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ON THE ZERO-DIVISOR GRAPH OF A RING

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Let R be a commutative ring with identity, $Z(R)$ its set of zero-divisors, and $Nil(R)$ its ideal of nilpotent elements. The zero-divisor graph of R is $\Gamma(R) = Z(R)\setminus\{0\}$, with distinct vertices x and y adjacent if and only if $xy = 0$. In this article, we study $\Gamma(R)$ for rings R with nonzero zero-divisors which satisfy certain divisibility conditions between elements of R or comparability conditions between ideals or prime ideals of R. These rings include chained rings, rings R whose prime ideals contained in $Z(R)$ are linearly ordered, and rings R such that $\{0\} \neq Nil(R) \subseteq zR$ for all $z \in Z(R)\setminus Nil(R)$.

Key Words: Chained rings; Linearly ordered prime ideals; ϕ-Rings; Zero-divisor graph.

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1. INTRODUCTION

Let R be a commutative ring with 1, and let $Z(R)$ be its set of zero-divisors. The *zero-divisor graph* of R, denoted by $\Gamma(R)$, is the (undirected) graph with vertices $Z(R)^* = Z(R)\setminus\{0\}$, the set of nonzero zero-divisors of R, and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$. Note that $\Gamma(R)$ is the empty graph if and only if R is an integral domain and that a nonempty $\Gamma(R)$ is finite if and only if R is finite and not a field (Anderson and Livingston, 1999, Theorem 2.2). This concept is due to Beck (1988), who let all the elements of R be vertices and was mainly interested in colorings. Our present definition and emphasis on the interplay between ring-theoretic properties of R and graph-theoretic properties of $\Gamma(R)$ are from Anderson and Livingston (1999).

In this article, we study $\Gamma(R)$ for several classes of rings which generalize valuation domains to the context of rings with zero-divisors. These are rings with nonzero zero-divisors that satisfy certain divisibility conditions between elements or comparability conditions between ideals or prime ideals. In Sections 2 and 3, we consider rings R such that the prime ideals of R contained in $Z(R)$ are linearly ordered. In particular, we compute the diameter and girth for $\Gamma(R)$ and $\Gamma(R[X])$.

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In Section 4, we specialize to the case where R is a chained ring. In the final section, we investigate $\Gamma(R)$ for rings R such that $\{0\} \neq Nil(R) \subseteq zR$ for all $z \in Z(R)\backslash Nil(R)$.

We assume throughout that all rings are commutative with $1 \neq 0$. If R is a ring, then $\dim(R)$ denotes its (Krull) dimension, $T(R)$ its total quotient ring, $U(R)$ its group of units, $Z(R)$ its set of zero-divisors, $Nil(R)$ its ideal of nilpotent elements, $N(R) = \{x \in R \mid x^2 = 0\} \subseteq Nil(R)$, and $Rad(I) = \{x \in R \mid x^n \in I$ for some integer $n \geq 1\}$ for I an ideal of R. We say that R is *reduced* if $Nil(R) = \{0\}$. For $A, B \subseteq R$, let $A^* = A\backslash\{0\}$ and $(A : B) = \{x \in R \mid xB \subseteq A\}$. We let $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{Z}_{(p)}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{F}_q$ denote the rings of integers, integers modulo n, integers localized at the prime ideal $p\mathbb{Z}$, rational numbers, real numbers, and the finite field with q elements, respectively. In the next six paragraphs, we recall some background material. To avoid any trivialities when $\Gamma(R)$ is the empty graph, we implicitly assume when necessary that R is not an integral domain. For any undefined ring-theoretic concepts or terminology, see Huckaba (1988) or Kaplansky (1974).

Let G be a graph. We say that G is *connected* if there is path between any two distinct vertices of G. At the other extreme, we say that G is *totally disconnected* if no two vertices of G are adjacent. For vertices x and y of G, we define $d(x, y)$ to be the length of a shortest path from x to y in G ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path). The *diameter* of G is $\text{diam}(G) = sup\{d(x, y) \mid x$ and y are vertices of $G\}$. The *girth* of G, denoted by $\text{gr}(G)$, is the length of a shortest cycle in G ($\text{gr}(G) = \infty$ if G contains no cycles). Then $\Gamma(R)$ is connected with $\text{diam}(\Gamma(R)) \leq 3$ (Anderson and Livingston, 1999, Theorem 2.3) and $\text{gr}(\Gamma(R)) \leq 4$ if $\Gamma(R)$ contains a cycle (Mulay, 2002, (1.4)). Thus $\text{diam}(\Gamma(R)) = 0, 1, 2,$ or 3, and $\text{gr}(\Gamma(R)) = 3, 4,$ or ∞. For other articles on zero-divisor graphs, see Anderson and Naseer (1993), Anderson (2008), Anderson et al. (2001, 2003), Anderson and Mulay (2007), Axtel et al. (2005), Axtel and Stickles (2006), DeMeyer and Schneider (2002), Lucas (2006), Mulay (2002), Redmond (2007), and Smith (2003). In particular, a list of all the zero-divisor graphs with up to 14 vertices is given in Redmond (2007). A general reference for graph theory is Bollaboás (1979).

Recall from Hedstrom and Houston (1978) that an integral domain R with quotient field K is called a *pseudo-valuation domain (PVD)* if every prime ideal P of R is *strongly prime*, in the sense that whenever $x, y \in K$ and $xy \in P$, then $x \in P$ or $y \in P$. This concept was extended to rings with zero-divisors in Badawi et al. (1995), where R is called a *pseudo-valuation ring (PVR)* if every prime ideal P of R is *strongly prime*, in the sense that xP and yR are comparable (under inclusion) for all $x, y \in R$. Any valuation domain is a PVD, and it was shown in Badawi et al. (1995) that an integral domain is a PVD if and only if it is a PVR. It is known that a ring R is a PVR if and only if for all $x, y \in R$, we have either $x \mid y$ or $y \mid xz$ for every nonunit $z \in R$ (Badawi et al., 1995, Theorem 5). We say that a ring R is a *chained ring* if the (principal) ideals of R are linearly ordered (by inclusion), equivalently, if either $x \mid y$ or $y \mid x$ for all $x, y \in R$. By our earlier comments, a chained ring is a PVR.

Another generalization of pseudo-valuation rings is given in Badawi (1999b). Recall from Dobbs (1976) and Badawi (1999a) that a prime ideal P of a ring R is called a *divided prime ideal* R if $P \subseteq xR$ for all $x \in R\backslash P$. Thus a divided prime ideal of R is comparable with every ideal of R. We say that a ring R is a *divided ring* if every prime ideal of R is divided; so the prime ideals in a divided ring are linearly ordered. Let $\mathcal{H} = \{R \mid R$ is a ring and $Nil(R)$ is a divided prime ideal of $R\}$. Note that an integral domain or a PVR is in $\mathcal{H}$. For any ring $R \in \mathcal{H}$, the ring homomorphism $\phi = \phi_R : T(R) \longrightarrow R_{Nil(R)}$, given by $\phi(x/y) = x/y$ for all $x \in R$ and $y \in R\backslash Z(R)$, was introduced in Badawi (1999b). Then $\phi|_R : R \longrightarrow R_{Nil(R)}$ is a ring homomorphism satisfying $\phi(x) = x/1$ for all $x \in R$ and $T(\phi(R)) = R_{Nil(R)}$.

Let $R \in \mathscr{H}$, and put $K = R_{Nil(R)}$. As in Badawi (1999b), a prime ideal Q of $\phi(R)$ is said to be *K-strongly prime* if whenever $x, y \in K$ and $xy \in Q$, then either $x \in Q$ or $y \in Q$. A prime ideal P of R is said to be a *ϕ-strongly prime ideal* of R if $\phi(P)$ is a K-strongly prime ideal of $\phi(R)$. It is known that the prime ideals of $\phi(R)$ are the sets that are (uniquely) expressible as $\phi(P)$ for some prime ideal P of R (cf. Badawi, 1999b, Lemma 2.5), the key fact being that $Ker(\phi) \subseteq Nil(R)$. If every prime ideal of R is a ϕ-strongly prime ideal, then R is called a *ϕ-pseudo-valuation ring* (*ϕ-PVR*). It was shown in Badawi (2002, Proposition 2.9) that a ring $R \in \mathscr{H}$ is a ϕ-PVR if and only if $R/Nil(R)$ is a PVD. A PVR is a ϕ-PVR, but an example of a ϕ-PVR which is not a PVR was given in Badawi (2000). Also, a ϕ-PVR is a divided ring Badawi (1999b, Proposition 4), and thus the prime ideals in a ϕ-PVR (or a PVR) are linearly ordered. In particular, a ϕ-PVR, and hence a PVR or a chained ring, is quasilocal.

Observe that if $Nil(R)$ is a divided prime ideal of R, then $Nil(R)$ is also the nilradical of $T(R)$ and $Ker(\phi)$ is a common ideal of R and $T(R)$. Other useful features of each ring $R \in \mathscr{H}$ include the following: (i) $\phi(R) \in \mathscr{H}$; (ii) $T(\phi(R)) = R_{Nil(R)}$ has only one prime ideal, namely, $Nil(\phi(R))$; (iii) $\phi(R)$ is naturally isomorphic to $R/Ker(\phi)$; (iv) $Z(\phi(R)) = Nil(\phi(R)) = \phi(Nil(R)) = Nil(R_{Nil(R)})$; and (v) $R_{Nil(R)}/Nil(\phi(R)) = T(\phi(R))/Nil(\phi(R))$ is the quotient field of $\phi(R)/Nil(\phi(R))$. For further studies on rings in the class $\mathscr{H}$, see Anderson and Badawi (2004), Anderson and Badawi (2005), Badawi (1999b), Badawi (2000), Badawi (2001), Badawi (2002), Badawi and Dobbs (2006), and Badawi and Lucas (2006).

Throughout this article, we will use the technique of idealization of a module to construct examples. Recall that for an R-module B, the *idealization of B over R* is the ring formed from $R \times B$ by defining addition and multiplication as $(r, a) + (s, b) = (r + s, a + b)$ and $(r, a)(s, b) = (rs, rb + sa)$, respectively. A standard notation for this "idealized ring" is $R(+)B$; see Huckaba (1988) for basic properties of rings resulting from the idealization construction. In particular, note that the ideal $I = \{0\}(+)B$ of $T = R(+)B$ satisfies $I^2 = \{0\}$; so $I \subseteq Nil(T)$. The zero-divisor graph $\Gamma(R(+)B)$ has recently been studied in Anderson and Mulay (2007) and Axtel and Stickles (2006).

2. LINEARLY ORDERED PRIMES

In this section, we investigate the zero-divisor graph of a ring R such that the prime ideals of R contained in $Z(R)$ are linearly ordered. These are precisely the rings R such that the prime ideals of $T(R)$ are linearly ordered, and include chained rings, divided rings, PVRs, ϕ-PVRs, rings with $Z(R) = Nil(R)$, and zero-dimensional quasilocal rings. For these rings, we show that $\text{diam}(\Gamma(R)) \leq 2$ and $\text{gr}(\Gamma(R)) = 3$ or ∞. We start with the following lemma (cf. Anderson, 2008, Lemma 3.1; Lucas, 2006, Lemma 2.3).

Lemma 2.1. *Let R be a ring, and let $x, y \in Nil(R)^*$ be distinct with $xy \neq 0$. Then $(0 : (x, y)) \neq \{0\}$, and moreover, there is a path of length 2 from x to y in $Nil(R)^* \subseteq \Gamma(R)$. In particular, if $Z(R) = Nil(R)$, then* $\text{diam}(\Gamma(R)) \leq 2$.

Proof. Since $xy \neq 0$ and $x \in Nil(R)^*$, let n (≥ 2) be the least positive integer such that $x^n y = 0$. Also, since $x^{n-1}y \neq 0$ and $y \in Nil(R)^*$, let m (≥ 2) be the least positive integer such that $x^{n-1}y^m = 0$. Then $0 \neq x^{n-1}y^{m-1} \in Nil(R)$ and $x^{n-1}y^{m-1} \in (0 : (x, y))$. Thus $x - x^{n-1}y^{m-1} - y$ is a path of length 2 from x to y in $Nil(R)^*$. The "in particular" statement is clear. □

When $Z(R) = Nil(R)$, it is easy to explicitly describe the diameter of $\Gamma(R)$; and moreover, $\operatorname{diam}(\Gamma(R)) \neq 3$ in this case. We record this as our first theorem (cf. Lucas, 2006, Theorem 2.6). Note that in this case, $Nil(R)$ is the unique minimal prime ideal of R and is the only prime ideal of R contained in $Z(R)$; so this is the simplest case where the prime ideals of R contained in $Z(R)$ are linearly ordered.

Theorem 2.2. *Let R be a ring with $Z(R) = Nil(R) \neq \{0\}$. Then exactly one of the following three cases must occur.*

(1) $|Z(R)^*| = 1$. *In this case, R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, and* $\operatorname{diam}(\Gamma(R)) = 0$;
(2) $|Z(R)^*| \geq 2$ *and* $Z(R)^2 = \{0\}$. *In this case, $\Gamma(R)$ is a complete graph, and* $\operatorname{diam}(\Gamma(R)) = 1$;
(3) $Z(R)^2 \neq \{0\}$. *In this case,* $\operatorname{diam}(\Gamma(R)) = 2$.

Proof. (1) If $|Z(R)^*| = 1$, then $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ (Beck, 1988, Proposition 2.2). Thus $\operatorname{diam}(\Gamma(R)) = 0$.

(2) If $Z(R)^2 = \{0\}$, then $xy = 0$ for all $x, y \in Z(R)$. Thus $\Gamma(R)$ is a complete graph with $\operatorname{diam}(\Gamma(R)) = 1$ since $|Z(R)^*| \geq 2$.

(3) Suppose that $Z(R)^2 \neq \{0\}$. Then $\Gamma(R)$ is not complete (Anderson and Livingston, 1999, Theorem 2.8), and thus $\operatorname{diam}(\Gamma(R)) \geq 2$. Hence $\operatorname{diam}(\Gamma(R)) = 2$ by Lemma 2.1. □

Thus when studying the diameter of the zero-divisor graph of a ring R, the interesting case is when $Nil(R) \subsetneq Z(R)$. We next give several lemmas. Note that in Lemma 2.4 we need only assume that $x \in Z(R) \backslash N(R)$, where $N(R) = \{x \in R \mid x^2 = 0\}$.

Lemma 2.3. *Let R be a ring with $x \in Nil(R)^*$ and $y \in Z(R)^*$. Then $d(x, y) \leq 2$ in $\Gamma(R)$.*

Proof. We may assume that $x \neq y$ and $xy \neq 0$. Since $y \in Z(R)^*$ and $xy \neq 0$, there is a $z \in Z(R)^* \backslash \{x\}$ such that $yz = 0$. Let n be the least positive integer such that $x^n z = 0$ (such an n exists since $x \in Nil(R)^*$). Then $x - x^{n-1}z - y$ is a path of length 2 from x to y (if $n = 1$, then $x^{n-1}z = z$). Thus $d(x, y) \leq 2$ in $\Gamma(R)$. □

Lemma 2.4. *Let R be a ring with $x \in Z(R) \backslash Nil(R)$ and $y \in Z(R)^*$ such that $x \mid zy^n$ for some integer $n \geq 1$ and $z \in R \backslash Z(R)$. Then $d(x, y) \leq 2$ in $\Gamma(R)$.*

Proof. We may assume that $x \neq y$ and $xy \neq 0$. Since $x \in Z(R) \backslash Nil(R)$ and $xy \neq 0$, there is a $w \in Z(R)^* \backslash \{x, y\}$ such that $xw = 0$. Since $x \mid zy^n$ with $z \in R \backslash Z(R)$ and $xw = 0$, we conclude that $y^n w = 0$. Let k be the least positive integer such that $y^k w = 0$. Then $x - y^{k-1}w - y$ is a path of length 2 from x to y. Thus $d(x, y) \leq 2$ in $\Gamma(R)$. □

By Badawi (1995, Theorem 1), the prime ideals of R are linearly ordered if and only if the radical ideals of R are linearly ordered, if and only if for all $x, y \in R$, there is an integer $n = n(x, y) \geq 1$ such that either $x \mid y^n$ or $y \mid x^n$. This result easily extends to the prime ideals of R contained in $Z(R)$.

Theorem 2.5. *Let R be a ring.*

(1) *The prime ideals of R contained in $Z(R)$ are linearly ordered if and only if for all $x, y \in Z(R)$, there is an integer $n = n(x, y) \geq 1$ and an element $z \in R \backslash Z(R)$ such that either $x \mid zy^n$ or $y \mid zx^n$.*
(2) *The radical ideals of R contained in $Z(R)$ are linearly ordered if and only if for all $x, y \in Z(R)$, there is an integer $n = n(x, y) \geq 1$ such that either $x \mid y^n$ or $y \mid x^n$.*
(3) *If the prime ideals of R contained in $Z(R)$ are linearly ordered, then $Nil(R)$ and $Z(R)$ are prime ideals of R.*

Proof. (1) Note that the prime ideals of R contained in $Z(R)$ are linearly ordered if and only if the prime ideals of $T(R)$ are linearly ordered, if and only if for all $x, y \in T(R)$, there is an integer $n = n(x, y) \geq 1$ such that either $x \mid y^n$ or $y \mid x^n$ in $T(R)$ (Badawi, 1995, Theorem 1). The result now easily follows.

(2) Suppose that the radical ideals of R contained in $Z(R)$ are linearly ordered. Let $x, y \in Z(R)$. Then $Rad(xR), Rad(yR) \subseteq Z(R)$; so we may assume that $Rad(xR) \subseteq Rad(yR)$. Thus $x \in Rad(yR)$; so $y|x^n$ for some integer $n \geq 1$. Conversely, let $I, J \subseteq Z(R)$ be radical ideals of R. If I and J are not comparable, pick $x \in I \backslash J$ and $y \in J \backslash I$. If $x \mid y^n$, then $y^n \in xR \subseteq I$, and hence $y \in I$, a contradiction.

(3) Suppose that the prime ideals of R contained in $Z(R)$ are linearly ordered. Then $Nil(R)$ is an intersection of linearly ordered prime ideals of R since each minimal prime ideal of R is contained in $Z(R)$ (Huckaba, 1988, Theorem 2.1), and thus $Nil(R)$ is prime. Also, $Z(R)$ is the union of linearly ordered prime ideals of R (Kaplansky, 1974, p. 3), and hence $Z(R)$ is prime. □

Since $Z(R)$ is a union of prime ideals of R (Kaplansky, 1974, p. 3), $Z(R)$ is a prime ideal of R if and only if it is an ideal of R. If $\dim(R) = 0$ (e.g., R is finite) and the prime ideals of R contained in $Z(R)$ are linearly ordered, then R is quasilocal with $Z(R) = Nil(R)$ its unique prime ideal. If $Nil(R) \subsetneq Z(R)$ and $Nil(R)$ is a prime ideal of R, then $\dim(R) \geq 1$ and $\Gamma(R)$ must be infinite. For in this case, R is not an integral domain, and thus if $\Gamma(R)$ is finite, then R must also be finite (Anderson and Livingston, 1999, Theorem 2.2), contradicting $\dim(R) \geq 1$. In particular, if the prime ideals of R contained in $Z(R)$ are linearly ordered and $Nil(R) \subsetneq Z(R)$, then $\Gamma(R)$ is infinite. It is clear that if the radical ideals of R contained in $Z(R)$ are linearly ordered, then the prime ideals of R contained in $Z(R)$ are also linearly ordered. However, we next give an example where the prime ideals of R contained in $Z(R)$ are linearly ordered, but the radical ideals of R contained in $Z(R)$ are not linearly ordered, and hence the prime ideals of R are not linearly ordered.

Example 2.6. Let $D = \mathbb{Z} + X\mathbb{Q}[[X]]$, and let $I = \mathbb{Z}_{(2)}X + X^2\mathbb{Q}[[X]]$ be an ideal of D. Set $R = D/I$. Then $Z(R) = (2\mathbb{Z} + X\mathbb{Q}[[X]])/I = 2R = \text{ann}_R(\frac{1}{2}X + I)$, $N(R) = Nil(R) = X\mathbb{Q}[[X]]/I$, and $Nil(R)^2 = \{0\}$. The prime ideals of R contained in $Z(R)$, namely, $Z(R)$ and $Nil(R)$, are linearly ordered. But the radical ideals of R contained in $Z(R)$ are not linearly ordered since the two radical ideals $(6\mathbb{Z} + X\mathbb{Q}[[X]])/I$ and $(10\mathbb{Z} + X\mathbb{Q}[[X]])/I$ are not comparable. Thus the prime ideals of R are also not linearly ordered; for example, $(2\mathbb{Z} + X\mathbb{Q}[[X]])/I$ and $(3\mathbb{Z} + X\mathbb{Q}[[X]])/I$ are not comparable. We have $\text{diam}(\Gamma(R)) = 2$ by Theorem 2.7, and $\text{gr}(\Gamma(R)) = 3$ by Theorem 2.12. Also note that $R \cong \mathbb{Z}(+)(\mathbb{Q}/\mathbb{Z}_{(2)})$.

The prime ideals of R contained in $Z(R)$ are linearly ordered if and only if the prime ideals of $T(R)$ are linearly ordered. Moreover, $\Gamma(R) \cong \Gamma(T(R))$ (Anderson et al., 2003, Theorem 2.2). Thus we can often reduce to the case where the prime ideals of R are linearly ordered. Note that a reduced ring R with its prime ideals contained in $Z(R)$ linearly ordered is an integral domain. Also observe that a nonreduced ring R has $\Gamma(R)$ complete if and only if $Z(R)^2 = \{0\}$ (Anderson and Livingston, 1999, Theorem 2.8), i.e., if $xy = 0$ for all $x, y \in Z(R)$ with $x \neq y$, then $x^2 = 0$ for all $x \in Z(R)$. So if R is a nonreduced ring with $Z(R)^2 = \{0\}$, then $\{0\} \neq N(R) = Nil(R) = Z(R)$ and $\text{diam}(\Gamma(R)) \leq 1$, with equality when $|Z(R)^*| \geq 2$. We are now ready for the first of the two main results of this section.

Theorem 2.7. *Let R be a ring with $Z(R)^2 \neq \{0\}$ such that the prime ideals of R contained in $Z(R)$ are linearly ordered. Then* $\text{diam}(\Gamma(R)) = 2$.

Proof. By the above comments, R is not reduced. So $\Gamma(R)$ is not a complete graph and $\text{diam}(\Gamma(R)) \geq 2$. Let $x, y \in Z(R)^*$ be distinct with $xy \neq 0$. If $x, y \in Nil(R)$, then $d(x, y) = 2$ by Lemma 2.1. If $x \in Nil(R)$ and $y \in Z(R)\backslash Nil(R)$, then $d(x, y) = 2$ by Lemma 2.3. Finally, suppose that $x, y \in Z(R)\backslash Nil(R)$. Since the prime ideals of R contained in $Z(R)$ are linearly ordered, there is an integer $n \geq 1$ and an element $z \in R\backslash Z(R)$ such that either $x \mid zy^n$ or $y \mid zx^n$ by Theorem 2.5(1). We may assume that $x \mid zy^n$ for some integer $n \geq 1$ and $z \in R\backslash Z(R)$. Thus $d(x, y)2$ by Lemma 2.4. Hence $\text{diam}(\Gamma(R)) \leq 2$, and thus $\text{diam}(\Gamma(R)) = 2$ since $\text{diam}(\Gamma(R)) \geq 2$. □

Corollary 2.8. *If R is any of the following types of rings with $Z(R)^2 \neq \{0\}$, then* $\text{diam}(\Gamma(R)) = 2$.

(1) *R is a ring such that the prime ideals of R are linearly ordered*;
(2) *R is a divided ring*;
(3) *R is a PVR*;
(4) *R is a ϕ-PVR*;
(5) *R is a chained ring.*

In view of Theorem 2.7 and Lucas (2006, Theorem 2.6(3)), we have the following corollary.

Corollary 2.9. *Let R be a ring with $Z(R)^2 \neq \{0\}$ such that the prime ideals of R contained in $Z(R)$ are linearly ordered. Then $Z(R)$ is an* (*prime*) *ideal of R and each pair of distinct zero-divisors of R has a nonzero annihilator.*

Our next example illustrates what can happen when the prime ideals of R contained in $Z(R)$ are not linearly ordered.

Example 2.10. (a) Let $D = \mathbb{R}[[X, Y]] + ZK[[Z]]$, where K is the quotient field of $\mathbb{R}[[X, Y]]$, and let $I = ZD$. Set $R = D/I$. Then R is quasilocal with maximal ideal $Z(R)((X, Y) + ZK[[Z]])/I$, $N(R) = Nil(R) = ZK[[Z]]/I$, $Nil(R)^2 = \{0\}$, and $((X) + ZK[[Z]])/I$ and $((Y) + ZK[[Z]])/I$ are incomparable prime ideals of R contained in $Z(R)$. One can easily show that $\mathrm{diam}(\Gamma(R)) = 3$ and $\mathrm{gr}(\Gamma(R)) = 3$. Also see Example 5.3(b).

(b) Let $R = \mathbb{Z}_2 \times \mathbb{Z}_4$. Then $N(R) = Nil(R) = \{0\} \times \{0, 2\} \subsetneq Z(R) = P \cup Q$, where $P = \mathbb{Z}_2 \times \{0, 2\}$ and $Q = \{0\} \times \mathbb{Z}_4$ are incomparable primes ideals of R contained in $Z(R)$. One can easily show that $\mathrm{diam}(\Gamma(R)) = 3$ and $\mathrm{gr}(\Gamma(R)) = \infty$.

We conclude this section with a discussion of the girth of $\Gamma(R)$ when the prime ideals of R contained in $Z(R)$ are linearly ordered. We first handle the case where $Z(R) = Nil(R)$. In this case, $\mathrm{gr}(\Gamma(R)) \neq 4$, and we can explicitly say when the girth is either 3 or ∞. Note that in Theorem 2.11, $\mathrm{gr}(\Gamma(R)) = \infty$ if and only if $\Gamma(R)$ is a finite star graph. (Recall that a graph is a *star graph* if it has a vertex which is adjacent to every other vertex and this is the only adjacency relation. We consider a singleton graph to be a star graph.)

Theorem 2.11. *Let R be a ring with $Z(R) = Nil(R) \neq \{0\}$. Then exactly one of the following four cases must occur*:

(1) $|Z(R)^*| = 1$. *In this case, R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, and* $\mathrm{gr}(\Gamma(R)) = \infty$;
(2) $|Z(R)^*| = 2$. *In this case, R is isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^2)$, and* $\mathrm{gr}(\Gamma(R)) = \infty$;
(3) $|Z(R)^*| = 3$. *If R is isomorphic to $\mathbb{Z}_8$, $\mathbb{Z}_2[X]/(X^3)$, or $\mathbb{Z}_4[X]/(2X, X^2 - 2)$, then* $\mathrm{gr}(\Gamma(R)) = \infty$. *Otherwise, R is isomorphic to $\mathbb{Z}_2[X, Y]/(X, Y)^2$, $\mathbb{Z}_4[X]/(2, X)^2$, $\mathbb{Z}_4[X]/(X^2 + X + 1)$, or $\mathbb{F}_4[X]/(X^2)$; and in this case,* $\mathrm{gr}(\Gamma(R)) = 3$;
(4) $|Z(R)^*| \geq 4$. *In this case,* $\mathrm{gr}(\Gamma(R)) = 3$.

Proof. By Anderson and Mulay (2007, Theorem 2.3), $\mathrm{gr}(\Gamma(R)) \neq 4$ when $Z(R) = Nil(R)$. Thus $\mathrm{gr}(\Gamma(R)) = 3$ or ∞. The theorem then follows from Anderson and Mulay (2007, Theorem 2.5, Remark 2.6(a)), and Anderson et al. (2001, Example 2.1). □

We next handle the $Nil(R) \subsetneq Z(R)$ case when $Nil(R)$ a prime ideal of R (cf. Remark 2.13(b)). In this case, we have already observed that $\Gamma(R)$ is infinite. The next theorem, together with Theorem 2.11, completely characterizes $\mathrm{gr}(\Gamma(R))$ in terms of $|Nil(R)^*|$ when the prime ideals of R contained in $Z(R)$ are linearly ordered. In particular, we have $\mathrm{gr}(\Gamma(R)) = 3$ or ∞, with $\mathrm{gr}(\Gamma(R)) = \infty$ if and only if $\Gamma(R)$ is a star graph.

Theorem 2.12. *Let R be a ring such that $Nil(R)$ is a prime ideal of R and $Nil(R) \subsetneq Z(R)$. In particular, this holds when the prime ideals of R contained in $Z(R)$ are linearly ordered and $Nil(R) \subsetneq Z(R)$. Then* $\mathrm{gr}(\Gamma(R)) = 3$ *or* ∞. *Moreover,* $\mathrm{gr}(\Gamma(R)) = \infty$ *if and only if $|Nil(R)^*| = 1$; and in this case, $\Gamma(R)$ is an infinite star graph.*

Proof. Since $\Gamma(R)) \cong \Gamma(T(R))$ (Anderson et al., 2003, Theorem 2.2), we may assume that $R = T(R)$. Note that R is not reduced; so if $\mathrm{gr}(\Gamma(R)) = 4$, then $R \cong D \times B$, where D is an integral domain and $B = \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ by Anderson and Mulay (2007, Theorem 2.3). In this case, $Nil(R) \cong \{0\} \times \mathbb{Z}_2$ is not a prime ideal of R. So we must have $\mathrm{gr}(\Gamma(R)) = 3$ or ∞. The "in particular" statement follows from Theorem 2.5(3). The "moreover" statement follows from Anderson and Mulay (2007, Theorem 2.5, Remark 2.6(a)). □

Remark 2.13. (a) $\Gamma(R)$ is a finite star graph if and only if either $R \cong \mathbb{F}_q \times \mathbb{Z}_2$ for some finite field $\mathbb{F}_q$ (when R is reduced), or R is one of the 7 rings with $\mathrm{gr}(\Gamma(R)) = \infty$ given in Theorem 2.11 (Anderson and Livingston, 1999, Theorem 2.13, DeMeyer and Schneider, 2002, Corollary 1.11).

If $\Gamma(R)$ is an infinite star graph, then either $R \cong D \times \mathbb{Z}_2$ for D an integral domain (when R is reduced), or $Nil(R)$ is a prime ideal of R with $|Nil(R)^*| = 1$ and $Z(R)$ is a prime ideal of R (DeMeyer and Schneider, 2002, Theorem 1.12 or Mulay, 2002, (2.1)). For example, if $R = \mathbb{Z}(+)\mathbb{Z}_2$ $(\cong \mathbb{Z}[X]/(2X, X^2))$, then $\Gamma(R)$ is an infinite star graph with center $(0, 1)$ and the prime ideals of R contained in $Z(R)$ are linearly ordered.

(b) The hypothesis that $Nil(R)$ is a prime ideal of R is needed in Theorem 2.12. For example, let $R = \mathbb{Z}_3 \times \mathbb{Z}_3$. Then $Nil(R) \subsetneq Z(R)$, $Nil(R)$ is not a prime ideal of R, and $\mathrm{gr}(\Gamma(R)) = 4$.

(c) It is instructive to give an elementary, self-contained proof of Theorem 2.12. If $|Nil(R)^*| = 1$, then $\mathrm{gr}(\Gamma(R)) = \infty$ since $\Gamma(R)\backslash Nil(R)$ is totally disconnected (Theorem 3.5(1)). So suppose that $|Nil(R)^*| \geq 2$, and let $z \in Z(R)\backslash Nil(R)$. Then there is a $w \in Nil(R)^*$ with $zw = 0$. First suppose that $w^2 \neq 0$, and let m (≥ 3) be the least positive integer such that $w^m = 0$. Thus $w^{m-1} \neq w$, and hence $z - w - w^{m-1} - z$ is a cycle of length 3. Now suppose that $w^2 = 0$, and let $d \in Nil(R)^*\backslash\{w\}$. Assume that $wd \neq 0$. Since wd and w are distinct and nonzero, we conclude that $z - w - wd - z$ is a cycle of length 3. Now assume that $wd = 0$ and $w^2 = 0$. If $zd = 0$, then $z - w - d - z$ is a cycle of length 3. Thus we may assume that $zd \neq 0$. If $zd = w$, then $zd^2 = wd = 0$, and hence $w - z^2 - d - w$ is a cycle of length 3. Thus we assume that zd and w are distinct and nonzero. Let n be the least positive integer such that $zd^n = 0$. Assume $n > 2$. Then it is clear that $d \neq zd^{n-1}$. If $zd^{n-1} \neq w$, then $w - zd^{n-1} - d - w$ is a cycle of length 3. Assume that $zd^{n-1} = w$. Then $z^2d^{n-1} = zw = 0$. Since $zw = 0$, d^{n-1} and w are distinct and nonzero, and thus $w - z^2 - d^{n-1} - w$ is a cycle of length 3. Now assume that $n = 2$ and $zd \neq w$. Then $zd^2 = 0$. If $zd \neq d$, then $w - zd - d - w$ is a cycle of length 3. Thus assume that $zd = d$. Hence $d^2 = zd^2 = 0$. Since $zw = 0$ and $zd \neq 0$, we have $w + d \neq 0$. Hence w, d, and $w + d$ are all distinct. Since $w^2 = d^2 = wd = 0$, $w - w + d - d - w$ is a cycle of length 3. Thus $\mathrm{gr}(\Gamma(R)) = 3$.

3. LINEARLY ORDERED PRIMES-II

In this section, we continue the investigation of $\Gamma(R)$ when the prime ideals of R contained in $Z(R)$ are linearly ordered. We show that for such rings R, $\Gamma(R)\backslash Nil(R)$ is totally disconnected, every finite set of vertices of $\Gamma(R)\backslash Nil(R)$ is adjacent to a common vertex of $Nil(R)^*$, and $\Gamma(R)\backslash Nil(R)$ is infinite when $Nil(R) \subsetneq Z(R)$. We also determine $\mathrm{diam}(\Gamma(R[X]))$ and $\mathrm{gr}(\Gamma(R[X]))$. Our first goal is to show that such a ring R is a McCoy ring, where a ring R is called a *McCoy ring* if every finitely generated ideal of R contained in $Z(R)$ has a nonzero annihilator.

Lemma 3.1. *Let R be a ring such that the prime ideals of R contained in $Z(R)$ are linearly ordered, and let $z_1, \ldots, z_n \in Z(R)$. Then there is an integer i, $1 \leq i \leq n$, a positive integer m, and an $s \in R\backslash Z(R)$ such that $z_i \mid sz_k^m$ for every integer k, $1 \leq k \leq n$.*

Proof. Let $T = T(R)$. Then the prime ideals of T are linearly ordered. Thus $Rad(z_1T), \ldots, Rad(z_nT)$ are prime ideals of T, and hence are linearly ordered. Thus there is an integer i, $1 \leq i \leq n$, such that $Rad(z_kT) \subseteq Rad(z_iT)$ for every integer k, $1 \leq k \leq n$. Hence there are positive integers $m_1, \ldots, m_n$ and $s_1, \ldots, s_n \in R\backslash Z(R)$ such that $z_i \mid s_i z_k^{m_k}$ for every integer k, $1 \leq k \leq n$. Let $s = s_1 \ldots s_n \in R\backslash Z(R)$ and $m = \max\{m_1, \ldots, m_n\}$. Then $z_i \mid s{z_k}^m$ for every integer k, $1 \leq k \leq n$, as desired. □

Theorem 3.2. *Let R be a ring such that the prime ideals of R contained in $Z(R)$ are linearly ordered. Then R is a McCoy ring.*

Proof. Let $I = (z_1, \ldots, z_n)$ be a nonzero finitely generated ideal of R contained in $Z(R)$. By Lemma 3.1, we may assume that there is a positive integer m and an $s \in R\backslash Z(R)$ such that $z_1 \mid sz_k^m$ for every integer k, $2 \leq k \leq n$. Let $w \in Z(R)^*$ such that $z_1 w = 0$. Thus there is an integer $m_2 \geq 0$ such that $z_2^{m_2} w \neq 0$ and $z_2^{m_2} w z_2 = 0$. Hence $0 \neq z_2^{m_2} w \in (0 : (z_1, z_2))$. Since $z_2^{m_2} w z_1 = 0$ and $z_1 \mid sz_3^m$, there is an integer $m_3 \geq 0$ such that $z_3^{m_3} z_2^{m_2} w \neq 0$ and $z_3^{m_3} z_2^{m_2} w z_3 = 0$. Thus $0 \neq z_3^{m_3} z_2^{m_2} w \in (0 : (z_1, z_2, z_3))$. Continuing in this manner, we can construct a $0 \neq z_n^{m_n} z_{n-1}^{m_{n-1}} \cdots z_2^{m_2} w \in (0 : (z_1, z_2, z_3, \ldots, z_n))$. Hence R is a McCoy ring. □

Corollary 3.3. *Let R be a ring such that the prime ideals of R contained in $Z(R)$ are linearly ordered, and let $x_1, \ldots, x_n \in Z(R)\backslash Nil(R)$. Then there is a $y \in Nil(R)^*$ such that $x_i y = 0$ for every integer i, $1 \leq i \leq n$.*

Proof. There is a $y \in Z(R)^*$ such that each $x_i y = 0$ since R is a McCoy ring and $Z(R)$ is an ideal of R. Moreover, $y \in Nil(R)$ since $x_1 \notin Nil(R)$ and $Nil(R)$ is a prime ideal of R by Theorem 2.5(3). □

Remark 3.4. If R is a McCoy ring and $Z(R)$ is an ideal of R, then clearly $\mathrm{diam}(\Gamma(R)) \leq 2$. This observation, together with Theorem 3.2, gives another proof of Theorem 2.7. However, note that $R = \mathbb{Z}_2 \times \mathbb{Z}_4$ is a McCoy ring with $\mathrm{diam}(\Gamma(R)) = 3$ (cf. Example 2.10(b)).

We next show that the subgraph $\Gamma(R)\backslash Nil(R)$ of $\Gamma(R)$ is infinite and totally disconnected when $Nil(R)$ is a prime ideal of R and $Nil(R) \subsetneq Z(R)$ (i.e., when $\Gamma(R)\backslash Nil(R)$ is nonempty). This fact gives another proof of the "moreover" statement of Theorem 2.12, namely, that $\Gamma(R)$ is an infinite star graph when $Nil(R)$ is a prime ideal of R, $Nil(R) \subsetneq Z(R)$, and $|Nil(R)^*| = 1$.

Theorem 3.5. *Let R be a ring.*

(1) *$\Gamma(R)\backslash Nil(R)$ is totally disconnected if and only if $Nil(R)$ is a prime ideal of R.*
(2) *If $Nil(R)$ is a prime ideal of R and $Nil(R) \subsetneq Z(R)$, then $Z(R)\backslash Nil(R)$ is infinite.*

In particular, $\Gamma(R)\backslash Nil(R)$ is infinite and totally disconnected when the prime ideals of R contained in $Z(R)$ are linearly ordered and $Nil(R) \subsetneq Z(R)$.

Proof. (1) Suppose that $\Gamma(R)\backslash Nil(R)$ is totally disconnected. Let $xy \in Nil(R)$ with $x, y \notin Nil(R)$. Then $x^n y^n = 0$ for some positive integer n. Thus $x^n, y^n \in Z(R)\backslash Nil(R)$ and $x^n \neq y^n$ since $x, y \notin Nil(R)$. But then x^n and y^n are adjacent in $\Gamma(R)\backslash Nil(R)$, a contradiction. Hence $Nil(R)$ is a prime ideal of R. The converse is clear.

(2) Let $x \in Z(R)\backslash Nil(R)$. Suppose that $x^n = x^m$ for some integers $n > m \geq 1$. Then $x^m(1 - x^{n-m}) = 0 \in Nil(R)$ and $x \notin Nil(R)$ implies $1 - x^{n-m} \in Nil(R)$ since $Nil(R)$ is prime. Thus $x^{n-m} = 1 - (1 - x^{n-m}) \in U(R)$, and hence $x \in U(R)$, a contradiction. Thus $Z(R)\backslash Nil(R)$ is infinite.

The "in particular" statement holds since in this case $Nil(R)$ is a prime ideal of R by Theorem 2.5(3). □

Combining Lemma 2.1, Theorem 3.5, and Corollary 3.3, we have the following structure theorem for $\Gamma(R)$ when the prime ideals of R contained in $Z(R)$ are linearly ordered. Then $Nil(R)^*$ is a subgraph of $\Gamma(R)$ of diameter at most 2, $\Gamma(R)\backslash Nil(R)$ is infinite and totally disconnected when $Nil(R) \subsetneq Z(R)$, and for each finite set of vertices $Y \subseteq \Gamma(R)\backslash Nil(R)$, there is a vertex $y \in Nil(R)^*$ which is adjacent to every element of Y.

Our next goal is to investigate $\text{diam}(\Gamma(R[X]))$ when the prime ideals of R contained in $Z(R)$ are linearly ordered. The diameter of $\Gamma(R[X])$ has recently been studied in Axtel et al. (2005), Anderson and Mulay (2007), and Lucas (2006). In particular, Lucas (2006, Theorems 3.4 and 3.6) give nice characterizations of $\text{diam}(\Gamma(R[X]))$. If $Z(R)^2 = \{0\}$ (i.e., $\Gamma(R)$ is a complete graph), then $Z(R[X])^2 = \{0\}$; so $\Gamma(R[X])$ is a complete graph with $\text{diam}(\Gamma(R[X])) = 1$. McCoy's Theorem for polynomial rings states that $f(X) \in Z(R[X])$ if and only if $rf(X) = 0$ for some $0 \neq r \in R$, i.e., $Z(R[X]) \subseteq Z(R)[X]$. Thus $Z(R[X])$ is an ideal of $R[X]$ if and only if R is a McCoy ring and $Z(R)$ is an ideal of R (Lucas, 2006, Theorem 3.3), and in this case, $Z(R[X]) = Z(R)[X]$.

Theorem 3.6. *Let R be a ring such that the prime ideals of R contained in* $Z(R)$ *are linearly ordered.*

(1) $Z(R[X])$ *is an* (*prime*) *ideal of* $R[X]$.
(2) *If R is not an integral domain and* $Z(R)^2 = \{0\}$, *then* $\text{diam}(\Gamma(R[X])) = 1$.
(3) *If* $Z(R)^2 \neq \{0\}$, *then* $\text{diam}(\Gamma(R[X])) = 2$.

Proof. Part (1) follows from Theorem 3.2 and Lucas (2006, Theorem 3.3). We have already observed part (2) above. Part (3) follows from Theorem 3.2, Corollary 2.9, and Lucas (2006, Theorem 3.4(3)). □

Corollary 3.7. *If R is any of the following types of rings with* $Z(R)^2 \neq \{0\}$, *then* $\text{diam}(\Gamma(R[X])) = 2$.

(1) *R is a ring such that the prime ideals of R are linearly ordered.*
(2) *R is a divided ring.*
(3) *R is a PVR.*
(4) *R is a* ϕ*-PVR.*
(5) *R is a chained ring.*

Corollary 3.8. *Let R be a nonreduced ring such that the prime ideals of R contained in $Z(R)$ are linearly ordered. Then exactly one of the following four cases must occur*:

(1) $|Z(R)^*| = 1$. *In this case, R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[Y]/(Y^2)$,* $\text{diam}(\Gamma(R)) = 0$, *and* $\text{diam}(\Gamma(R[X])) = 1$;
(2) $|Z(R)^*| \geq 2$, $Z(R) = Nil(R)$, *and* $Z(R)^2 = \{0\}$. *In this case,* $\text{diam}(\Gamma(R)) = \text{diam}(\Gamma(R[X])) = 1$;
(3) $Z(R) = Nil(R)$ *and* $Z(R)^2 \neq \{0\}$. *In this case,* $\text{diam}(\Gamma(R)) = \text{diam}(\Gamma(R[X])) = 2$;
(4) $Nil(R) \subsetneq Z(R)$. *In this case,* $\text{diam}(\Gamma(R)) = \text{diam}(\Gamma(R[X])) = 2$.

Proof. This follows directly from Theorems 2.2 and 3.6. □

The following example illustrates the four cases stated in Corollary 3.8. In each case, the ring R is actually a chained ring. The routine details are left to the reader.

Example 3.9. (a) Let $R = \mathbb{Z}_4$. Then R is a chained ring with $|Z(R)^*| = 1$. Thus $\text{diam}(\Gamma(R)) = 0$ and $\text{diam}(\Gamma(R[X])) = 1$.

(b) Let $R = \mathbb{Z}_9$. Then R is a chained ring with $|Z(R)^*| = 2$, $Z(R) = Nil(R) = N(R)$, and $Z(R)^2 = \{0\}$. Thus $\text{diam}(\Gamma(R)) = \text{diam}(\Gamma(R[X])) = 1$.

(c) Let $R = \mathbb{Z}_8$. Then R is a chained ring with $N(R) \subsetneq Nil(R) = Z(R)$ and $Z(R)^2 \neq \{0\}$. Thus $\text{diam}(\Gamma(R)) = \text{diam}(\Gamma(R[X])) = 2$.

(d) Let $D = \mathbb{Z}_{(2)} + X\mathbb{Q}[[X]]$ and $I = XD\mathbb{Z}_{(2)}X + X^2\mathbb{Q}[[X]]$. Set $R = D/I$. Then D is a valuation domain; so R is a chained ring. Note that $Z(R) = (2\mathbb{Z}_{(2)} + X\mathbb{Q}[[X]])/I = 2R$ and $N(R) = Nil(R) = X\mathbb{Q}[[X]]/I$; so $Nil(R) \subsetneq Z(R)$ and $Nil(R)^2 = \{0\}$. Thus $\text{diam}(\Gamma(R)) = \text{diam}(\Gamma(R[X])) = 2$.

Unlike the case for the diameter of the zero-divisor graph of a polynomial ring as in Corollary 3.8, the girth case is very easy. The girth of $\Gamma(R[X])$ and $\Gamma(R[[X]])$ has been studied in Axtel et al. (2005) and Anderson and Mulay (2007), and a complete characterization is given in Anderson and Mulay (2007, Theorem 3.2). For any nonreduced ring R, we always have $\text{gr}(\Gamma(R[X])) = \text{gr}(\Gamma(R[[X]])) = 3$ by Anderson and Mulay (2007, Lemma 3.1) (since $aX - aX^2 - aX^3 - aX$ forms a triangle for any $a \in N(R)^*$).

4. CHAINED RINGS

In this section, we investigate $\Gamma(R)$ when R is a chained ring. This is probably the nicest case where the prime ideals of R contained in $Z(R)$ are linearly ordered since in a chained ring all the ideals are linearly ordered. A typical example of a chained ring is a homomorphic image of a valuation domain. In particular, $\mathbb{Z}_n$ is a chained ring if and only if n is a prime power. In fact, it was an open question (attributed to Kaplansky) if every chained ring is the homomorphic image of a valuation domain (cf. Huckaba, 1988, Chapter V). However, an example in Fuchs and Salce (1985) shows that this is not true in general. It will turn out that the subset $N(R) = \{x \in R \mid x^2 = 0\}$ of $Nil(R)$ will play a major role in describing $\Gamma(R)$ when R is a chained ring. Note that if R is a chained ring, then $N(R) = \{0\}$ if and only if $Z(R) = \{0\}$. Also note that for any ring R, we have $N(R) = Nil(R)$ when $Nil(R)^2 = \{0\}$, and $N(R) = \{0\}$ if and only if $Nil(R) = \{0\}$. We start with several lemmas. In some cases, these results are special cases of ones from previous sections; however, the proofs are much easier in the chained ring setting.

Lemma 4.1. *Let R be a ring, $N(R) = \{x \in R \mid x^2 = 0\}$, and $x \in Nil(R)\backslash N(R)$. Then $xy = 0$ for some $y \in N(R)^*\backslash\{x\}$.*

Proof. Let n (≥ 3) be the least positive integer such that $x^n = 0$, and let $y = x^{n-1}$. Then $xy = x^n = 0$, $y = x^{n-1} \neq 0$, and $y^2 = (x^{n-1})^2 = x^{2n-2} = 0$ because $2n-2 \geq n$ since $n \geq 3$. Clearly $x \neq y$ since $x^2 \neq 0$. □

Thus any vertex of the subgraph $Nil(R)\backslash N(R)$ of $\Gamma(R)$ is adjacent to a vertex of $N(R)^*$. We next show, among other things, that for a chained ring R, any vertex of $\Gamma(R)\backslash N(R)$ is adjacent to a vertex of $N(R)^*$ and any two vertices of $N(R)^*$ are adjacent.

Lemma 4.2. *Let R be a chained ring, $N(R) = \{x \in R \mid x^2 = 0\}$, and $x, y \in R$.*

(1) *If $xy = 0$, then either $x \in N(R)$ or $y \in N(R)$.*
(2) *If $x, y \in N(R)$, then $xy = 0$.*
(3) *If $x, y \in Z(R)\backslash N(R)$, then $xy \neq 0$.*
(4) *If $x \in Z(R)^*$, then $xy = 0$ for some $y \in N(R)^*$.*
(5) *If $x_1, \ldots, x_n \in Z(R)^*$, then there is a $y \in N(R)^*$ such that $x_i y = 0$ for every integer i, $1 \leq i \leq n$.*
(6) *$N(R)$ is an ideal of R.*
(7) *$N(R)$ is a prime ideal of R if and only if $N(R) = Nil(R)$.*

Proof. (1) Suppose that $x|y$. Then $y = rx$ for some $r \in R$; so $y^2 = rxy = 0$.

(2) Suppose that $x \mid y$. Then $y = rx$ for some $r \in R$, and hence $xy = rx^2 = 0$.

(3) This follows from part (1).

(4) If $x \in N(R)^*$, then let $y = x$. If $x \in Z(R)\backslash N(R)$, then $xy = 0$ for some $0 \neq y \in R$. By part (3) above, we must have $y \in N(R)$.

(5) There is an integer j, $1 \leq j \leq n$, such that $x_j \mid x_i$ for all i, $1 \leq i \leq n$. By part (4) above, there is a $y \in N(R)^*$ such that $x_j y = 0$; so $x_i y = 0$ for all i, $1 \leq i \leq n$.

(6) Clearly, $xN(R) \subseteq N(R)$ for all $x \in R$; so we need only show that $N(R)$ is closed under addition. Let $x, y \in N(R)$. Then $x^2 = y^2 = 0$, and $xy = 0$ by part (2) above. Thus $(x+y)^2 = x^2 + 2xy + y^2 = 0$, and hence $x + y \in N(R)$.

(7) This is clear since $Nil(R)$ is the unique minimal prime ideal of R. □

One can ask if part (5) above extends to any subset of $Z(R)^*$. Of course, if $X \subseteq xR$ and $yx = 0$, then $yX = \{0\}$. So if $X \subseteq Z(R)^*$ and $X \subseteq xR$ for some $x \in Z(R)^*$, then $yX = \{0\}$ for some $y \in N(R)^*$. Our next remark addresses this question.

Remark 4.3. (a) Let $D = V + XK[[X]]$, where V is a valuation domain with nonzero maximal ideal M and quotient field K; so D is also a valuation domain.

Let $I = XD = VX + X^2K[[X]]$, and set $R = D/I$. Then R is a chained ring with maximal ideal $Z(R) = (M + XK[[X]])/I$ and $N(R) = Nil(R) = XK[[X]]]/I$. Note that there is a $y \in N(R)^*$ such that $yZ(R) = \{0\}$ if and only if there is a $y \in M^{-1}\backslash V$. (So for $\dim(V) = 1$, this happens if and only if V is a DVR.)

(b) If R is a chained ring, then $N(R) = \{x \in R \mid x^2 = 0\}$ is an ideal of R by Lemma 4.2(6). In general, $N(R)$ need not be an ideal of R (see Examples 5.5 and 5.6). However, if $\mathrm{char}(R) = 2$, then $N(R)$ is an ideal of R. Also note that if $2 \in U(R)$ and $N(R)$ is an ideal of R, then $xy = 0$ for all $x, y \in N(R)$.

By Theorem 3.5(1), $\Gamma(R)\backslash Nil(R)$ is totally disconnected when R is a chained ring. Lemma 4.2(3) yields the following stronger result (also see Example 5.5).

Theorem 4.4. *Let R be a chained ring and $N(R) = \{x \in R \mid x^2 = 0\}$. Then $\Gamma(R)\backslash N(R)$ is totally disconnected.*

Our next result is a special case of Theorem 2.7, but we give a proof in the spirit of this section. We can also explicitly say when $\mathrm{diam}(\Gamma(R))$ is 0, 1, or 2.

Theorem 4.5. *Let R be a chained ring. Then* $\mathrm{diam}(\Gamma(R)) \leq 2$.

Proof. We may assume that $|Z(R)^*| \geq 2$. Let $N(R) = \{x \in R \mid x^2 = 0\}$, and let $x, y \in Z(R)^*$ be distinct. If $x, y \in N(R)$, then $xy = 0$ by Lemma 4.2(2), and thus $d(x, y) = 1$. If $x \in N(R)$ and $y \notin N(R)$, then $yz = 0$ for some $z \in N(R)^*$ by Lemma 4.2(4), and hence $xz = 0$ by Lemma 4.2(2). Thus $d(x, y) \leq 2$. Finally, let $x \notin N(R)$ and $y \notin N(R)$. Then $xz = yz = 0$ for some $z \in N(R)^*$ by Lemma 4.2(5). Thus $d(x, y) \leq 2$, and hence $\mathrm{diam}(\Gamma(R)) \leq 2$. □

Theorem 4.6. *Let R be a chained ring with $Z(R) \neq \{0\}$, and let $N(R) = \{x \in R \mid x^2 = 0\}$. Then exactly one of the following three cases must occur:*

(1) $|Z(R)^*| = 1$. *In this case, R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, and* $\mathrm{diam}(\Gamma(R)) = 0$;
(2) $|Z(R)^*| \geq 2$ *and $N(R) = Z(R)$. In this case,* $\mathrm{diam}(\Gamma(R)) = 1$;
(3) $N(R) \subsetneq Z(R)$. *In this case,* $\mathrm{diam}(\Gamma(R)) = 2$.

Proof. The first part follows from Beck (1988, Proposition 2.2). The other two follow directly from Lemma 4.2 and Theorem 4.5. □

Let R be a chained ring with $N(R) = \{x \in R \mid x^2 = 0\}$. It is now easy to describe the structure of $\Gamma(R)$. First, observe that $N(R)^*$ is a complete subgraph of $\Gamma(R)$ by Lemma 4.2(2), $\Gamma(R)\backslash N(R)$ is totally disconnected by Lemma 4.2(3), and $\Gamma(R)\backslash N(R)$ is infinite if $Nil(R) \subsetneq Z(R)$. Moreover, for any finite set of vertices $Y \subseteq \Gamma(R)\backslash N(R)$, there is a vertex $z \in N(R)^*$ adjacent to every element in Y by Lemma 4.2(5). In particular, $\Gamma(R)$ is complete if and only if $Z(R) = N(R)$. Note that this description of $\Gamma(R)$ recovers Theorem 4.6. Also note that $Nil(R)^*$ need not be a complete subgraph of $\Gamma(R)$ (e.g., when R is the chained ring $\mathbb{Z}_{16}$).

The structure of $\Gamma(R)$ described in the preceding paragraph also extends to $\Gamma(R[X])$ when R is a chained ring. Note that when R is a chained ring, we have $N(R[X]) = N(R)[X]$, $Nil(R[X]) = Nil(R)[X]$, and $Z(R[X]) = Z(R)[X]$ (of course, $Nil(R[X]) = Nil(R)[X]$ holds for any ring R). These statements are easy to verify directly, or just note that for any $0 \neq f \in R[X]$, we have $f = af^*$, where $a \in R$ and $f^* \in R[X]$ has unit content. Then $f \in N(R[X])$ (resp., $Nil(R[X])$, $Z(R[X])$) if and only if $a \in N(R)$ (resp., $Nil(R)$, $Z(R)$). Thus $N(R[X])^*$ is a complete subgraph of $\Gamma(R[X])$, $\Gamma(R[X])\backslash N(R[X])$ is totally disconnected, and for any finite set of vertices $Y \subseteq \Gamma(R[X])\backslash N(R[X])$, there is a vertex $f \in N(R[X])^*$ which is adjacent to every element in Y when R is a chained ring. Moreover, $N(R[X])^*$ and $\Gamma(R[X])\backslash N(R[X])$ are both infinite when R is a nonreduced chained ring. This observation shows that $\mathrm{diam}(\Gamma(R[X])) = 1$ when $Z(R)^2 = \{0\}$ and $\mathrm{diam}(\Gamma(R[X])) = 2$ when $Z(R)^2 \neq \{0\}$.

The above description of $\Gamma(R)$ also enables us to easily determine $\mathrm{gr}(\Gamma(R))$ when R is a chained ring (cf. Theorem 2.12). Note that $\Gamma(R)$ is a finite star graph in the first three cases of the next theorem, but it is not possible to have $\Gamma(R)$ be an infinite star graph when R is a chained ring (cf. Theorem 2.12).

Theorem 4.7. *Let R be a chained ring with $N(R) = \{x \in R \mid x^2 = 0\} \neq \{0\}$. Then exactly one of the following five cases must occur*:

(1) $|N(R)^*| = 1$ *and* $N(R) = Z(R)$. *In this case, R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, and* $\mathrm{gr}(\Gamma(R)) = \infty$;
(2) $|N(R)^*| = 1$ *and* $N(R) \subsetneq Z(R)$. *In this case, R is isomorphic to $\mathbb{Z}_8$, $\mathbb{Z}_2[X]/(X^3)$, or $\mathbb{Z}_4[X]/(2X, X^2 - 2)$, and* $\mathrm{gr}(\Gamma(R)) = \infty$;
(3) $|N(R)^*| = 2$ *and* $N(R) = Z(R)$. *In this case, R is isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^2)$, and* $\mathrm{gr}(\Gamma(R)) = \infty$;
(4) $|N(R)^*| = 2$ *and* $N(R) \subsetneq Z(R)$. *In this case,* $\mathrm{gr}(\Gamma(R)) = 3$;
(5) $|N(R)^*| \geq 3$. *In this case,* $\mathrm{gr}(\Gamma(R)) = 3$.

Proof. If $|N(R)^*| \geq 3$, then clearly $\mathrm{gr}(\Gamma(R)) = 3$ by Lemma 4.2(2). Suppose that $|N(R)^*| = 2$; say $N(R)^* = \{x, y\}$. If $y \neq -x$, then $x + y$ is a third nonzero element of $N(R)$, a contradiction. Thus $y = -x$; so $\mathrm{ann}_R(x) = \mathrm{ann}_R(y)$. If there is a $z \in Z(R)\backslash N(R)$, then $x - y - z - x$ is a triangle by Lemma 4.2(4); so $\mathrm{gr}(\Gamma(R)) = 3$. Otherwise, $Z(R) = N(R)$, and thus $\mathrm{gr}(\Gamma(R)) = \infty$. Finally, suppose that $|N(R)^*| = 1$, say $N(R) = \{0, x\}$. If $Z(R) = N(R)$, then $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ by Beck (1988, Proposition 2.2). In this case, $\mathrm{gr}(\Gamma(R)) = \infty$. So suppose that $N(R) \subsetneq Z(R)$. By parts (3) and (4) of Lemma 4.2, $\Gamma(R)$ is a star graph with center x. Thus $|R| = 8$, $|R| = 9$, or $|R| > 9$ and $Nil(R) = \{0, x\}$ by Anderson et al. (2003, Lemma 3.7). The $|R| > 9$ case cannot happen. For in this case, $Nil(R) = N(R) = xR$ is a prime ideal of R. Let $y \in Z(R)^*\backslash\{x\}$. Then $xR \subsetneq yR$; so $x = yr$ for some $0 \neq r \in R$. Hence $r \in xR = \{0, x\}$ since xR is a prime ideal of R; so $r = x$. Thus $x = yx$, and hence $x(1 - y) = 0$. But R is quasilocal; so $1 - y \in U(R)$, and thus $x = 0$, a contradiction. If $|R| = 8$, then $R \cong \mathbb{Z}_8$, $\mathbb{Z}_2[X]/(X^3)$, or $\mathbb{Z}_4[X]/(2X, X^2 - 2)$; and if $|R| = 9$, then $R \cong \mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^3)$ by Anderson et al. (2003, Corollary 3.11). As each of these rings is a chained ring, the result follows. □

We close this section with several examples.

Example 4.8. (a) Let R be the (nonreduced) chained ring $\mathbb{Z}_{p^n}$, where p is prime and $n \geq 2$. Then $\mathrm{diam}(\Gamma(R)) = 0$ if and only if $p = 2$ and $n = 2$, $\mathrm{diam}(\Gamma(R)) = 1$ if and only if $p > 2$ and $n = 2$, and $\mathrm{diam}(\Gamma(R)) = 2$ if and only if $n \geq 3$.

We have $\mathrm{gr}(\Gamma(R)) = \infty$ if either $p = 2$ and $2 \leq n \leq 3$ or $p = 3$ and $n = 2$; otherwise, $\mathrm{gr}(\Gamma(R)) = 3$.

(b) We have $N(R) \subseteq Nil(R) \subseteq Z(R)$ for any ring R. We give examples to show that all four cases for inclusion or proper inclusion are possible when R is a chained ring. The easy details are left to the reader. Recall that $\{0\} \neq Nil(R) \subsetneq Z(R)$ forces a chained ring R to be infinite, and thus so is $\Gamma(R)$. (i) Let $R = \mathbb{Z}_4$. Then $N(R) = Nil(R) = Z(R)$. (ii) Let $R = (\mathbb{Z}_{(2)} + X\mathbb{Q}[[X]])/(X)$. Then $N(R) = Nil(R) \subsetneq Z(R)$. (iii) Let $R = \mathbb{Z}_8$. Then $N(R) \subsetneq Nil(R) = Z(R)$. (iv) Let $R = (\mathbb{Z}_{(2)} + X\mathbb{Q}[[X]])/(X^2)$. Then $N(R) \subsetneq Nil(R) \subsetneq Z(R)$.

(c) Let R_1 and R_2 be chained rings and $R = R_1 \times R_2$. Then $N(R) = N(R_1) \times N(R_2)$ and R is never a chained ring since the ideals $(1, 0)R$ and $(0, 1)R$ are not comparable. Note that $N(R)^*$ is still a complete subgraph of $\Gamma(R)$ and any $(x, y) \in \Gamma(R)$ is still adjacent to some element of $N(R)^*$, but $\Gamma(R)\backslash N(R)$ is not totally disconnected since $(0, 1)$ and $(1, 0)$ are adjacent.

(d) We have already observed that for a chained ring R, its zero-divisor graph $\Gamma(R)$ is complete if and only if $Z(R) = N(R)$. However, if R is not a chained ring, then $Z(R) = N(R)$ does not imply that $\Gamma(R)$ is complete. For example, let $R\mathbb{Z}_2[X, Y]/(X^2, Y^2) = \mathbb{Z}_2[x, y]$. Then R is not a chained ring since the ideals xR and yR are not comparable. However, $N(R) = Nil(R) = Z(R) = \{0, x, y, x + y, xy, x + xy, y + xy, x + y + xy\}$, but $\Gamma(R)$ is not complete since $xy \neq 0$. Note that the prime ideals of R are (trivially) linearly ordered, $\mathrm{diam}(\Gamma(R)) = 2$, and $\mathrm{gr}(\Gamma(R)) = 3$.

(e) A ring R such that $Nil(R)^*(= N(R)^*)$ is a complete subgraph of $\Gamma(R)$ and $\Gamma(R)\backslash Nil(R)$ is totally disconnected, but R is not a chained ring. Let D be an integral domain which is not a valuation domain, and let K be the quotient field of D. Set $R = D(+)(K/D)$; for example, let $R = \mathbb{Z}(+)(\mathbb{Q}/\mathbb{Z})$. Note that $N(R) = Nil(R) = \{0\}(+)(K/D) \subsetneq Z(R) = (D\backslash U(D))(+)(K/D)$ and $Nil(R)^2 = \{0\}$. Thus one can easily verify that R satisfies the desired conditions.

5. $\Gamma(R)$ WHEN $R \in \mathcal{H}$

In this final section, we are interested in the case where the ring R satisfies $\{0\} \neq Nil(R) \subseteq zR$ for all $z \in Z(R)\backslash Nil(R)$. In particular, this condition holds when $R \in \mathcal{H}$ is not an integral domain (i.e., when $Nil(R)$ is a nonzero divided prime ideal of R; so $\{0\} \neq Nil(R) \subseteq zR$ for all $z \in R\backslash Nil(R)$). We start by showing that if $\{0\} \neq Nil(R) \subseteq zR$ for all $z \in Z(R)\backslash Nil(R)$, then $Nil(R)$ is a prime ideal of R (cf. the proof of Anderson and Badawi, 2001, Proposition 5.1), and that $Nil(R)$ is a divided prime ideal of R when $Nil(R) \subsetneq Z(R)$.

Theorem 5.1. *Let R be a ring with $\{0\} \neq Nil(R) \subseteq zR$ for all $z \in Z(R)\backslash Nil(R)$.*

(1) *$Nil(R)$ is a prime ideal of R.*
(2) *$Nil(R) \subseteq \bigcap_{n\geq 1} z^n R$ for all $z \in Z(R)\backslash Nil(R)$.*
(3) *If $Nil(R) \subsetneq Z(R)$, then $Nil(R)$ is a divided prime ideal of R.*

Proof. (1) If $Nil(R) = Z(R)$, then $Nil(R)$ is a prime ideal of R. So we may assume that $Nil(R) \subsetneq Z(R)$ and $Nil(R) \subseteq zR$ for all $z \in Z(R)\backslash Nil(R)$. Suppose that

$Nil(R)$ is not prime. Then there are $x, y \in Z(R)\backslash Nil(R)$ with $xy \in Nil(R)$. Thus $x^2 \in Z(R)\backslash Nil(R)$, and hence $Nil(R) \subseteq x^2R$. Thus $xy = x^2d$ for some $d \in R$, and hence $y - xd \notin Nil(R)$ since $xd \in Nil(R)$ and $y \notin Nil(R)$. Since $(y - xd)x0$, we have $y - xd \in Z(R)\backslash Nil(R)$. Thus $Nil(R) \subseteq (y - xd)R$, and hence $xNil(R) \subseteq x(y - xd)R = \{0\}$. Let $0 \neq z \in Nil(R) \subseteq x^2R$. Then $z = x^2r$ for some $r \in R$, and $xr \in Nil(R)$. Thus $z = x(xr) = 0$, a contradiction. Hence $Nil(R)$ is a prime ideal of R.

(2) Let $z \in Z(R)\backslash Nil(R)$. Then $z^n \in Z(R)\backslash Nil(R)$ for all integers $n \geq 1$ since $Nil(R)$ is a prime ideal of R by part (1), and thus $Nil(R) \subseteq z^nR$ for all integers $n \geq 1$. Hence $Nil(R) \subseteq \bigcap_{n\geq 1} z^nR$.

(3) Let $z \in R\backslash Nil(R)$ and $w \in Z(R)\backslash Nil(R)$. Then $wz \in Z(R)\backslash Nil(R)$, and thus $Nil(R) \subseteq wzR \subseteq zR$. Hence $Nil(R)$ is a divided prime ideal of R. □

Corollary 5.2. *The following statements are equivalent for a ring R:*

(1) $\{0\} \neq Nil(R) \subseteq zR$ *for all* $z \in Z(R)\backslash Nil(R)$ *and* $Nil(R) \subsetneq Z(R)$;
(2) $R \in \mathcal{H}$ *and* $Nil(R) \subsetneq Z(R)$.

The simplest example of a ring R with $\{0\} \neq Nil(R) \subseteq zR$ for all $z \in Nil(R)\backslash Nil(R)$ and $Nil(R) \subsetneq Z(R)$ is a nondomain chained ring R with $\dim(R) \geq 1$. We next give two examples to show that the condition $\{0\} \neq Nil(R) \subseteq zR$ for all $z \in Z(R)\backslash Nil(R)$ neither implies nor is implied by the condition that the prime ideals of R contained in $Z(R)$ are linearly ordered. We also show that the $Nil(R) \subsetneq Z(R)$ hypothesis is needed in part (3) of Theorem 5.1.

Example 5.3. (a) Let $R = \mathbb{Z}(+)\mathbb{Z}_2$. Then $N(R) = Nil(R) = \{0\}(+)\mathbb{Z}_2$ and $Z(R) = 2\mathbb{Z}(+)\mathbb{Z}_2$. Thus the prime ideals of R contained in $Z(R)$, namely $Nil(R)$ and $Z(R)$, are linearly ordered, but $Nil(R) \not\subseteq (2, 0)R$ for $(2, 0) \in Z(R)\backslash Nil(R)$.

(b) Let $R = \mathbb{Z}(+)(\mathbb{Q}/\mathbb{Z})$. Then $N(R) = Nil(R) = \{0\}(+)(\mathbb{Q}/\mathbb{Z})$ and $Z(R) = (\mathbb{Z}\backslash\{1, -1\})(+)(\mathbb{Q}/\mathbb{Z})$. Thus the prime ideals of R contained in $Z(R)$ are not linearly ordered, but $Nil(R) \subseteq zR$ for all $z \in Z(R)\backslash Nil(R)$; so $R \in \mathcal{H}$. We have $\operatorname{diam}(\Gamma(R)) = 3$ since $d((2, 0), (3, 0)) = 3$. Also note that R is a McCoy ring, $\operatorname{gr}(\Gamma(R)) = 3$, and $R \cong (\mathbb{Z} + X\mathbb{Q}[[X]])/(X)$.

(c) Let $R = \mathbb{Z}_4[X]$ (or $\mathbb{Z}_4[[X]]$). Then $N(R) = Nil(R) = Z(R) = 2R$; so $\{0\} \neq Nil(R) \subseteq zR$ for all $z \in Z(R)\backslash Nil(R)$. But $Nil(R)$ is not divided since $Nil(R) = 2R \not\subseteq XR$.

Suppose that $R \in \mathcal{H}$ with $Nil(R) \subsetneq Z(R)$. Then we have already observed that $Z(R)\backslash Nil(R)$ must be infinite (Theorem 3.5(2)). In fact, both $Nil(R)$ and $Z(R)\backslash Nil(R)$ are infinite.

Theorem 5.4. *Let $R \in \mathcal{H}$ with $Nil(R) \subsetneq Z(R)$.*

(1) *If $xy = 0$ for $x \in Z(R)\backslash Nil(R)$ and $y \in R$, then $y \in N(R) \subseteq Nil(R)$ and $yNil(R) = \{0\}$. Thus $\operatorname{ann}_R(x) \subseteq \operatorname{ann}_R(Nil(R))$.*
(2) *$Nil(R)$ is infinite.*
(3) *$\Gamma(R)\backslash Nil(R)$ is infinite and totally disconnected.*

Proof. (1) Suppose that $xy = 0$ for $x \in Z(R)\backslash Nil(R)$ and $y \in R$. Then $y \in Nil(R)$ since $Nil(R)$ is a prime ideal of R. Then $Nil(R) \subseteq xR$ since $Nil(R)$ is a divided prime ideal, and thus $yNil(R) \subseteq xyR = \{0\}$. In particular, $y^2 = 0$; so $y \in N(R)$.

(2) Let $x \in Z(R)\backslash Nil(R)$. We have $xz = 0$ for some $z \in Nil(R)^*$. Then for each integer $n \geq 1$, we have $z = z_n x^n$ for some $z_n \in R$ by Theorem 5.1(2). Note that $z_n \in Nil(R)^*$ since $Nil(R)$ is a prime ideal of R and $x^n \notin Nil(R)$. If $z_n = z_m$ for some integers $n > m \geq 1$, then $z = x^n z_n = x^n z_m = x^{n-m}(x^m z_m) = x^{n-m} z = 0$, a contradiction. Thus $Nil(R)$ is infinite.

(3) Since $Nil(R)$ is a prime ideal of R, the graph $\Gamma(R)\backslash Nil(R)$ is totally disconnected by Theorem 3.5(1) and infinite by Theorem 3.5(2). □

We can now describe the structure of $\Gamma(R)$ when $R \in \mathcal{H}$ and $Nil(R) \subsetneq Z(R)$. The subgraph $\Gamma(R)\backslash Nil(R)$ is infinite and totally disconnected, $Nil(R)^*$ is infinite, and for each vertex $x \in \Gamma(R)\backslash Nil(R)$, there is a vertex $y \in Nil(R)^*$ such that y is adjacent to x and to all other elements of $Nil(R)^*$.

Since $N(R) \subseteq Nil(R)$, the graph $\Gamma(R)\backslash Nil(R)$ is totally disconnected when $\Gamma(R)\backslash N(R)$ is totally disconnected (so this happens when R is a chained ring). However, our next example shows that we may have $\Gamma(R)\backslash Nil(R)$ totally disconnected, but $\Gamma(R)\backslash N(R)$ is not totally disconnected for a ring $R \in \mathcal{H}$ with the prime ideals of R contained in $Z(R)$ linearly ordered.

Example 5.5. Let $D = \mathbb{Z}_{(2)} + X\mathbb{R}[[X]]$ and $I = X^2 D = \mathbb{Z}_{(2)}X^2 + X^3\mathbb{R}[[X]]$. Set $R = D/I$. Then R is quasilocal with maximal ideal $Z(R) = (2\mathbb{Z}_{(2)} + X\mathbb{R}[[X]])/I = 2R$ and $Nil(R) = X\mathbb{R}[[X]]/I$. Note that R is not a chained ring and the prime ideals of R contained in $Z(R)$, namely $Nil(R)$ and $Z(R)$, are linearly ordered. Let $f = \pi X + I$ and $g = \pi^{-1}X + I$. Then $f, g \in Nil(R)\backslash N(R)$, but $fg = X^2 + I = 0$; so $\Gamma(R)\backslash N(R)$ is not totally disconnected. Also $N(R)$ is not an ideal of R and $N(R)^2 \neq \{0\}$(and hence $Nil(R)^2 \neq \{0\}$) since $f = \sqrt{2}X + I$, $g = \sqrt{3}X + I \in N(R)$, but $f + g \notin N(R)$ and $fg = \sqrt{6}X^2 + I \neq 0$. It is easy to check that $R \in \mathcal{H}$.

The next example shows that Theorem 5.4(1) need not hold if we only assume that the prime ideals of R contained in $Z(R)$ are linearly ordered.

Example 5.6. Let $D = \mathbb{Q}[X, Y, Z]_{(X,Y,Z)}$ and $I = (X^2, Y^2, XZ)_{(X,Y,Z)}$. Set $R = D/I = \mathbb{Q}[x, y, z]$. Then $Nil(R) = (x, y) \subsetneq (x, y, z) = Z(R)$. The prime ideals of R contained in $Z(R)$, namely, $Nil(R)$ and $Z(R)$, are linearly ordered. Then $z \in Z(R)\backslash Nil(R)$ and $xz = 0$, but $xNil(R) \neq \{0\}$ since $xy \neq 0$. Note that $N(R)$ is not an ideal of R and $Nil(R)^2 \neq \{0\}$.

Observe that if $R \in \mathcal{H}$ and $Nil(R) \subsetneq Z(R)$, then $Ker(\phi) = \{w \in Nil(R) \mid zw = 0$ for some $z \in Z(R)\backslash Nil(R)\} \subseteq Nil(R)$. Thus $Ker(\phi)^*$ is precisely the set of vertices of $\Gamma(R)$ which are adjacent to some vertex of $\Gamma(R)\backslash Nil(R)$. Clearly, $Nil(R) \subseteq Ker(\phi)$ when $\phi(R)$ is an integral domain, and thus $Ker(\phi) = Nil(R)$ when $\phi(R)$ is an integral domain.

Corollary 5.7. *Let $R \in \mathcal{H}$ with $Nil(R) \subsetneq Z(R)$. Then $Nil(R)Ker(\phi) = \{0\}$, and thus $Ker(\phi)^2 = \{0\}$ (so $Ker(\phi) \subseteq N(R)$). In particular, when $\phi(R)$ is an integral domain, then $Nil(R)^2 = \{0\}$, and hence $Nil(R)^*$ is a complete subgraph of $\Gamma(R)$.*

Proof. Let $y \in Ker(\phi)$. Then there is a $z \in Z(R)\backslash Nil(R)$ with $zy = 0$. Thus $y\,Nil(R) = \{0\}$ by Theorem 5.4(1), and hence $Nil(R) = Ker(\phi) = \{0\}$. Thus $Ker(\phi)^2 = \{0\}$ since $Ker(\phi) \subseteq Nil(R)$. Now suppose that $\phi(R)$ is an integral domain. Then $Nil(R)Ker(\phi)$, and hence $Nil(R)^2 = \{0\}$. Thus $Nil(R)^*$ is a complete subgraph of $\Gamma(R)$. □

Remark 5.8. The proof of Theorem 5.4(2) actually shows that $Ker(\phi)$ is infinite since z and each z_n are in $Ker(\phi)$. Thus by the above corollary, $Ker(\phi)^*$ is an infinite complete subgraph of $\Gamma(R)$ when $R \in \mathcal{H}$ and $Nil(R) \subsetneq Z(R)$. Also $Ker(\phi) \subseteq N(R) \subseteq Nil(R)$; so all three are infinite when $R \in \mathcal{H}$ and $Nil(R) \subsetneq Z(R)$.

The following is an example of a ring $R \in \mathcal{H}$ with $Nil(R) \subsetneq Z(R)$ and $Nil(R)^2 = \{0\}$, but $\phi(R)$ is not an integral domain.

Example 5.9. Let $R = \mathbb{Z}(+)(\mathbb{R}/\mathbb{Z}_{(2)})$. Then $N(R) = Nil(R) = \{0\}(+)(\mathbb{R}/\mathbb{Z}_{(2)})$, $Nil(R)^2 = \{0\}$, $Z(R) = 2\mathbb{Z}(+)(\mathbb{R}/\mathbb{Z}_{(2)})$, and $Ker(\phi) = \{0\}(+)(\mathbb{Q}/\mathbb{Z}_{(2)})$. Thus $R \in \mathcal{H}$ and $Ker(\phi) \subsetneq Nil(R) \subsetneq Z(R)$; so $\phi(R)$ is not an integral domain. In fact, $\phi(R) \cong R/Ker(\phi) \cong \mathbb{Z}(+)(\mathbb{R}/\mathbb{Q})$. Note that $Nil(R)^*$ (and hence $Ker(\phi)^*$) is a complete subgraph of $\Gamma(R)$, and $\Gamma(R)\backslash Nil(R)$ is totally disconnected by Theorem 5.4(3). However, $\Gamma(R)\backslash Ker(\phi)$ is not totally disconnected; for example, $(0, \pi + \mathbb{Z}_{(2)})$ and $(0, \pi^{-1} + \mathbb{Z}_{(2)})$ are adjacent in $\Gamma(R)\backslash Ker(\phi)$ (cf. Theorem 5.10).

We next give another characterization for when $\phi(R)$ is an integral domain in terms of complete and totally disconnected subgraphs of $\Gamma(R)$.

Theorem 5.10. *The following statements are equivalent for a ring $R \in \mathcal{H}$ with $Nil(R) \subsetneq Z(R)$:*

(1) $\phi(R)$ *is an integral domain*;
(2) $Nil(R) = Ker(\phi)$;
(3) $Ker(\phi)^*$ *is a complete subgraph of* $\Gamma(R)$ *and* $\Gamma(R)\backslash Ker(\phi)$ *is totally disconnected*;
(4) $\Gamma(R)\backslash Ker(\phi)$ *is totally disconnected.*

Proof. (1) $\Leftrightarrow$ (2) This is clear.

(2) $\Rightarrow$ (3) This follows from Theorem 5.4(3) and Corollary 5.7.

(3) $\Rightarrow$ (4) This is also clear.

(4) $\Rightarrow$ (2) We always have $Ker(\phi) \subseteq Nil(R)$ since $R \in \mathcal{H}$. Suppose that there is a $w \in Nil(R)\backslash Ker(\phi)$, and let $z \in Z(R)\backslash Nil(R)$. Then $zw \in Nil(R)\backslash Ker(\phi)$; so $zw \neq 0$. For if $zw \in Ker(\phi)$, then $tzw = 0$ for some $t \in Z(R)\backslash Nil(R)$. Thus $w \in Ker(\phi)$ since $tz \in Z(R)\backslash Nil(R)$, a contradiction. Also $zw \neq w$. For if $zw = w$, then $(z-1)w = 0$, and hence $z - 1 \in Z(R)^*$. Also $z - 1 \notin Nil(R)$ since $z - 1 \in Nil(R)$ implies that $z = 1 + (z-1) \in U(R)$, a contradiction. But then $z - 1 \in Z(R)\backslash Nil(R)$ and $(z-1)w = 0$; so $w \in Ker(\phi)$, a contradiction. If $w^2 = 0$, then $w - zw$ is an edge in $\Gamma(R)\backslash Ker(\phi)$, a contradiction. Hence we may assume that $w^2 \neq 0$. Let m (≥ 3) be the least positive integer such that $w^m = 0$. If $w^{m-1} \notin Ker(\phi)$, then $w - w^{m-1}$ is an edge in $\Gamma(R)\backslash Ker(\phi)$, which is again a contradiction. Thus let k, $1 \leq k \leq m-1$,

be the least positive integer such that $w^k \in Ker(\phi)$, and let $d \in Z(R)\setminus Nil(R)$ such that $dw^k = 0$. Then $k \geq 2$ since $w \notin Ker(\phi)$. Also $dw^{k-1} \notin Ker(\phi)$. For if $dw^{k-1} \in Ker(\phi)$, then $tdw^{k-1} = 0$ for some $t \in Z(R)\setminus Nil(R)$. Hence $w^{k-1} \in Ker(\phi)$ since $td \in Z(R)\setminus Nil(R)$, a contradiction. Since $w \neq dw^{k-1}$ because $w^2 \neq 0$, we have that $w - dw^{k-1}$ is an edge in $\Gamma(R)\setminus Ker(\phi)$, a contradiction. Hence $Ker(\phi) = Nil(R)$. □

Example 5.3(b) shows that a ring $R \in \mathscr{H}$ with $Nil(R) \subsetneq Z(R)$ may have $\text{diam}(\Gamma(R)) = 3$. Thus any of the possible diameters, $0, 1, 2$, or 3, may be realized by a ring in $\mathscr{H}$. However, if $R \in \mathscr{H}$ and $Nil(R) \subsetneq Z(R)$, then $\text{diam}(\Gamma(R))$ is either 2 or 3. For if $\text{diam}(\Gamma(R)) = 0$ or 1, then $Z(R)^2 = \{0\}$, and thus $Nil(R) = Z(R)$.

We end the article with the analog of Theorem 2.12 for rings in $\mathscr{H}$. Note that the $\text{gr}(\Gamma(R)) = \infty$ case is not possible since $\Gamma(R)$ cannot be an infinite star graph.

Theorem 5.11. *Let $R \in \mathscr{H}$ with $Nil(R) \subsetneq Z(R)$. Then* $\text{gr}(\Gamma(R)) = 3$.

Proof. The theorem follows directly from Theorems 2.12 and 5.4(2). □

As an alternate proof of the above theorem, just note that $Ker(\phi)^*$ is an infinite complete subgraph of $\Gamma(R)$ when $R \in \mathscr{H}$ and $Nil(R) \subsetneq Z(R)$ by Remark 5.8; so $\text{gr}(\Gamma(R)) = 3$.

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