem 2.6] and it is clear that \( R \) has finitely many prime ideals. □

Observe that if \( R \in \mathcal{H} \), then \( \phi(R) \in \mathcal{H}_0 \). Hence in view of Theorem 2.2, Corollary 2.3, and Proposition 2.4, we have the following corollary.

**Corollary 2.6.** Let \( R \in \mathcal{H} \). Then all the following statements hold:

1. Each \( \alpha \in R_{\text{Nil}(R)} \) is the root of a polynomial in \( \phi(R)[X] \) with unit coefficient if and only if the integral closure of \( \phi(R) \) (in \( R_{\text{Nil}(R)} \)) is a Prüfer ring. In particular, a \( \phi \)-integrally closed ring \( R \in \mathcal{H} \) is a \( \phi \)-Prüfer ring if and only if each \( \alpha \in T(R) \) is the root of a polynomial in \( \phi(R)[X] \) with unit coefficient.
2. If \( \text{d.c.c} \) is satisfied in \( [\phi(R), R_{\text{Nil}(R)}] \), then \( \phi(R)' \) is a Prüfer ring. In particular, the \( \phi \)-integral closure of a \( \phi \)-FC-ring in \( \mathcal{H} \) is a Prüfer ring.
3. If \( R \) is a \( \phi \)-FC-ring, then \( R_{\text{Nil}(R)} \) is strongly affine over \( \phi(R) \); hence the integral closure of \( \phi(R) \) (inside \( R_{\text{Nil}(R)} \)) is a finite \( \phi(R) \)-module.

**Theorem 2.7.** Let \( R \in \mathcal{H} \). The following statements hold:

1. \( R \) is a \( \phi \)-FC-ring if and only if \( R/\text{Nil}(R) \) is an FC-domain.
2. \( R \) is a \( \phi \)-FO-ring if and only if \( R/\text{Nil}(R) \) is an FO-domain.

**Proof.** (1) Suppose that \( R \) is a \( \phi \)-FC-ring. Then \( \phi(R) \) is an FC-ring. Let \( D = \phi(R)/\text{Nil}(\phi(R)) \). Since \( T(D) = T(\phi(R))/\text{Nil}(\phi(R)) = R_{\text{Nil}(R)}/\text{Nil}(\phi(R)) \), we conclude that \( D \) is an FC-domain by Lemma 2.1. Since \( D \) is ring-isomorphic to \( R/\text{Nil}(R) \) by [3, Lemma 2.5], we conclude that \( R/\text{Nil}(R) \) is an FC-domain. Conversely, suppose that \( F = R/\text{Nil}(R) \) is an FC-domain. Again, by Lemma 2.1 \( \phi(R) \) is an FC-ring, and thus \( R \) is a \( \phi \)-FC-ring.

(2) Just use a similar argument as in (1). □

Let \( R \in \mathcal{H} \). Then \( R \) is a \( \phi \)-Prüfer ring if and only if \( R/\text{Nil}(R) \) is a Prüfer domain by [3, Theorem 2.6]. In view of Theorem 2.5, for a ring \( R \in \mathcal{H} \) we have the following implications:

\( R \) is a \( \phi \)-Prüfer ring with finitely many prime ideals \( \iff \) \( R \) is a \( \phi \)-FC and a \( \phi \)-integrally closed ring \( \iff \) \( R \) is a \( \phi \)-FO and a \( \phi \)-integrally closed ring.
The following result is a generalization of [22, Corollary 1.6].

**Corollary 2.8.** A $\phi$-FC-ring in $\mathcal{H}$ has finitely many prime ideals.

**Proof.** Let $D = R/\text{Nil}(R)$. Since $D$ is an FC-domain by Theorem 2.7, $D$ has finitely many prime ideals by [22, Corollary 1.6], and hence it is clear that $R$ has finitely many prime ideals. \qed

The following is an example of a non-domain FC-ring $R \in \mathcal{H}_0$ that is not an FO-ring.

**Example 2.9.** Let $J$ be the FC-domain that is not an FO-domain constructed in [22, Example 1.7] and let $L$ be the quotient field of $J$. Set $R = J(+)L$. It is easily verified that $Z(R) = \text{Nil}(R) = \{0\}(+)L$ is a divided prime ideal of $R$, and hence $R \in \mathcal{H}_0$. Since $R/\text{Nil}(R)$ is ring-isomorphic to $J$, we conclude that $R/\text{Nil}(R)$ is an FC-domain that is not an FO-domain. Hence $R$ is an FC-ring that is not an FO-ring by Lemma 2.1.

The following result is a generalization of [22, Theorem 2.3].

**Theorem 2.10.** Let $R \in \mathcal{H}_0$. Then $R$ is an FC-ring if and only if a.c.c. and d.c.c. hold in both $[R, R']$ and $[R', T(R)]$.

**Proof.** Let $D = R/\text{Nil}(R)$. Then $D$ is an integral domain with quotient field $T(R)/\text{Nil}(R)$ and $D' = R'/\text{Nil}(R)$. Suppose that $R$ is an FC-ring. Then $D$ is an FC-domain by Lemma 2.1. Thus a.c.c. and d.c.c. hold in both $[D, D']$ and $[D', T(D)]$ by [22, Theorem 2.3], and hence a.c.c. and d.c.c. hold in both $[R, R']$ and $[R', T(R)]$ by Lemma 2.1. Conversely, suppose that a.c.c. and d.c.c. hold in both $[D, D']$ and $[D', T(D)]$ by Lemma 2.1. Thus $D$ is an FC-domain by [22, Theorem 2.3]. Hence $R$ is an FC-ring by Lemma 2.1. \qed

In view of Theorems 2.7, 2.10, and [22, Theorem 2.4], we have the following corollary.

**Corollary 2.11.** Let $R \in \mathcal{H}$. The following statements are equivalent:

1. $R$ is a $\phi$-FC-ring;
2. a.c.c and d.c.c hold in both $[R/\text{Nil}(R), (R/\text{Nil}(R))']$ and $[(R/\text{Nil}(R))', R_{\text{Nil}(R)}/\text{Nil}(R_{\text{Nil}(R)})]$.

The following result is a generalization of [22, Theorem 2.3].
Theorem 2.12. Suppose that $R \in \mathcal{H}$ has finitely many maximal ideals. Then $R$ is a $\phi$-FC-ring if and only if $R_M$ is a $\phi$-FC-ring for each maximal ideal $M$ of $R$.

Proof. Set $D = R/\text{Nil}(R)$. Suppose that $R$ is a $\phi$-FC-ring. Let $M$ be a maximal ideal of $R$. Since $D$ is an FC-domain by Theorem 2.7, $D_{M/\text{Nil}(R)} = R_{M/\text{Nil}(R)}$ is an FC-domain by [22, Theorem 2.4]. Hence $R_M$ is a $\phi$-FC-ring by Theorem 2.7. Conversely, suppose that $R_M$ is a $\phi$-FC-ring for each maximal ideal $M$ of $R$. Hence $R_{M/\text{Nil}(R)} = D_{M/\text{Nil}(R)}$ is an FC-domain by Theorem 2.7 for each maximal ideal $M$ of $R$. Thus, $D = R/\text{Nil}(R)$ is an FC-domain by [22, Theorem 2.4], and hence $R$ is a $\phi$-FC ring by Theorem 2.7.

Corollary 2.13. Suppose that $R \in \mathcal{H}_0$ has finitely many maximal ideals. Then $R$ is an FC-ring if and only if $R_M$ is an FC-ring for each maximal ideal $M$ of $R$.

The following result is a generalization of [22, Theorem 2.14].

Theorem 2.14. Let $R \in \mathcal{H}_0$ and let $C$ be the conductor of $R$ in $R'$. Then $R$ is an FC-ring if and only if the following three conditions are satisfied:

1. $R'$ is a Prüfer ring with finitely many prime ideals.
2. $R'$ is a finite $R$-module.
3. $R/C$ is an Artinian ring.

Proof. Let $D = R/\text{Nil}(R)$. Suppose that $R$ is an FC-ring. Then the conditions (1) and (2) hold by Theorem 2.5, Corollary 2.8, and Proposition 2.4. Let $J$ be the conductor of $D$ in $D'$. Then $J = C/\text{Nil}(R)$ by Lemma 2.1. Since $D$ is an FC-domain by Lemma 2.1 and $R/C \cong R/\text{Nil}(R) \cong D/J$, we conclude that $D/J$ is an Artinian ring by [22, Theorem 2.14], and hence $R/C$ is an Artinian ring. Conversely, suppose that the conditions (1), (2), and (3) hold. Since $J = C/\text{Nil}(R)$ is the conductor of $D$ in $D'$ and $R/C \cong D/J$, $D/J$ is an Artinian ring. Since $R'$ is a finite $R$-module and $D' = R'/\text{Nil}(R)$, we conclude that $D'$ is a finite $D$-module. Since $R'$ is a Prüfer ring with finitely many prime ideals, $D$ is a Prüfer domain with finitely many prime ideals by [3, Theorem 2.6]. Thus $D$ is an FC-domain by [22, Theorem 2.14]. Hence $R$ is an FC-ring by Lemma 2.1.

In view of Theorem 2.14 and Theorem 2.7, we have the following corollary.

Corollary 2.15. Let $R \in \mathcal{H}$, $D = R/\text{Nil}(R)$, and let $C$ be the conductor of $\phi(R)$ in $\phi(R)'$. The following statements are equivalent:
(1) \( R \) is a \( \phi \)-FC-ring.

(2) The following three conditions are satisfied:
   (a) \( D' \) is a Prüfer ring with finitely many prime ideals.
   (b) \( D' \) is a finite \( D \)-module.
   (c) \( D/N \) is an Artinian ring, where \( N \) is the conductor of \( D \) in \( D' \).

Combining Theorems 2.10, Corollary 2.13, and Theorem 2.14 we arrive at the following corollary.

**Corollary 2.16.** Let \( R \in \mathcal{H}_0 \), and let \( C \) be the conductor of \( R \) in \( R' \). The following statements are equivalent:

(1) \( R \) is an \( FC \)-ring.
(2) a.c.c. and d.c.c. hold in both \([R,R']\) and \([R',T(R)]\).
(3) \( \text{Max}(R) \) is finite and \( R_M \) is an FC-ring for each maximal ideal \( M \) of \( R \).
(4) The following three conditions are satisfied:
   (a) \( R' \) is a Prüfer ring with finite spectrum;
   (b) \( R' \) is finite \( R \)-module;
   (c) \( R/C \) is an Artinian ring.

The following result is a generalization of [26, Corollary 3.4].

**Theorem 2.17.** Let \( R \in \mathcal{H}_0 \) be a Prüfer ring. If \( R \) is an FC-ring, then each maximal chain \( R = R_0 \subset R_1 \subset R_2 \cdots \subset R_n = T(R) \) of overrings of \( R \) has length \( n = |\text{Spec}(R)| - 1 \).

**Proof.** Let \( D = R/\text{Nil}(R) \). Then \( D \) is a Prüfer domain by [3, Theorem 2.6]. Let \( R = R_0 \subset R_1 \subset R_2 \cdots \subset R_n = T(R) \) be a maximal chain of overrings of \( R \). Since \( T(D) = T(R)/\text{Nil}(R) \), \( D = R/\text{Nil}(R) \subset R_1/\text{Nil}(R) \subset R_2/\text{Nil}(R) \cdots \subset R_n/\text{Nil}(R) = T(D) \) is a maximal chain of overrings of \( D \), and hence it has length \( |\text{Spec}(D)| - 1 \) by [26, Corollary 3.4]. Since \( |\text{Spec}(D)| = |\text{Spec}(R)| \), we conclude that the maximal chain \( R = R_0 \subset R_1 \subset R_2 \cdots \subset R_n = T(R) \) of overrings of \( R \) has length \( |\text{Spec}(R)| - 1 \).

\( \square \)

**Corollary 2.18.** Let \( R \in \mathcal{H} \) be a \( \phi \)-Prüfer ring. If \( R \) is a \( \phi \)-FC-ring, then the following statements hold:

(1) Each maximal chain \( \phi(R) = R_0 \subset R_1 \subset R_2 \cdots \subset R_n = R_{\text{Nil}(R)} \) of overrings of \( \phi(R) \) has length \( n = |\text{Spec}(R)| - 1 \).
(2) Each maximal chain \( R/\text{Nil}(R) = R_0 \subset R_1 \subset R_2 \cdots \subset R_n = R_{\text{Nil}(R)}/\text{Nil}(R_{\text{Nil}(R)}) \) of overrings of \( R/\text{Nil}(R) \) has length \( n = |\text{Spec}(R)| - 1 \).
\[ n = | \text{Spec}(R) | - 1. \]

**Proof.** Just observe that \( \phi(R) \in \mathcal{H}_0 \) and \(| \text{Spec}(R) | = | \text{Spec}(\phi(R)) | = | \text{Spec}(R/\text{Nil}(R)) | \) by [16, Lemma 2.1]. \( \square \)

The following result is a generalization of [17, Theorem 3.6 and Proposition 3.8].

**Theorem 2.19.** Let \( R \in \mathcal{H} \) be of finite Krull dimension \( d \geq 1 \). The following statements are equivalent:

1. \( R \) is a \( \phi \)-chained ring;
2. \( R/\text{Nil}(R) \) is a valuation domain;
3. \(| [R/\text{Nil}(R), R_{N_{\text{Nil}}(R)}/\text{Nil}(R_{N_{\text{Nil}}(R)})] | = d + 1 \);
4. \(| [\phi(R), R_{N_{\text{Nil}}(R)}] | = d + 1 \);
5. For each chain of overrings \( \phi(R) = R_0 \subset R_1 \subset R_2 \cdots \subset R_n = R_{N_{\text{Nil}}(R)} \) of \( \phi(R) \), we have \( n \leq d \);
6. For each chain of overrings \( R/\text{Nil}(R) = R_0 \subset R_1 \subset R_2 \cdots \subset R_n = R_{N_{\text{Nil}}(R)}/\text{Nil}(R_{N_{\text{Nil}}(R)}) \) of \( R/\text{Nil}(R) \), we have \( n \leq d \).

**Proof.** Let \( D = R/\text{Nil}(R) \) and \( F = \phi(R)/\text{Nil}(\phi(R)) \). Then \( T(D) \cong T(F) = R_{N_{\text{Nil}}(R)}/\text{Nil}(R_{N_{\text{Nil}}(R)}) \). \( (1) \iff (2) \). See [3, Lemma 2.7]. \( (2) \Rightarrow (3) \). Since \( D \) is ring-isomorphic to \( F \) by [3, Lemma 2.5], \( F \) is a valuation domain and the Krull dimension of \( F \) is \( d \). Hence \(| [F, T(F)] | = | [D, T(D)] | = d + 1 \) by [17, Theorem 3.6 and Proposition 3.8]. \( (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \). These implications are clear since there is a one-to-one correspondence between the overrings of \( F \) and the overrings of \( \phi(R) \). \( (6) \Rightarrow (1) \). By [17, Theorem 3.6 and Proposition 3.8], \( D \) is a valuation domain, and thus \( R \) is a \( \phi \)-chained ring by [3, Lemma 2.7]. \( \square \)

In the following result, we show that a \( \phi \)-FC-ring is a pullback of an FC-domain.

**Theorem 2.20.** Let \( R \in \mathcal{H} \). Then \( R \) is a \( \phi \)-FC-ring if and only if \( \phi(R) \) is ring-isomorphic to a ring \( A \) obtained from the following pullback diagram:

\[
\begin{array}{ccc}
A & \longrightarrow & A/M \\
\downarrow & & \downarrow \\
T & \longrightarrow & T/M
\end{array}
\]

where \( T \) is a zero-dimensional quasilocal ring with maximal ideal \( M \), \( A/M \) is an FC-subring of \( T/M \), the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

**Proof.** Suppose \( \phi(R) \) is ring-isomorphic to a ring \( A \) obtained from the given diagram. Then \( A \in \mathcal{H} \) and \( \text{Nil}(A) = Z(A) = M \). Since \( A/M \) is an FC-domain, \( A \) is a \( \phi \)-FC-ring by Theorem 2.7(1), and thus \( R \) is a \( \phi \)-FC-ring.
Conversely, suppose that $R$ is a $\phi$-FC-ring. Then, letting $T = R_{\text{Nil}(R)}$, $M = \text{Nil}(R_{\text{Nil}(R)})$, and $A = \phi(R)$ yields the desired pullback diagram.

It is clear that if $R \in \mathcal{H}$ is a $\phi - FC$-ring, then $R$ is an $FC$-ring. The following is an example of an $FC$-ring $R \in \mathcal{H}$ but $R$ is not a $\phi - FC$-ring.

Example 2.21. Let $D$ be a Prüfer domain with infinitely many maximal ideals and let $K$ be the quotient field of $D$. Set $R = D(+(K/D))$. It is easily verified that $R \in \mathcal{H}$ and every nonunit of $R$ is a zero-divisor of $R$. Thus $R = T(R)$, so $R$ is $\phi$-integrally closed. Hence $R$ is an $FC$-ring. Since $R/\text{Nil}(R)$ is ring-isomorphic to $D$, we conclude that $R/\text{Nil}(R)$ is not an $FC$-domain by Corollary ??, and thus $R$ is not a $\phi - FC$-ring by Theorem 2.7(1).

3. $\phi$-FO-Extension

The results in this section are parallel to those for FC-extension in the previous section and the proofs are similar too. Hence we will only state the results of this section without giving proofs.

The following result is a generalization of [22, Theorem 3.1], also see [1, Theorem 2.6].

Theorem 3.1. Let $R \in \mathcal{H}_0$. Then $R$ is an FO-ring if and only if each of the sets $[R, R']$, and $[R', T(R)]$ is finite.

The following result is a generalization of [22, Theorem 3.2].

Theorem 3.2. Let $R \in \mathcal{H}_0$ with finitely many maximal ideals. Then $R$ is an FO-ring if and only if $R_M$ is an FO-ring for each maximal ideal $M$ of $R$.

Anderson, Dobbs, and Mullins [1] and [2] investigated finiteness of $[R, S]$ for a ring extension $R \subseteq S$. If $[R, S]$ is finite, they say $R \subseteq S$ has FIP. The following result is a generalization of [22, Theorem 3.4]

Theorem 3.3. Let $R \in \mathcal{H}_0$, and let $C$ be the conductor of $R$ in $R'$. Then $R$ is an FO-ring if and only if $R'$ is a Prüfer ring with finitely many prime ideals and the extension $R/C \subseteq R'/C$ has FIP.

Combining Theorem 3.1, 3.2, and 3.3 we arrive at the following corollary.

Corollary 3.4. Let $R \in \mathcal{H}_0$, and let $C$ be the conductor of $R$ in $R'$. The following statements are equivalent:

1. $R$ is an FO-ring;
(2) $R$ has finitely many maximal ideals and $R_M$ is an FO-ring for each maximal ideal $M$ of $R$;
(3) $R'$ is a Prüfer ring with finitely many prime ideals and $R/C \subset R'/C$ has FIP.

A similar argument as in Theorem 2.20, one can easily verify the following result.

**Corollary 3.5.** Let $R \in \mathcal{H}$. Then $R$ is a $\phi$-FO-ring if and only if $\phi(R)$ is ring-isomorphic to a ring $A$ obtained from the following pullback diagram:

$$
\begin{array}{ccc}
A & \rightarrow & A/M \\
\downarrow & & \downarrow \\
T & \rightarrow & T/M
\end{array}
$$

where $T$ is a zero-dimensional quasilocal ring with maximal ideal $M$, $A/M$ is an FO-subring of $T/M$, the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

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**References**


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