On Divided Rings and $\phi$-Pseudo-Valuation Rings

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Abstract. Let $R$ be a commutative ring with 1 and $T(R)$ be its total quotient ring such that $Nil(R)$ is a divided prime ideal of $R$. Then $R$ is called a $\phi$-chained ring ($\phi$-CR) if for every $x, y \in R \setminus Nil(R)$ either $x | y$ or $y | x$. Also, $R$ is called a $\phi$-pseudo-valuation ring ($\phi$-PVR) if for every $x, y \in R \setminus Nil(R)$ either $x | y$ or $y | x$ for each nonzero $m \in R$. We show that a ring $R$ is a $\phi$-PVR iff $Nil(R)$ is a divided prime ideal and $R/Nil(R)$ is a pseudo-valuation domain. Also, we show that every covering of a quasi-local ring $R$ with maximal ideal $M$ is a $\phi$-PVR iff $R[u]$ is quasi-local for each $u \in (M : M) \setminus R$ if every covering of $R$ is quasi-local iff every $\phi$-CR between $R$ and $T(R)$ other than $(M : M)$ is of the form of $R_P$ for some nonmaximal prime ideal $P$ of $R$. Among other results, we show that if $B$ is an covering of a $\phi$-PVR and $I$ is a proper ideal of $B$, then there is a $\phi$-CR between $B$ and $T(R)$ such that $I\cap C \neq C$. Also, we show that the integral closure $R'_{\text{int}}$ of $R$ in $T(R)$ is the intersection of all the $\phi$-CRs between $R$ and $T(R)$.

1. Introduction

We assume throughout that all rings are commutative with $1 \neq 0$. We begin by recalling some background material. As in [15], an integral domain $R$, with quotient field $K$, is called a pseudo-valuation domain (PVD) in case each prime ideal $P$ of $R$ is strongly prime, in the sense that $xy \in P, x \in K, y \in K$ implies that either $x \in P$ or $y \in P$. In [5], Anderson, Dobbs and the author generalized the study of pseudo-valuation domains to the context of arbitrary rings (possibly with nonzero zero divisors). Recall from [5] that a prime ideal $P$ of $R$ is said to be strongly prime (in $R$) if $aP$ and $bR$ are comparable (under inclusion) for all $a, b \in R$. A ring $R$ is called a pseudo-valuation ring (PVR) if each prime ideal of $R$ is strongly prime. A PVR is necessarily quasilocal [5, Lemma 1(b)]; a chained ring is a PVR [5, Corollary 4]; and an integral domain is a PVR if and only if it is a PVD (cf. [1, Proposition 3.1], [2, Proposition 4.2], and [6, Proposition 3]). Recall from [7] and [13] that a prime ideal $P$ of $R$ is called divided if it is comparable (under inclusion) to every ideal of $R$. A ring $R$ is called a divided ring if every prime ideal of $R$ is divided.

In [8], the author gave another generalization of PVDs to the context of arbitrary rings (possibly with nonzero zero divisors). As in [8], for a ring $R$ with total quotient ring $T(R)$ such that $Nil(R)$ is a divided prime ideal of $R$, let $\phi : T(R) \rightarrow K := R_{\text{Nil}(R)}$ such that $\phi(a/b) = a/b$ for every $a \in R$ and every $b \in R \setminus Z(R)$. Then $\phi$ is a ring homomorphism from $T(R)$ into $K$, and $\phi$ restricted to $R$ is also a ring homomorphism from $R$ into $K$ given by $\phi(x) = x/1$ for every $x \in R$. A prime ideal $Q$ of $\phi(R)$ is called a $K$-strongly prime ideal if $xy \in Q, x \in K, y \in K$ implies that either $x \in Q$ or $y \in Q$. If each prime ideal of $\phi(R)$ is K-strongly prime, then $\phi(R)$...
is called a $K$-pseudo-valuation ring ($K$-PVR). A prime ideal $P$ of $R$ is called a $\phi$-strongly prime ideal if $\phi(P)$ is a $K$-strongly prime ideal of $\phi(R)$. If each prime ideal of $R$ is $\phi$-strongly prime, then $R$ is called a $\phi$-pseudo-valuation ring ($\phi$-PVR). For an equivalent characterization of a $\phi$-PVR, see Proposition 1.1(5). It was shown in [9, Theorem 2.6] that for each $n \geq 0$ there is a $\phi$-PVR of Krull dimension $n$ that is not a PVR. Also, recall from [10], that a ring $R$ is called a $\phi$-chained ring ($\phi$-CR) if $\text{Nil}(R)$ is a divided prime ideal of $R$ and for every $x \in R_{\text{Nil}(R)} \setminus \phi(R)$, we have $x^{-1} \in \phi(R)$. For an equivalent characterization of a $\phi$-CR, see Lemma 3.9. A $\phi$-CR is a divided ring [10, Corollary 3.3(2)], and hence is quasi-local. It was shown in [10, Theorem 2.7] that for each $n \geq 0$ there is a $\phi$-CR of Krull dimension $n$ that is not a chained ring.

In this paper, we show that a quasi-local ring $R$ with maximal ideal $M$ is a $\phi$-PVR iff $R[u]$ is quasi-local for each $u \in (M : M) \setminus R$ iff every overring of $R$ is quasi-local iff every overring contained in $(M : M)$ is quasi-local iff each $\phi$-CR between $R$ and $T(R)$ other than $(M : M)$ is of the form $R[p]$ for some nonmaximal prime ideal $P$ of $R$. Among other results, we show that if $B$ is an overring of a $\phi$-PVR and $I$ is a proper ideal of $B$, then there is a $\phi$-CR $C$ between $B$ and $T(R)$ such that $IB \neq B$. Also, we show that the integral closure of $R$ in $T(R)$ is the intersection of all the $\phi$-CRs between $R$ and $T(R)$.

The following notations will be used throughout. Let $R$ be a ring. Then $T(R)$ denotes the total quotient ring of $R$, $\text{Nil}(R)$ denotes the set of nilpotent elements of $R$, and $Z(R)$ denotes the set of zero-divisors of $R$. If $I$ is an ideal of $R$, then $\text{Rad}(I)$ denotes the radical ideal of $I$ in $R$.

We summarize some basic properties of PVRs and $\phi$-PVRs in the following proposition.

**Proposition 1.1.**
1. A PVR is a divided ring [5, Lemma 1], and hence is quasi-local.
2. A $\phi$-PVR is a divided ring [8, Proposition 4], and hence is quasi-local.
3. An integral domain is a PVR iff it is a $\phi$-PVR iff it is a PVD ([1, Proposition 3.1], [2, Proposition 4.2], [6, Proposition 3], and [8]).
4. A ring $R$ is a PVR if and only if for every $a, b \in R$, either $a | b$ in $R$ or $b | ac$ in $R$ for each nonunit $c \in R$ [5, Theorem 5].
5. A ring $R$ is a $\phi$-PVR if and only if $\text{Nil}(R)$ is a divided prime ideal of $R$ and for every $a, b \in R \setminus \text{Nil}(R)$, either $a | b$ in $R$ or $b | ac$ in $R$ for every nonunit $c \in R$ [8, Corollary 7(2)].
6. If $R$ is a PVR or a $\phi$-PVR, then $\text{Nil}(R)$ and $Z(R)$ are divided prime ideals of $R$ ([5], [8]).

**2. Divided Rings and $\phi$-PVRs**

**Definition.** A proper ideal $I$ of a ring $R$ is called a divided ideal if $I$ is comparable (under inclusion) to every principal ideal of $R$; equivalently, if $I$ is comparable to every ideal of $R$. If every prime ideal of $R$ is divided, then $R$ is called a divided ring.
In view of the proof of [9, Proposition 2.1], we see that the result in [9, Proposition 2.1] is still valid if we only assume that the ring $D$ is a divided domain. Hence, we state the following result without proof.

**PROPOSITION 2.1.** [9, Proposition 2.1] Let $D$ be a divided domain with maximal ideal $M$ and Krull dimension $n$, say $M = P_n \supset P_{n-1} \supset \ldots \supset P_1 \supset \{0\}$, where the $P_i$'s are the distinct prime ideals of $D$. Let $i, m, d \geq 1$ such that $1 \leq i \leq m \leq n$. Choose $x \in D$ such that $\text{Rad}([x]) = P_i$. Let $Q := P_m$ and $J := x^{d+1}D_q$. Then:

1. $J$ is an ideal of $D$ and $\text{Rad}(J) = P_i$.
2. $R := D/J$ is a divided ring with maximal ideal $M/J$, $Z(R) = P_m/J$, and $\text{Nil}(R) = P_i/J$. Furthermore, $w := x + J \in \text{Nil}(R)$ and $w^d \neq 0$ in $R$.
3. $\dim(R) = n - i$.
4. If $i < m < n$, then $\text{Nil}(R)$ is properly contained between $Z(R)$ and $M/J$.

Recall that a prime ideal $P$ of a ring $A$ is called branched if $\text{Rad}(I) = P$ for some primary ideal $I \neq P$ of $A$. It is well-known that a prime ideal $P$ of a Prüfer domain $D$ is branched iff $\text{Rad}(I) = P$ for some ideal $I \neq P$ of $D$. In the following result we will show that this result is still valid for divided rings.

**PROPOSITION 2.2.** Let $R$ be a divided ring, and let $P$ be a prime ideal of $R$ such that $P \neq \text{Nil}(R)$. Then $P$ is branched if and only if $\text{Rad}(I) = P$ for some ideal $I \neq P$ of $R$.

**Proof.** Suppose that $\text{Rad}(I) = P$ for some ideal $I \neq P$ of $R$. It is clear that $\text{Rad}(IP) \subset P$. Let $x \in P$. Since $\text{Rad}(I) = P$, $x^n \in I$ for some $n \geq 1$. Hence, $x^{n+1} \in IP$. Thus, $P \subset \text{Rad}(IP)$. Now, we show that $IP$ is a primary ideal of $R$. Suppose that $xy \in IP$ for some $x, y \in R$ and $x \notin P$. Since $xy \in IP$, $xy = i_1 p_1 + \ldots + i_n p_n$, where each $i_k \in I$ and each $p_k \in P$, $1 \leq k \leq n$. Since $P$ is a divided prime ideal and $x \notin P$, $p_k = q_k x$ for some $q_k \in P$ for each $p_k \in P$. Thus, $x(y - (i_1 q_1 + \ldots + i_n q_n)) = 0$. Since $\text{Nil}(R)$ is a prime ideal of $R$ and $x \notin \text{Nil}(R)$, $y - (i_1 q_1 + \ldots + i_n q_n) = w \in \text{Nil}(R)$. Since $\text{Rad}(IP) = P \neq \text{Nil}(R)$, there is a $d \in IP \setminus \text{Nil}(R)$. Hence, $\text{Nil}(R) \subset (d) \subset IP$. Since $i_1 q_1 + \ldots + i_n q_n \in IP$ and $w \in \text{Nil}(R) \subset IP$, $y \in IP$. Thus, $IP$ is a primary ideal of $R$. 

In light of the proof of the above proposition, we have the following corollary.

**COROLLARY 2.3.** Let $R$ be a ring such that $\text{Nil}(R)$ is a divided prime ideal of $R$, and let $P$ be a divided prime ideal of $R$ such that $P \neq \text{Nil}(R)$. Then $P$ is branched if and only if $\text{Rad}(I) = P$ for some ideal $I \neq P$ of $R$.

**PROPOSITION 2.4.** Let $R$ be a ring such that $\text{Nil}(R)$ is a divided prime ideal of $R$. Suppose that $I$ is a proper ideal of $R$ such that $I$ contains a nonnilpotent of $R$ and for some $N \geq 1$, $I^n$ is a divided ideal of $R$ for each $n \geq N$. Then $P = \bigcap_{n \geq 1} I^n$ is a divided prime ideal of $R$.

**Proof.** Since $\text{Nil}(R)$ is a divided ideal and $I$ contains a nonnilpotent of $R$, $\text{Nil}(R) \subset I^n$ for each $n \geq N$. Hence, $\text{Nil}(R) \subset P$. Now, suppose that $xy \in P$ for some $x, y \in R$ and suppose that $x \notin P$. Hence, $x \notin I^n$ for some $m \geq N$. Hence, $I^m \subset (x)$. Thus,
for each $k \geq 1$ we have $xy \in I^{m+k} \subseteq xI^k$. Hence, for each $k \geq 1$, there is a $d_k \in I^k$

such that $xy = xd_k$. Thus, $x(y - d_k) = 0$ for each $k \geq 1$. Since $\text{Nil}(R) \subseteq P$ and $\text{Nil}(R)$ is a prime ideal of $R$ and $x \not\in P$, we have $y - d_k = w_k \in \text{Nil}(R)$. Hence, $y = d_k + w_k \in I^k$. Hence, $y \in P$. Thus, $P$ is a prime ideal of $R$. Now, we show that $P$ is divided. Let $x \not\in P$. Then $x \not\in I^m$ for some $m \geq N$. Hence, $P \cap I^m \subseteq (x)$. □

In view of the above proposition, we have the following corollary.

COROLLARY 2.5. Let $R$ be a ring such that $\text{Nil}(R)$ is a divided prime ideal of $R$, and let $I$ be a proper ideal of $R$ such that $I$ contains a nonunit of $R$. Then the following statements are equivalent:

1. $I^n = I^m$ for some positive integers $n \neq m$ and $I^n$ is a divided ideal of $R$.
2. $I$ is a divided prime ideal of $R$ and $I = I^2$.

In the following result, we give a characterization of $\phi$-PVRs in terms of divided ideals.

PROPOSITION 2.6. Let $R$ be a quasilocal ring with maximal ideal $M$. Then the following statements are equivalent:

1. $R$ is a $\phi$-PVR.
2. $aM$ is a divided ideal of $R$ for each $a \in R \setminus \text{Nil}(R)$.

Proof. (1) $\Rightarrow$ (2). Let $a \in R \setminus \text{Nil}(R)$ and $b \not\in aM$. If $b = ar$ for some unit $r$ of $R$, then $b | am$ for every $m \in M$. Otherwise, $a \not| b$ in $R$. Thus, $b | am$ for each $m \in M$ by Proposition 1.1(5). Hence, $aM \subseteq (b)$.

(2) $\Rightarrow$ (1). Let $w \in \text{Nil}(R)$ and $a \in R \setminus \text{Nil}(R)$. If $a$ is a unit of $R$, then $a | w$ in $R$. Hence, assume that $a$ is a nonunit of $R$. Since $w \not| a^2$, $aM \not\subseteq (w)$. Hence, $w \in aM$. Therefore, $a | w$. Thus, $\text{Nil}(R)$ is a divided ideal of $R$. Hence, $\text{Nil}(R)$ is a prime ideal of $R$ by [3, Proposition 5.1]. Now, let $a, b \in R \setminus \text{Nil}(R)$. Then either $b \in aM$ or $aM \subseteq (b)$. Hence, either $a | b$ or $b | am$ for each $m \in M$. Thus, $R$ is a $\phi$-PVR by Proposition 1.1(5). □

The following result follows directly from the definition of strongly prime ideal as in [5] and the fact that a quasilocal ring with maximal ideal $M$ is a PVR if and only if $M$ is strongly prime [5, Theorem 2].

PROPOSITION 2.7. For a quasilocal ring $R$ with maximal ideal $M$, the following statements are equivalent:

1. $R$ is a PVR.
2. $aM$ is a divided ideal for each $a \in M$.

An element $d$ in a ring $R$ is called a proper divisor of $s \in R$ if $s = dm$ for some nonunit $m \in R$. The proof of the following result is very similar to that in [11, Proposition 4], but here we make use of the above proposition.

PROPOSITION 2.8. A ring $R$ is a $\phi$-PVR if and only if $\text{Nil}(R)$ is a divided prime ideal of $R$ and for every $a, b \in R \setminus \text{Nil}(R)$, either $b | a$ in $R$ or $d | b$ in $R$ for each proper divisor $d$ of $a$. 
Proof. Suppose that $R$ is a $\phi$-PVR with maximal ideal $M$. Then $\text{Nil}(R)$ is a divided prime ideal of $R$ by Proposition 1.1(6). Let $a, b \in R \setminus \text{Nil}(R)$, and suppose that $b \not| a$ in $R$. Let $d$ be a proper divisor of $a$. Since $\text{Nil}(R)$ is a divided ideal of $R$ and $b \not| a$, we conclude that $d \not\in \text{Nil}(R)$. Thus, since $dM$ is a divided ideal by Proposition 2.6 and $b \not| a$ in $R$, $b \in dM$. Conversely, suppose that $b \not| a$ in $R$ for some $a, b \in R \setminus \text{Nil}(R)$. We need to show that $a \mid bm$ for each nonunit $m \in R$. Suppose that $a \not| bm$ for some nonunit $m \in R$. Since $b$ is a proper divisor of $bm$, $b \mid a$ which is a contradiction. Hence, $a \mid bm$ for each nonunit $m \in R$. Thus, $R$ is a $\phi$-PVR by Proposition 1.1(5).

In the following result, we make a connection between $\phi$-PVR’s and PVR’s.

**PROPOSITION 2.9.** A ring $R$ is a $\phi$-PVR if and only if $\text{Nil}(R)$ is a divided prime ideal of $R$ and $R/\text{Nil}(R)$ is a PVR.

**Proof.** Suppose that $R$ is a $\phi$-PVR. Then $\text{Nil}(R)$ is a divided prime ideal of $R$ by Proposition 1.1(6). By applying Proposition 1.1(4) to the ring $R/\text{Nil}(R)$, one can conclude that $R/\text{Nil}(R)$ is a PVR. Conversely, suppose that $\text{Nil}(R)$ is a divided prime ideal of $R$ and $R/\text{Nil}(R)$ is a PVR. Let $a, b \in R \setminus \text{Nil}(R)$ and $c$ be a nonunit of $R$. Then it is easy to see that $c + \text{Nil}(R)$ is a nonunit of $R/\text{Nil}(R)$. Hence, by Proposition 1.1(4) either $a + \text{Nil}(R) \mid b + \text{Nil}(R)$ in $R/\text{Nil}(R)$ or $b + \text{Nil}(R) \mid ac + \text{Nil}(R)$ in $R/\text{Nil}(R)$. Suppose that $a + \text{Nil}(R) \mid b + \text{Nil}(R)$ in $R/\text{Nil}(R)$. Then $b = ak + w$ in $R$ for some $w \in \text{Nil}(R)$ and $k \in R$. Since $\text{Nil}(R)$ is a divided prime ideal of $R$ and $a \notin \text{Nil}(R)$, $a \mid w$. Thus, $a \mid b$ in $R$. Now, assume that $b + \text{Nil}(R) \mid ac + \text{Nil}(R)$ in $R/\text{Nil}(R)$. Then by an argument similar to the one just given we conclude that $b \mid ac$ in $R$. Thus, $R$ is a $\phi$-PVR by Proposition 1.1(5).

3. $\phi$-PVRS and $\phi$-CRS

Let $VD$ denote a valuation domain and $CR$ denote a chained ring. We then have the following implications, none of which are reversible.

$$VD \Rightarrow PV D \Rightarrow PVR \Rightarrow \phi - PVR$$

AND

$$VD \Rightarrow CR \Rightarrow \phi - CR \Rightarrow \phi - PVR.$$

We start with the following lemma.

**LEMMA 3.1.** Let $R$ be a $\phi$-PVR, and let $P$ be a prime ideal of $R$. Then $x^{-1}P \subseteq P$ for each $x \in T(R) \setminus R$.

**Proof.** Let $x = a/b \in T(R) \setminus R$ for some $a \in R$ and for some $b \in R \setminus Z(R)$. Since $b \not| a$ in $R$ and $Z(R)$ is a divided prime ideal by Proposition 1.1(6), we conclude that $a \in R \setminus Z(R)$. Hence, $x^{-1} = b/a \in T(R)$. Now, let $p \in P$. Then $x(x^{-1}p) = p \in P$. Hence, $\phi(x^{-1}p) = \phi(x)\phi(x^{-1}p) = \phi(p) \in \phi(P)$. Since $\phi(P)$ is a $K$-strongly prime ideal of $\phi(R)$ and by [8, Proposition 3(3)] $\phi(x) \notin \phi(P)$, we conclude that $\phi(x^{-1}p) \in \phi(P)$. Thus, $\phi(x^{-1}p) = \phi(q)$ for some $q \in P$. Hence, $x^{-1}p - q \in \text{Ker}(\phi)$. Since $q \in P$, $\text{Ker}(\phi) \subseteq \text{Nil}(R)$ by [8, Proposition 2(1)], and $\text{Nil}(R) \subseteq P$, we conclude that $x^{-1}p \in P$. \qed
PROPOSITION 3.2. Let $R$ be a $\phi$-PVR and $z \in T(R) \setminus R$ be integral over $R$. Then there is a minimal monic polynomial $f(x) \in R[x]$ such that $f(z) = 0$ and all nonzero coefficients of $f(x)$ are units in $R$. Furthermore, if $g(x)$ is a minimal monic polynomial in $R[x]$ such that $g(z) = 0$, then $g(0)$ is a unit in $R$.

Proof. Let $g(x)$ be a minimal monic polynomial in $R[x]$ such that $g(z) = 0$. Suppose that $a_0$, the constant term of $g(x)$, is a nonunit of $R$. Since $z \in T(R) \setminus R$ is integral over $R$, $z^{-1} \not\in R$. Hence, by Lemma 3.1, $z^{-1}a_0 = m$ is a nonunit of $R$. Thus, $mz = a_0$. Hence, we can replace the constant term $a_0$ in $g(x)$ with $mz$. Thus, we may factor $x$ from all terms of $g(x)$ and get a monic polynomial $H(x)$ of less degree than $g(x)$ such that $H(z) = 0$, a contradiction. Hence, $a_0$ is a unit in $R$.

Now, assume that $c_0x^k$ is a term in $g(x)$ such that $c_0$ is a nonunit of $R$. Since $z^k$ is integral over $R$, $z^{-k} \not\in R$. Hence, by Lemma 3.1, $c_0z^{-k} = s$ is a nonunit of $R$. Thus, we may replace the term $c_0z^{-k}$ in $g(x)$ with $s$. Since $s$ is a nonunit of $R$ and $a_0$ is a unit in $R$ and $R$ is quasilocal, $s + a_0$ is a unit in $R$. Continuing in this manner, we get a minimal monic polynomial $f(x)$ such that $f(z) = 0$ and all nonzero coefficients of $f(x)$ are units in $R$. The remaining part of the Proposition follows directly from the first part of our proof.

It is well-known ([16],[5],[8],[11]) that the integral closure of a PVR is a PVR. In view of the above result, one can give an alternative proof of this fact. For a ring $R$, let $R'$ denotes the integral closure of $R$ in $T(R)$.

PROPOSITION 3.3. Let $R$ be a $\phi$-PVR with maximal ideal $M$, and let $B$ be an overring of $R$ such that $B \subset R'$. Then $B$ is a $\phi$-PVR with maximal ideal $M$.

Proof. Let $x \in B \setminus R$. Hence, $x^{-1} \in R'$ by Proposition 3.2. Thus, $x^{-1} \in R[x] \subset B$ by [18, Theorem 15]. Hence, $x$ is a unit in $B$. Since $1/s$ is never integral over $R$ for any $s \in M$ and any $x \in B \setminus R$ is a unit in $B$, $M$ is the maximal ideal of $B$. Thus, by applying Proposition 1.1(5) to the ring $B$, we conclude that $B$ is a $\phi$-PVR with maximal ideal $M$.

PROPOSITION 3.4. Let $R$ be a $\phi$-PVR with maximal ideal $M$, and let $B$ be an overring of $R$. Then the following statements are equivalent:

1. $B = R_P$ is a $\phi$-CR for some nonmaximal prime ideal $P$ of $R$.
2. $IB = B$ for some proper ideal $I$ of $R$.
3. $1/s \in B$ for some nonzero divisor $s \in M$.

Proof. (1) $\Rightarrow$ (2). No comments.

(2) $\Leftrightarrow$ (3). This is clear by [10, Proposition 3.6].

(3) $\Rightarrow$ (1). Suppose that $B$ contains an element of the form $1/s$ for some nonzero divisor $s \in M$. Then by [10, Proposition 3.8] $B$ is a $\phi$-CR, and hence is quasilocal. Thus, let $N$ be the maximal ideal of $B$, and let $P = N \cap R$. Since $s \not\in P$, $P$ is a nonmaximal prime ideal of $R$. Clearly, $Z(R) \subset P$. Hence, $R_P \subset B$. Now, let $x \in B \setminus R$. If $x^{-1} \in R$, then $x = 1/d$ for some $d \in R \setminus P$. Thus, $x \in R_P$. Thus, assume that $x^{-1} \not\in R$. Hence, $xs = m \in M$ by Lemma 3.1. Thus, $x = m/s \in R_P$. Hence, $B \subset R_P$. □
The following result is a generalization of [10, Theorem 3].

**COROLLARY 3.5.** Let \( R \) be a \( \phi \)-PVR with maximal ideal \( M \), and let \( B \) be an overring of \( R \) such that \( B \) is a \( \phi \)-CR with maximal ideal \( N \). If \( P = N \cap R \neq M \), then \( B = R_P \).

**Proof.** Since \( P \neq M \), \( B \) contains an element of the form \( 1/s \) for some nonzerodivisor \( s \in M \). Hence, by the above proposition, the proof is complete. \( \square \)

The proof of the following result is similar to that in [4, Theorem 2.1]. Hence, we invite the reader to finish the proof.

**PROPOSITION 3.6.** Let \( R \) be a \( \phi \)-PVR with maximal ideal \( M \) and \( u \in (M : M) \setminus R \). Then \( R[u] \) is a \( \phi \)-PVR if and only if \( R[u] \) is quasi-local. Furthermore, if \( R[u] \) is quasi-local for some \( u \in (M : M) \setminus R \), then \( R[u] \) is a \( \phi \)-PVR with maximal ideal \( M \).

**PROPOSITION 3.7.** Let \( R \) be a \( \phi \)-PVR with maximal ideal \( M \). If \( C \) is an overring of \( R \) such that \( C \) does not contain an element of the form \( 1/s \) for some nonzerodivisor \( s \in M \), then \( C \subset (M : M) \).

**Proof.** Let \( x \in C \setminus R \). By hypothesis, \( x^{-1} \notin R \). Hence, \( xM \subset M \) by Lemma 3.1. Thus, \( x \in (M : M) \). \( \square \)

**COROLLARY 3.8.** Let \( R \) be a \( \phi \)-PVR with maximal ideal \( M \). Then every overring of \( R \) is a \( \phi \)-PVR if and only if \( R[u] \) is quasi-local for each \( u \in (M : M) \setminus R \).

**Proof.** Suppose that \( R[u] \) is quasi-local for each \( u \in (M : M) \setminus R \). Let \( C \) be an overring of \( R \). If \( C \) contains an element of the form \( 1/s \) for some nonzerodivisor \( s \in M \), then \( C \) is a \( \phi \)-PVR by Proposition 3.4. Hence, assume that \( C \) does not contain an element of the form \( 1/s \) for some nonzerodivisor \( s \in M \). Hence, \( C \subset (M : M) \) by Proposition 3.7. Let \( u \in C \setminus R \). Then \( R[u] \) is quasi-local by hypothesis. Hence, by Proposition 3.6, \( M \) is the maximal ideal of \( R[u] \). Thus, \( u^{-1} \in R[u] \subset C \). Hence, \( M \) is the maximal ideal of \( C \). Thus, by applying Proposition 1.1(5) to the ring \( C \), we conclude that \( C \) is a \( \phi \)-PVR. \( \square \)

We recall the following result.

**LEMMA 3.9.** [10, Proposition 2.3] A ring \( R \) is a \( \phi \)-CR if and only if \( \text{Nil}(R) \) is a divided prime ideal of \( R \) and for every \( a, b \in R \setminus \text{Nil}(R) \), either \( a \mid b \) in \( R \) or \( b \mid a \) in \( R \).

Recall that an ideal of \( R \) is called regular if it contains a nonzerodivisor of \( R \). If every regular ideal of \( R \) is generated by its set of nonzerodivisors, then \( R \) is called a Marot ring. Also, recall that a ring \( R \) has few zerodivisors if \( Z(R) \) is a finite union of prime ideals. We have the following result which is a generalization of [10, Proposition 6].

**PROPOSITION 3.10.** Let \( R \) be a \( \phi \)-PVR. Then:

1. \( R \) is a Marot ring.
2. If \( R \neq T(R) \), then \( T(R) \) is a \( \phi \)-CR.
Proof. (1). Since $Z(R)$ is a prime ideal of $R$ by Proposition 1.1(6), $R$ has few zero divisors. Hence, $R$ is a Marot ring by [17, Theorem 7.2].

(2). Since $\text{Nil}(R)$ is a divided prime ideal of $R$, $\text{Nil}(R) = \text{Nil}(R)$. Now, let $x, y \in T(R) \setminus \text{Nil}(R)$. Then $x = a/s$ and $y = b/s$ for some $a, b \in R \setminus \text{Nil}(R)$ and $s \in R \setminus Z(R)$. By Lemma 3.9, we need to show that either $x | y$ in $T(R)$ or $y | x$ in $T(R)$. If $a | b$ in $R$, then $x | y$ in $T(R)$. Hence, assume that $a \nmid b$ in $R$. Since $R$ is a $\phi$-PVR and $R \neq T(R)$, $b | ad$ in $R$ for some $d \in M \setminus Z(R)$. Thus, $ad = bc$ for some $c \in R$. Thus, $a/s = (b/s)(c/d)$. Thus, $y | x$ in $T(R)$. \qed

REMARK 3.11. Let $R$ be a $\phi$-PVR with maximal ideal $M$ such that $M$ contains a nonzerodivisor of $R$, and let $I$ be a proper ideal of $R$. Then, since $V = (M : M)$ is a $\phi$-CR with maximal ideal $M$, it is easy to see that there is a $\phi$-CR $V$ between $R$ and $T(R)$ such that $IV \neq V$.

The proof of the following result starts exactly as in [18, Theorem 56].

THEOREM 3.12. Let $R$ be a $\phi$-PVR with maximal ideal $M$ such that $M$ contains a nonzerodivisor of $R$, let $C$ be an overring of $R$ ($R \subset C \subset T(R)$), and let $I$ be a proper ideal of $C$. Then there exists a $\phi$-CR $B$ such that $C \subset B \subset T(R)$ and $IB \neq B$.

Proof. Consider all pairs $(C_\alpha, I_\alpha)$, where $C_\alpha$ is a ring between $C$ and $T(R)$, and $I_\alpha \subsetneq C_\alpha, I \subsetneq I_\alpha$. We partially order the pairs by decreasing inclusion to mean both $C_\alpha \supsetneq C_\beta$ and $I_\alpha \supsetneq I_\beta$. Zorn’s Lemma is applicable to yield a maximal pair $(B, J)$. To show that $B$ is a $\phi$-CR, by Lemma 3.9, we only need to show that $\text{Nil}(B)$ is a divided prime ideal of $B$ and for every $a, b \in B$ either $a | b$ in $B$ or $b | a$ in $B$. Clearly, $IB \neq B, C \subset B \subset T(T)$, and $\text{Nil}(B) = \text{Nil}(R)$ is a divided prime ideal of $B$. Let $x \in T(R) \setminus R$. Since $R$ is a divided ring by Proposition 1.1(2) and $x \notin R$, $x = a/b$ for some nonzerodivisors $a, b$ of $R$. Hence, $x$ is a unit in $T(R)$. Thus, $JB[x] \neq B[x]$ or $JB[x^{-1}] \neq B[x^{-1}]$ by [18, Theorem 55]. Hence, by the maximality of the pair $(B, J)$, either $x \in B$ or $x^{-1} \in B$. Thus, if $x, y \in B \setminus R$, then $x | y$ or $y | x$ in $B$. Now, let $a, b \in R$ and suppose that $a \nmid b$ in $R$. Since $R$ is a $\phi$-PVR and $M$ contains a nonzerodivisor of $R$, $b \mid as$ for some nonzerodivisor $s \in M$. Thus, $as = bc$ for some $c \in M$. Suppose that $c \in Z(R)$. Since $Z(R)$ is a divided prime ideal of $R$ and $s \notin Z(R)$, $s \notin c$ in $R$. Hence, $b \mid a$ in $R$ and therefore $b \mid c$ in $R$. Now, assume that $c \notin Z(R)$. If $s | c$ in $R$, then, once again, $b \mid a$ in $R$ and we are done. Thus, suppose that $s \nmid c$ in $R$. Then $x = c/s \in T(R) \setminus R$, and hence either $x \in B$ or $x^{-1} \in B$ as we have shown earlier in the proof. Thus, either $b | a$ in $B$ or $a | b$ in $B$. $\text{Finally, suppose that } a \in R$ and $b \in B \setminus R$. Write $b = c/d$ for some $c \in R$ and $d \in R \setminus Z(R)$. Since $Z(R)$ is a divided ideal of $R$ by Proposition 1.1(6) and $b = c/d \notin R$, we conclude that $c \in R \setminus Z(R)$. If $a \in Z(R)$, then $c | a$ in $R$ and hence $b | a$ in $B$. Thus, assume that $a \notin Z(R)$. Let $x = ad/c$. If $x \notin R$, then $b | a$ in $B$. Otherwise, $x \in T(R) \setminus R$. Hence, either $x \in B$ or $x^{-1} \in B$ as we have shown earlier in the proof. Thus, either $b | a$ in $B$ or $a | b$ in $B$. Hence, $B$ is a $\phi$-CR by Lemma 3.9. \qed

PROPOSITION 3.13. Let $R$ be a $\phi$-PVR and $B$ be an overring of $R$ such that $B$ is a $\phi$-CR. Then $R' \subset B$. \qed
Proof. Deny. Then there is an \( x \in R' \setminus B \). Hence, since \( R' \) is a \( \phi \)-PVR with maximal ideal \( M \) by Proposition 3.3, \( x \) is a unit in \( R' \). Since \( x \not\in B \) and \( B \) is a \( \phi \)-CR, \( x^{-1} \in B \). Since \( x \in R' \), \( x \in R[x^{-1}] \) by [18, Theorem 15]. Hence, \( x \in R[x^{-1}] \subset B \), which is a contradiction. Thus, \( R' \supseteq B \).

**THEOREM 3.14.** Let \( R \) be a \( \phi \)-PVR with maximal ideal \( M \) such that \( M \) contains a nonzerodivisor. Then \( R' \) is the intersection of all the \( \phi \)-CRs between \( R \) and \( T(R) \).

Proof. By Proposition 3.13, \( R' \) is contained in the intersection of all the \( \phi \)-CRs between \( R \) and \( T(R) \). Let \( y \in \) the intersection of all the \( \phi \)-CRs between \( R \) and \( T(R) \); we must show that \( y \in R' \). Suppose not. By [18, Theorem 15], \( y \not\in C = R[y^{-1}] \). Let \( I = y^{-1}C \). Then \( I \) is a proper ideal of \( C \). By Theorem 3.12 there is a \( \phi \)-CR \( B \) between \( C \) and \( T(R) \) such that \( IB \neq B \). But by hypothesis \( y \in B \), and we have our contradiction.

The following result is a generalization of [12, Theorem 8].

**THEOREM 3.15.** Let \( R \) be a \( \phi \)-PVR with maximal ideal \( M \). Then every overring of \( R \) is a \( \phi \)-PVR if and only if every \( \phi \)-CR between \( R \) and \( T(R) \) other than \( (M : M) \) is of the form \( R_P \) for some nonmaximal prime ideal \( P \) of \( R \).

Proof. If \( T(R) = R \), then there is nothing to prove. Hence, assume that \( M \) contains a nonzerodivisor of \( R \). Suppose that every overring of \( R \) is a \( \phi \)-PVR. Then \( R' = (M : M) \) by [8, Proposition 15(1)]. Let \( C \) be an overring of \( R \) such that \( C \neq (M : M) \) and \( C \) is a \( \phi \)-CR. Since every overring of \( R \) not containing an element of the form \( 1/s \) for some nonzerodivisor \( s \) of \( R \) is contained in \( R' = (M : M) \) by Proposition 3.7 and hence is a \( \phi \)-PVR with maximal ideal \( M \) by Proposition 3.3 and \( (M : M) \) is the only \( \phi \)-CR between \( R \) and \( T(R) \) that has maximal ideal \( M \) by [10, Lemma 3.1(1)], \( C \not\subset R' = (M : M) \). Thus, \( C \) must contain an element of the form \( 1/s \) for some nonzerodivisor \( s \in M \). Hence, \( C = R_P \) for some nonmaximal prime ideal \( P \) of \( R \) by Proposition 3.4.

Conversely, suppose that every \( \phi \)-CR between \( R \) and \( T(R) \) other than \( (M : M) \) is of the form \( R_P \) for some nonmaximal prime ideal \( P \) of \( R \). Then \( (M : M) \) is contained in every \( \phi \)-CR between \( R \) and \( T(R) \). Hence, \( (M : M) \) is the intersection of all the \( \phi \)-CRs between \( R \) and \( T(R) \). Thus, by Theorem 3.14, \( R' = (M : M) \). Hence, every overring of \( R \) is a \( \phi \)-PVR by [8, Proposition 15(1)].

In light of [8, Proposition 15(1)] and the above Theorem, we have the following result.

**COROLLARY 3.16.** Let \( R \) be a \( \phi \)-PVR with maximal ideal \( M \) such that \( R' \neq (M : M) \). Then there is a \( \phi \)-CR that is properly contained between \( R' \) and \( (M : M) \).

Combining [8, Proposition 15(1)], Proposition 3.3, Proposition 3.4, Proposition 3.8, and Theorem 3.14, we arrive at the following result that is a generalization of ([4, Corollary 2.2], [12, Theorem 8], and [11, Corollary 17]).

**COROLLARY 3.17.** Let \( R \) be a \( \phi \)-PVR with maximal ideal \( M \). Then the following statements are equivalent:
1. Every overring of $R$ is a $\phi$-PVR.
2. $R[u]$ is a $\phi$-PVR for each $u \in (M : M) \setminus R$.
3. $R[u]$ is quasilocal for each $u \in (M : M) \setminus R$.
4. If $B$ is an overring of $R$ and $B \subseteq (M : M)$, then $B$ is a $\phi$-PVR with maximal ideal $M$.
5. If $B$ is an overring of $R$ and $B \subseteq (M : M)$, then $B$ is quasilocal.
6. Every overring of $R$ is quasilocal.
7. Every $\phi$-CR between $R$ and $T(R)$ other than $(M : M)$ is of the form $Rp$ for some nonmaximal prime ideal $P$ of $R$.
8. $R' = (M : M)$.

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