ON THE TOTAL GRAPH OF A COMMUTATIVE RING WITHOUT THE ZERO ELEMENT

DAVID F. ANDERSON
Department of Mathematics
University of Tennessee
Knoxville, TN 37996-1320, USA
anderson@math.utk.edu

AYMAN BADAWI
Department of Mathematics and Statistics
American University of Sharjah
P. O. Box 26666, Sharjah, United Arab Emirates
abadawi@aus.edu

Received 21 April 2011
Accepted 19 September 2011
Published 30 July 2012

Communicated by S. R. Lopez-Permouth

Let $R$ be a commutative ring with nonzero identity, and let $Z(R)$ be its set of zero-divisors. The total graph of $R$ is the (undirected) graph $T(\Gamma(R))$ with vertices all elements of $R$, and two distinct vertices $x$ and $y$ are adjacent if and only if $x + y \in Z(R)$. Let $Z(\Gamma(R))$ be the (induced) subgraph of $T(\Gamma(R))$ with vertices $Z(R) \setminus \{0\}$ and $R \setminus \{0\}$, respectively. We determine when $Z_0(\Gamma(R))$ and $T_0(\Gamma(R))$ are connected and compute their diameter and girth. We also investigate zero-divisor paths and regular paths in $T_0(\Gamma(R))$.

Keywords: Total graph; zero-divisor graph; total graph without zero.

Mathematics Subject Classification: 13A15, 05C99
$T_0(\Gamma(R))$ has vertices $Z(R)^* = Z(R)\backslash\{0\}$ (respectively, $R^* = R\backslash\{0\}$), and two distinct vertices $x$ and $y$ are adjacent if and only if $x + y \in Z(R)$. Note that $Z_0(\Gamma(R))$ is a finite nonempty graph if and only if $R$ is a finite ring that is not a field (cf. [6, Theorem 2.2]). In addition to $Z(\Gamma(R))$, the (induced) subgraphs $\text{Reg}(\Gamma(R))$ and $\text{Nil}(\Gamma(R))$ of $T(\Gamma(R))$, with vertices $\text{Reg}(R)$ and $\text{Nil}(R)$, respectively, were studied in [5]. The total graph has also been investigated in [1, 2, 14].

Recently, there has been considerable attention in the literature to associating graphs with algebraic structures (see [13]). Probably the most attention has been to the zero-divisor graph $\Gamma(R)$ for a commutative ring $R$. The set of vertices of $\Gamma(R)$ is $Z(R)^*$, and two distinct vertices $x$ and $y$ are adjacent if and only if $xy = 0$. So, in some sense, $Z_0(\Gamma(R))$ is the additive analog of $\Gamma(R)$. The concept of a zero-divisor graph goes back to Beck [7], who let all elements of $R$ be vertices and was mainly interested in colorings. Our definition was introduced by Anderson and Livingston in [6], where it was shown, among other things, that $\Gamma(R)$ is connected with $\text{diam}(\Gamma(R)) \in \{0, 1, 2, 3\}$ and $\text{gr}(\Gamma(R)) \in \{3, 4, \infty\}$. For a recent survey article on zero-divisor graphs, see [4].

In the second section, we determine when $Z_0(\Gamma(R))$ is connected and show that $\text{diam}(Z_0(\Gamma(R))) \in \{0, 1, 2, \infty\}$. In the third section, we show that $\text{gr}(Z_0(\Gamma(R))) \in \{3, \infty\}$ and explicitly calculate $\text{gr}(Z_0(\Gamma(R)))$. In both cases, our answers depend on whether or not $R$ is reduced and on the number of minimal prime ideals of $R$. In the fourth section, we consider the graph $T_0(\Gamma(R))$, show that $\text{diam}(T_0(\Gamma(R))) = \text{diam}(T(\Gamma(R)))$ when $|R| \geq 4$, and determine its girth. In the final section, we define and investigate zero-divisor paths and regular paths in $T_0(\Gamma(R))$.

Let $\Gamma$ be a graph. For vertices $x$ and $y$ of $\Gamma$, we define $d(x, y)$ to be the length of a shortest path from $x$ to $y$ ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no path). Then the diameter of $\Gamma$ is $\text{diam}(\Gamma) = \sup\{d(x, y) \mid x$ and $y$ are vertices of $\Gamma\}$. The girth of $\Gamma$, denoted by $\text{gr}(\Gamma)$, is the length of a shortest cycle in $\Gamma$ ($\text{gr}(\Gamma) = \infty$ if $\Gamma$ contains no cycles).

Throughout, $R$ will be a commutative ring with nonzero identity, $Z(R)$ its set of zero-divisors, $\text{Reg}(R) = R\backslash Z(R)$ its set of regular elements, $\text{Idem}(R)$ its set of idempotent elements, $\text{Nil}(R)$ its ideal of nilpotent elements, $U(R)$ its group of units, and total quotient ring $T(R) = R_{\text{Reg}(R)}$. For any $A \subseteq R$, let $A^* = A\backslash\{0\}$. We say that $R$ is reduced if $\text{Nil}(R) = \{0\}$ and that $R$ is quasilocal if $R$ has a unique maximal ideal. Let $\text{Spec}(R)$ denote the set of prime ideals of $R$, $\text{Max}(R)$ the set of maximal ideals of $R$, and $\text{Min}(R)$ the set of minimal prime ideals of $R$. Any undefined notation or terminology is standard, as in [10, 11], or [8].

We would like to thank the referee for a careful reading of the paper and several helpful suggestions.

2. The Diameter of $Z_0(\Gamma(R))$

In this section, we show that $Z_0(\Gamma(R))$ is connected unless $R$ is a reduced ring with exactly two minimal prime ideals. Moreover, if $Z_0(\Gamma(R))$ is connected, then
diam($Z_0(\Gamma(R))$) $\leq 2$. The case for $Z(\Gamma(R))$ is much simpler since every nonzero vertex in $Z(\Gamma(R))$ is adjacent to 0. If $Z(R)$ is an ideal of $R$, then $Z(\Gamma(R))$ is a complete graph [5, Theorem 2.1]; and if $Z(R)$ is not an ideal of $R$, then $Z(\Gamma(R))$ is connected with diam($Z(\Gamma(R))$) $= 2$ [5, Theorem 3.1].

We begin with a lemma containing several results which we will use throughout this paper.

**Lemma 2.1.** Let $R$ be a commutative ring.

(1) $Z(R)$ is a union of prime ideals of $R$.
(2) $P \subseteq Z(R)$ for every $P \in \text{Min}(R)$.
(3) $Z(R) = \cup \{P \mid P \in \text{Min}(R)\}$ if $R$ is reduced.
(4) Let $x \in Z(R)$ and $y \in \text{Nil}(R)$. Then $x + y \in Z(R)$.
(5) If $P_1, P_2, P_3$ are distinct minimal prime ideals of $R$, then $P_1 \cap P_2 \cap P_3 \subsetneq P_1 \cap P_2$.

**Proof.** For (1), see [11, Theorem 2 and Remarks]. Parts (2) and (3) may be found in [10, Theorem 2.1; 10, Corollary 2.4], respectively.

(4) By (1) above, $x \in P \subseteq Z(R)$ for some $P \in \text{Spec}(R)$. Since $y \in \text{Nil}(R) \subseteq P$, it follows that $x + y \in P \subseteq Z(R)$.

(5) If $P_1 \cap P_2 = P_1 \cap P_2 \cap P_3$, then $P_1 P_2 \subseteq P_1 \cap P_2 \subseteq P_3$. Thus either $P_1 \not\subseteq P_3$ or $P_2 \not\subseteq P_3$, a contradiction. $\Box$

We first study the case when $R$ is not reduced.

**Theorem 2.2.** Let $R$ be a non-reduced commutative ring. Then $Z_0(\Gamma(R))$ is connected with diam($Z_0(\Gamma(R))$) $\in \{0, 1, 2\}$.

**Proof.** Assume that $R$ is not reduced, and let $x, y \in Z(R)^*$ be distinct vertices of $Z_0(\Gamma(R))$. If either $x \in \text{Nil}(R)$ or $y \in \text{Nil}(R)$, then $x + y \in Z(R)$ by Lemma 2.1(4); so $x - y$ is an edge in $Z_0(\Gamma(R))$. Thus we may assume that $x \notin \text{Nil}(R)$, $y \notin \text{Nil}(R)$, and $x + y \notin Z(R)$. Let $0 \neq w \in \text{Nil}(R)$. Then $x - w - y$ is a path in $Z_0(\Gamma(R))$ by Lemma 2.1(4), and hence diam($Z_0(\Gamma(R))$) $\leq 2$. $\Box$

Note that $Z_0(\Gamma(R))$ is a complete graph if and only if $Z(R)$ is an ideal of $R$, and in this case, diam($Z_0(\Gamma(R))$) $\leq 1$. Also, $Z(R)$ is a union of prime ideals of $R$ by Lemma 2.1(1); so $Z(R)$ is an ideal of $R$ if and only if it is a prime ideal of $R$. Thus a non-reduced ring $R$ has diam($Z_0(\Gamma(R))$) $= 0$ if and only if $|Z(R)^*| = 1$, if and only if $R \cong \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2[X]/(X^2)$. Examples of non-reduced rings $R$ with either diam($Z_0(\Gamma(R))$) $= 1$ or diam($Z_0(\Gamma(R))$) $= 2$ are given in Example 2.9 (also see Theorem 2.8).

We next consider the case when $R$ is reduced. In this case, $R$ is an integral domain if and only if $|\text{Min}(R)| = 1$. If $R$ is an integral domain, then $Z_0(\Gamma(R))$ is the empty graph; so we assume that $|\text{Min}(R)| \geq 2$.

**Theorem 2.3.** Let $R$ be a reduced commutative ring with $|\text{Min}(R)| = 2$. Then $Z_0(\Gamma(R))$ is not connected.
Proof. Suppose that \( R \) is reduced and \( |\text{Min}(R)| = 2 \). Let \( P \) and \( Q \) be the minimal prime ideals of \( R \). Then \( \text{Nil}(R) = P \cap Q = \{0\} \), and \( Z(R) = P \cup Q \) by Lemma 2.1(3) since \( R \) is reduced. Let \( 0 \neq x \in P \) and \( 0 \neq y \in Q \). Then \( x + y \not\in Z(R) \); so there can be no path in \( Z_0(\Gamma(R)) \) from any \( a \in P^* \) to any \( b \in Q^* \). Thus, \( Z_0(\Gamma(R)) \) is not connected.

Note that the \( P^* \) and \( Q^* \) in the proof of Theorem 2.3 are the connected components of \( Z_0(\Gamma(R)) \), and each component is a complete subgraph of \( Z_0(\Gamma(R)) \). However, in this case, \( Z(R) \) is not an ideal of \( R \); so \( Z(\Gamma(R)) \) is connected with \( \text{diam}(Z(\Gamma(R))) = 2 \) when \( R \) is reduced and \( |\text{Min}(R)| = 2 \).

Theorem 2.4. Let \( R \) be a reduced commutative ring that is not an integral domain. Then \( Z_0(\Gamma(R)) \) is connected if and only if \( |\text{Min}(R)| \geq 3 \). Moreover, if \( Z_0(\Gamma(R)) \) is connected, then \( \text{diam}(Z_0(\Gamma(R))) \in \{1, 2\} \).

Proof. Suppose that \( Z_0(\Gamma(R)) \) is connected and \( R \) is reduced, but not an integral domain. Then \( |\text{Min}(R)| \geq 3 \) by Theorem 2.3. Conversely, suppose that \( R \) is reduced and \( |\text{Min}(R)| \geq 3 \). Let \( x, y \in Z(R)^* \) such that \( x + y \not\in Z(R) \) (thus \( x \neq y \)). Then there are minimal prime ideals \( P_1 \) and \( P_2 \) of \( R \) with \( x \in P_1 \) and \( y \in P_2 \) by Lemma 2.1(3), and \( P_1 \neq P_2 \) since \( x + y \not\in Z(R) \). Since \( |\text{Min}(R)| \geq 3 \), there is a \( Q \in \text{Min}(R) \setminus \{P_1, P_2\} \); so \( P_1 \cap P_2 \neq \{0\} \) by Lemma 2.1(5). Pick \( 0 \neq z \in P_1 \cap P_2 \). Then \( x - z - y \) is a path in \( Z_0(\Gamma(R)) \) from \( x \) to \( y \). Thus \( Z_0(\Gamma(R)) \) is connected with \( \text{diam}(Z_0(\Gamma(R))) \leq 2 \), and \( \text{diam}(Z_0(\Gamma(R))) \neq 0 \) since \( |Z(R)^*| \geq 2 \). Hence \( 1 \leq \text{diam}(Z_0(\Gamma(R))) \leq 2 \).

Corollary 2.5. Let \( R \) be a reduced commutative ring with \( 3 \leq |\text{Min}(R)| < \infty \). Then \( \text{diam}(Z_0(\Gamma(R))) = 2 \). In particular, \( \text{diam}(Z_0(\Gamma(R))) = 2 \) when \( R \) is a reduced Noetherian ring with \( |\text{Min}(R)| \geq 3 \).

Proof. We have \( 1 \leq \text{diam}(Z_0(\Gamma(R))) \leq 2 \) by Theorem 2.4. Also, \( \text{diam}(Z_0(\Gamma(R))) \leq 1 \) if and only if \( Z(R) \) is a prime ideal of \( R \). If \( R \) is reduced with \( \text{Min}(R) \) finite, then \( Z(R) \) is a prime ideal of \( R \) if and only if \( \text{Min}(R) = \{Z(R)\} \) by Lemma 2.1(3) and the Prime Avoidance Lemma [11, Theorem 81]. But \( |\text{Min}(R)| \geq 3 \); so \( \text{diam}(Z_0(\Gamma(R))) = 2 \). The “in particular” statement is clear since \( \text{Min}(R) \) is finite when \( R \) is Noetherian [11, Theorem 88].

Corollary 2.6. The following statements are equivalent for a commutative ring \( R \).

1. \( Z_0(\Gamma(R)) \) is not connected.
2. \( T(R) \) is a von Neumann regular ring with exactly two maximal ideals.
3. \( T(R) \) is isomorphic to \( K_1 \times K_2 \) for fields \( K_1 \) and \( K_2 \).

In particular, if \( R \) is a finite ring, then \( Z_0(\Gamma(R)) \) is connected unless \( R \cong K_1 \times K_2 \) for finite fields \( K_1 \) and \( K_2 \).

Proof. This follows directly from Theorems 2.3 and 2.4. The “in particular” statement is clear.
Let $R$ be a reduced commutative ring with $|\text{Min}(R)| \geq 3$. By Corollary 2.5, $\text{diam}(Z_0(\Gamma(R))) = 2$ if $\text{Min}(R)$ is finite. Note that $\text{diam}(Z_0(\Gamma(R))) = 1$ if and only if $Z(R)$ is an (prime) ideal of $R$; so if $R$ is reduced with $|\text{Min}(R)| \geq 3$ and $\text{diam}(Z_0(\Gamma(R))) = 1$, then both $\text{Min}(R)$ and $Z(R)$ must be infinite. An example of a reduced quasilocal commutative ring $R$ with nonzero maximal ideal $Z(R)$ is given in [3, Example 3.13] (cf. [12, Example 5.1]). For this ring $R$, both $\text{Min}(R)$ and $Z(R)$ are infinite, and $Z_0(\Gamma(R))$ is connected with $\text{diam}(Z_0(\Gamma(R))) = 1$.

The next two theorems summarize results about $\text{diam}(Z(\Gamma(R)))$ (mentioned earlier from [5]) and $\text{diam}(Z_0(\Gamma(R)))$ when $R$ is a finite commutative ring. Note that $\text{Max}(R) = \text{Min}(R)$ when $R$ is a finite commutative ring.

**Theorem 2.7.** Let $R$ be a finite commutative ring. Then $\text{diam}(Z(\Gamma(R))) \in \{0, 1, 2\}$. Moreover,

1. $\text{diam}(Z(\Gamma(R))) = 0$ if and only if $R$ is a field,
2. $\text{diam}(Z(\Gamma(R))) = 1$ if and only if $R$ is local and not a field, and
3. $\text{diam}(Z(\Gamma(R))) = 2$ if and only if $R$ is not local.

**Theorem 2.8.** Let $R$ be a finite commutative ring. Then $\text{diam}(Z_0(\Gamma(R))) \in \{0, 1, 2, \infty\}$. Moreover,

1. $Z_0(\Gamma(R))$ is the empty graph if and only if $R$ is a field,
2. $\text{diam}(Z_0(\Gamma(R))) = 0$ if and only if $R$ is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$,
3. $\text{diam}(Z_0(\Gamma(R))) = 1$ if and only if $R$ is a local ring with maximal ideal $M$ and $|M| \geq 3$,
4. $\text{diam}(Z_0(\Gamma(R))) = 2$ if and only if either $|\text{Max}(R)| \geq 3$ or $R$ is not reduced with $|\text{Max}(R)| = 2$, and
5. $\text{diam}(Z_0(\Gamma(R))) = \infty$ if and only if $R$ is reduced with $|\text{Max}(R)| = 2$.

We next illustrate the above results by computing $\text{diam}(Z_0(\Gamma(R)))$ for $R = \mathbb{Z}_n$ and $R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$. The details are left to the reader; they follow directly from Theorem 2.8.

**Example 2.9.** (a) (diam($Z_0(\Gamma(Z_n))$)) Let $R = \mathbb{Z}_n$ with $n \geq 2$ and $n$ not prime (note that $Z_0(\Gamma(Z_n))$ is the empty graph if $n$ is prime). Then $\text{diam}(Z_0(\Gamma(Z_4))) = 0$; $\text{diam}(Z_0(\Gamma(Z_{p^m}))) = 1$ if either $p = 2$ and $m \geq 3$, or $p \geq 3$ is prime and $m \geq 2$; $\text{diam}(Z_0(\Gamma(Z_{pq}))) = \infty$ for distinct primes $p$ and $q$; and $\text{diam}(Z_0(\Gamma(R))) = 2$ otherwise. (b) (diam($Z_0(\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}))$)) Let $R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ with $2 \leq n_1 \leq \cdots \leq n_k$ and $k \geq 2$. Then $\text{diam}(Z_0(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q))) = \infty$ for primes $p \leq q$; otherwise $\text{diam}(Z_0(\Gamma(R))) = 2$.

3. **The Girth of $Z_0(\Gamma(R))$**

In this section, we show that $\text{gr}(Z_0(\Gamma(R))) \in \{3, \infty\}$. If $Z(R)$ is an ideal of $R$, then it is clear that $\text{gr}(Z_0(\Gamma(R))) = \infty$ if $|Z(R)| \leq 3$ and $\text{gr}(Z_0(\Gamma(R))) = 3$ if
$|Z(R)| \geq 4$. Just as for the diameter in Sec. 2, our answer depends on the number of minimal prime ideals of $R$. If $Z(R)$ is an ideal of $R$, then, $\text{gr}(Z(\Gamma(R))) = \infty$ if $|Z(R)| \leq 2$ and $\text{gr}(Z(\Gamma(R))) = 3$ if $|Z(R)| \geq 3$. If $Z(R)$ is not an ideal of $R$, then $\text{gr}(Z(\Gamma(Z_2 \times Z_2))) = \infty$ and $\text{gr}(Z(\Gamma(R))) = 3$ if $R \neq Z_2 \times Z_2$ [5, Theorem 3.14(1)] (also, see Theorem 3.3(1)).

We first handle the case when $R$ is not reduced.

**Theorem 3.1.** Let $R$ be a non-reduced commutative ring. Then $\text{gr}(Z_0(\Gamma(R))) = \infty$ if and only if $R$ has a unique nonzero minimal prime ideal $P$ with $P = \text{Nil}(R) = Z(R)$ and $|P| \leq 3$ (i.e. $\text{gr}(Z_0(\Gamma(R))) = \infty$ if and only if $\text{Nil}(R) = Z(R)$ and $|\text{Nil}(R)| \leq 3$). Otherwise, $\text{gr}(Z_0(\Gamma(R))) = 3$. Moreover, $\text{gr}(Z_0(\Gamma(R))) = \infty$ if $|Z(R)| \leq 3$ and $\text{gr}(Z_0(\Gamma(R))) = 3$ if $|Z(R)| \geq 4$.

**Proof.** Suppose that $|\text{Min}(R)| \geq 2$. Let $P$ and $Q$ be distinct minimal prime ideals of $R$. Then $\{0\} \subseteq P \cap Q \subseteq P$, so $|P \cap Q| \geq 2$, and thus $|P| \geq 4$. Let $x, y, z \in P^*$ be distinct. Then $x - y - z - x$ is a triangle in $Z_0(\Gamma(R))$; so $\text{gr}(Z_0(\Gamma(R))) = 3$. Now suppose that $\text{Min}(R) = \{P\}$, and thus $\text{Nil}(R) = P$. If $\text{Nil}(R) \subseteq Z(R)$, then there is a prime ideal $Q$ of $R$ with $\{0\} \neq \text{Nil}(R) = P \subseteq Q \subseteq Z(R)$ by Lemma 2.1(1). As above, $|Q| \geq 4$; so again $\text{gr}(Z_0(\Gamma(R))) = 3$. If $\text{Nil}(R) = Z(R)$, then $\text{gr}(Z_0(\Gamma(R))) = 3$ if $|\text{Nil}(R)| \geq 4$ and $\text{gr}(Z_0(\Gamma(R))) = \infty$ if $|\text{Nil}(R)| \leq 3$. The “moreover” statement follows directly from the above arguments.

We next consider the case when $R$ is reduced.

**Theorem 3.2.** Let $R$ be a reduced commutative ring that is not an integral domain. Then $\text{gr}(Z_0(\Gamma(R))) = \infty$ if and only if $\text{Min}(R) = \{P, Q\}$ with $\max\{|P|, |Q|\} \leq 3$. Otherwise, $\text{gr}(Z_0(\Gamma(R))) = 3$. In particular, $\text{gr}(Z_0(\Gamma(R))) = 3$ when $|\text{Min}(R)| \geq 3$.

**Proof.** Suppose that $P_1, P_2, P_3$ are distinct minimal prime ideals of $R$. Then $\{0\} \subseteq P_1 \cap P_2 \cap P_3 \subseteq P_1 \cap P_2 \subseteq P_1$ by Lemma 2.1(5); so $|P_1 \cap P_2| \geq 2$, and thus $|P_1| \geq 4$. Let $x, y, z \in P_1^*$ be distinct. Then $x - y - z - x$ is a triangle in $Z_0(\Gamma(R))$; so $\text{gr}(Z_0(\Gamma(R))) = 3$ if $|\text{Min}(R)| \geq 3$. Thus we may assume that $|\text{Min}(R)| = 2$; say $\text{Min}(R) = \{P, Q\}$. As in the proof of Theorem 2.3, $P \cap Q = \{0\}$ and $Z(R) = P \cup Q$, and hence no $x \in P^*$ and $y \in Q^*$ are adjacent in $Z_0(\Gamma(R))$. Thus $\text{gr}(Z_0(\Gamma(R))) = 3$ if and only if either $|P| \geq 4$ or $|Q| \geq 4$. Otherwise, $\text{gr}(Z_0(\Gamma(R))) = \infty$. The “in particular” statement is clear.

Using earlier mentioned results from [5] and Theorems 3.1 and 3.2, we can give explicit calculations for $\text{gr}(Z_0(\Gamma(R)))$ and $\text{gr}(Z(\Gamma(R)))$.

**Theorem 3.3.** Let $R$ be a commutative ring. Then $\text{gr}(Z(\Gamma(R))) \in \{3, \infty\}$ and $\text{gr}(Z_0(\Gamma(R))) \in \{3, \infty\}$.

1. $\text{gr}(Z(\Gamma(R))) = \infty$ if and only if either $R$ is an integral domain or $R$ is isomorphic to $Z_4$, $Z_2[X]/(X^2)$, or $Z_2 \times Z_2$. Otherwise, $\text{gr}(Z(\Gamma(R))) = 3$. 

1250074-6
(2) $Z_0(\Gamma(R))$ is the empty graph if and only if $R$ is an integral domain. For $R$ not an integral domain, $\text{gr}(Z_0(\Gamma(R))) = \infty$ if and only if $R$ is isomorphic to $\mathbb{Z}_4$, $\mathbb{Z}_2[X]/(X^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_6$, $\mathbb{Z}_9$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, or $\mathbb{Z}_3[X]/(X^2)$. Otherwise, $\text{gr}(Z_0(\Gamma(R))) = 3$.

**Proof.** (1) First, suppose that $Z(R)$ is an ideal of $R$. If $|Z(R)| = 1$, then $R$ is an integral domain; so $|Z(\Gamma(R))| = 1$, and thus $\text{gr}(Z(\Gamma(R))) = \infty$. If $|Z(R)| = 2$, then $R$ is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$; so $|Z(\Gamma(R))| = 2$, and hence $\text{gr}(Z(\Gamma(R))) = \infty$. If $|Z(R)| \geq 3$, then $\text{gr}(Z(\Gamma(R))) = 3$ since $x - 0 - y - x$ is a triangle in $Z(\Gamma(R))$ for distinct $x, y \in Z(R)^*$. If $Z(R)$ is not an ideal of $R$, then $\text{gr}(Z(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2))) = \infty$ and $\text{gr}(Z(\Gamma(R))) = 3$ if $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$ [5, Theorem 3.14(1)]. Part (1) now follows directly from the above two cases.

(2) First, suppose that $R$ is not reduced. Then by Theorem 3.1, $\text{gr}(Z_0(\Gamma(R))) = \infty$ if and only if $\{0\} \neq \text{Nil}(R) = Z(R)$ and $|Z(R)| \leq 3$, and $\text{gr}(Z_0(\Gamma(R))) = 3$ otherwise. So in this case, $\text{gr}(Z_0(\Gamma(R))) = \infty$ if and only if $R$ is isomorphic to $\mathbb{Z}_4$, $\mathbb{Z}_2[X]/(X^2)$, $\mathbb{Z}_9$, or $\mathbb{Z}_3[X]/(X^2)$.

Next, suppose that $R$ is reduced and not an integral domain. Then by Theorem 3.2, $\text{gr}(Z_0(\Gamma(R))) = \infty$ if and only if $\text{Min}(R) = \{P, Q\}$ with $\max\{|P|, |Q|\} \leq 3$, and $\text{gr}(Z_0(\Gamma(R))) = 3$ otherwise. In the first case, we have $Z(R) = P \cup Q$ and $P \cap Q = \{0\}$ with $\max\{|P|, |Q|\} \leq 3$. In this case, $R$ is a reduced finite ring with two maximal ideals, each with two or three elements. Thus $\text{gr}(Z_0(\Gamma(R))) = \infty$ if and only if $R$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$. Part (2) now follows directly from the above two cases. □

We end this section with the analog of Example 2.9 for $\text{gr}(Z_0(\Gamma(R)))$ when $R = \mathbb{Z}_n$ or $R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$. The details are left to the reader; they follow directly from Theorem 3.3(2).

**Example 3.4.** (a) ($\text{gr}(Z_0(\Gamma(\mathbb{Z}_n)))$) Let $R = \mathbb{Z}_n$ with $n \geq 2$ and $n$ not prime (note that $Z_0(\Gamma(\mathbb{Z}_n))$ is the empty graph if $n$ is prime). Then $\text{gr}(Z_0(\Gamma(R))) = \infty$ if either $n = 4$, $n = 6$, or $n = 9$. Otherwise, $\text{gr}(Z_0(\Gamma(R))) = 3$.

(b) ($\text{gr}(Z_0(\Gamma(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})))$) Let $R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ with $2 \leq n_1 \leq \cdots \leq n_k$ and $k \geq 2$. Then $\text{gr}(Z_0(\Gamma(R))) = \infty$ if either $n_1 = n_2 = 2$, $n_1 = 2$ and $n_2 = 3$, or $n_1 = n_2 = 3$. Otherwise, $\text{gr}(Z_0(\Gamma(R))) = 3$.

4. $T_0(\Gamma(R))$

In this section, we study the graph $T_0(\Gamma(R))$. We show that $\text{diam}(T_0(\Gamma(R))) = \text{diam}(T(\Gamma(R)))$ if and only if $|R| \geq 4$. (Note that $|R| \leq 3$ if and only if $R$ is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.) We then explicitly compute $\text{gr}(T_0(\Gamma(R)))$. For $x, y \in R^*$, let $d_T(x, y)$ (respectively, $d_{T_0}(x, y)$) denote the distance from $x$ to $y$ in $T(\Gamma(R))$ (respectively, $T_0(\Gamma(R))$). We first show that these two distances are always equal.
Lemma 4.1. Let $R$ be a commutative ring and $x, y \in R^*$. Then $x, y$ are connected by a path in $T_0(\Gamma(R))$ if and only if $x, y$ are connected by a path in $T(\Gamma(R))$. Moreover, $d_{T_0}(x, y) = d_{T}(x, y)$ and $\text{diam}(T_0(\Gamma(R))) \leq \text{diam}(T(\Gamma(R)))$.

Proof. If $x, y$ are connected by a path in $T_0(\Gamma(R))$, then clearly $x, y$ are connected by a path in $T(\Gamma(R))$. Conversely, assume that $x - a_1 - \cdots - a_n - y$ is a shortest path from $x$ to $y$ in $T(\Gamma(R))$, and assume that $a_i = 0$ for some $i$ with $1 \leq i \leq n$. Then $a_{i-1}, a_{i+1} \in Z(R)^*$ and $a_{i-1} + a_{i+1} \in \text{Reg}(R)$ (let $a_0 = x$ and $a_{n+1} = y$). Let $z_i = -(a_{i-1} + a_{i+1})$. Then $x - a_1 - \cdots - a_{i-1} - z_i - a_{i+1} - \cdots - a_n - y$ is a shortest path from $x$ to $y$ in $T_0(\Gamma(R))$, and hence $x, y$ are connected by a path in $T_0(\Gamma(R))$. The “moreover” statement is clear. \qed

Recall that $T(\Gamma(R))$ is not connected if $Z(R)$ is an ideal of $R$ [5, Theorem 2.1]. If $Z(R)$ is not an ideal of $R$, then $T(\Gamma(R))$ is connected if and only if $(Z(R)) = R$ (i.e. $R$ is generated by $Z(R)$ as an ideal) [5, Theorem 3.3]. Moreover, in this case, $\text{diam}(T(\Gamma(R))) = n$, where $n \geq 2$ is the least positive integer such that $R = (z_1, \ldots, z_n)$ for some $z_1, \ldots, z_n \in Z(R)$ [5, Theorem 3.4]. Also, $\text{diam}(T(\Gamma(R))) = d_T(0,1)$ [5, Corollary 3.5(1)]. Thus $T(\Gamma(R))$ is connected if and only if $\text{diam}(T(\Gamma(R))) < \infty$.

Theorem 4.2. Let $R$ be a commutative ring.

1. If $|R| \leq 3$, then $T_0(\Gamma(R))$ is connected, but $T(\Gamma(R))$ is not connected.
2. If $|R| \geq 4$, then $T_0(\Gamma(R))$ is connected if and only if $T(\Gamma(R))$ is connected.

Proof. (1) If $|R| \leq 3$, then $R \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_3$. It is easily verified that (1) holds for these two rings.

(2) If $T(\Gamma(R))$ is connected, then $T_0(\Gamma(R))$ is also connected by Lemma 4.1. Conversely, assume that $T_0(\Gamma(R))$ is connected and $|R| \geq 4$. Then $R$ is not an integral domain; so there is an $x \in Z(R)^*$. Let $y \in R^*$. Then there is a path from $x$ to $y$ in $T_0(\Gamma(R))$. But $x$ is adjacent to 0 in $T(\Gamma(R))$; so there is a path from 0 to $y$ in $T(\Gamma(R))$. Thus $T(\Gamma(R))$ is also connected. \qed

Corollary 4.3. Let $R$ be a commutative ring. Then $T_0(\Gamma(R))$ is connected if and only if either $(Z(R)) = R$ or $R$ is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. Moreover, $T_0(\Gamma(R))$ is connected if and only if $\text{diam}(T_0(\Gamma(R))) < \infty$.

Proof. This follows directly from Theorem 4.2 and the discussion preceding Theorem 4.2. \qed

In general, there is no relationship between $\text{diam}(Z_0(\Gamma(R)))$ and $\text{diam}(T_0(\Gamma(R)))$. By Examples 2.9 and 4.6, we have $\text{diam}(Z_0(\Gamma(\mathbb{Z}_8))) = 1 < \infty = \text{diam}(T_0(\Gamma(\mathbb{Z}_8)))$, $\text{diam}(T_0(\Gamma(\mathbb{Z}_6))) = 2 < \infty = \text{diam}(Z_0(\Gamma(\mathbb{Z}_6)))$, and $\text{diam}(Z_0(\Gamma(\mathbb{Z}_{12}))) = 2 = \text{diam}(T_0(\Gamma(\mathbb{Z}_{12})))$. 1250074-8
Our next goal is to show that $\text{diam}(T_0(\Gamma(R))) = \text{diam}(T(\Gamma(R)))$ when $|R| \geq 4$. However, we will need the following lemma.

**Lemma 4.4.** Let $R$ be a commutative ring with $\text{diam}(T(\Gamma(R))) = n < \infty$, and let $s \in R^*$ and $u \in U(R)$ be distinct.

1. If $s \in Z(R)^*$, then $d_{T_0}(u, s) = d_T(u, s) \in \{n - 1, n\}$.
2. If $n$ is an even integer, then $d_{T_0}(u - s, s) = m = d_T(u - s, s)$ for some even integer $m \leq n$.
3. If $n$ is an odd integer and $u \neq -s$, then $d_{T_0}(u + s, s) = m = d_T(u + s, s)$ for some odd integer $m \leq n$.
4. If $n$ is an even integer, then $d_{T_0}(u - s, s) = n = d_T(u - s, s)$ for every $s \in Z(R)^*$.
5. If $n$ is an odd integer, then $d_{T_0}(u + s, s) = n = d_T(u + s, s)$ for every $s \in Z(R)^*$.

**Proof.** Observe that $n \geq 2$ by [5, Theorem 3.4].

1. Let $s - a_1 - \cdots - a_{m-1} - u$ be a shortest path from $s$ to $u$ in $T_0(\Gamma(R))$ of length $m$. Then $m = d_{T_0}(x, y) = d_T(x, y) \leq n$ by Lemma 4.1. Since $u \in \langle s, s + a_1, a_1 + a_2, \ldots, a_{m-1} + u \rangle$, we have $R = \langle s, s + a_1, a_1 + a_2, \ldots, a_{m-1} + u \rangle$. Since $R$ is generated by $m + 1$ elements of $Z(R)$ and $\text{diam}(\Gamma(R)) = n$, we have $n \leq m + 1$ by [5, Theorem 3.4]. Thus $m \leq n \leq m + 1$; so either $m = n - 1$ or $m = n$.

2. Let $n$ be an even integer. If $u - s = s$, then $d_{T_0}(u - s, s) = 0$. Thus we may assume that $u - s \neq s$, and hence $d_{T_0}(u - s, s) \geq 2$ since $(u - s) + s = u \notin Z(R)$. Let $m \geq 2$, and let $s - a_1 - \cdots - a_{m-1} - (u - s)$ be a shortest path from $s$ to $u - s$ in $T_0(\Gamma(R))$ of length $m$. Thus $m \leq n$. Suppose that $m$ is an odd integer. Since $u = s + a_1 - (a_1 + a_2) - \cdots - (a_{m-2} + a_{m-1}) + (a_{m-1} + (u - s))$, we have $R = \langle s + a_1, a_1 + a_2, a_2 + a_3, \ldots, a_{m-1} + (u - s) \rangle$ generated by $m$ elements of $Z(R)$. Hence $n \leq m$ by [5, Theorem 3.4]; so $m = n$, which is a contradiction since $n$ is an even integer. Thus $d_{T_0}(u - s, s) = m = d_T(u - s, s)$ for some even integer $m \leq n$.

3. Let $n$ be an odd integer and $s \neq -u$; so $u \neq u + s \in R^*$. If $u + 2s \in Z(R)$, then $d_{T_0}(u + s, s) = 1$. Thus we may assume that $u + 2s \notin Z(R)$, and hence $d_{T_0}(u + s, s) \geq 2$. Let $m \geq 2$, and let $s - a_1 - \cdots - a_{m-1} - (u + s)$ be a shortest path from $s$ to $u + s$ in $T_0(\Gamma(R))$ of length $m$. Thus $m \leq n$. Suppose that $m$ is an even integer. Since $-u = (s + a_1) - (a_1 + a_2) + \cdots + (a_{m-2} + a_{m-1}) - (a_{m-1} + (u + s))$, we have $R = \langle s + a_1, a_1 + a_2, a_2 + a_3, \ldots, a_{m-1} + (u + s) \rangle$ generated by $m$ elements of $Z(R)$. Hence $n \leq m$ by [5, Theorem 3.4]; so $m = n$, which is a contradiction since $n$ is an odd integer. Thus $d_{T_0}(u + s, s) = m = d_T(u + s, s)$ for some odd integer $m \leq n$.

4. Let $n$ be an even integer and $s \in Z(R)^*$. Then $u - s, s \in R^*$ are distinct and $(u - s) + s = u \notin Z(R)$; so $m = d_{T_0}(u - s, s)$ is an even positive integer by part (2) above. Let $s - a_1 - \cdots - a_{m-1} - (u - s)$ be a shortest path from $s$ to $u - s$ in $T_0(\Gamma(R))$ of length $m$. If $m = n$, then we are done; so assume that $m < n$. 

1250074-9
Since $u = 2s - (s + a_1) + (a_1 + a_2) - \cdots - (a_{m-2} + a_{m-1}) + (a_{m-1} + (u - s)),$ we have $R = (s, s + a_1, a_1 + a_2, a_2 + a_3, \ldots, a_{m-1} + (u - s))$ is generated by $m + 1$ elements of $Z(R).$ Hence $n \leq m + 1$ by [5, Theorem 3.4]. Thus $n = m + 1,$ which is a contradiction since $n$ is an even integer and $m + 1$ is an odd integer. Thus $d_{T_0}(u - s, s) = n = d_T(u - s, s).

(5) Let $n$ be an odd integer and $s \in Z(R)^*.$ Thus $u + s, s \in R^*$ are distinct and $2s + u \notin Z(R)$ (for if $2s + u \in Z(R),$ then $R = (s, 2s + u),$ and hence diam$(T(\Gamma(R))) = 2$ by [5, Theorem 3.4]); so $d = d_{T_0}(u + s, s) \geq 3$ is an odd integer by part (3) above. Let $s - a_1 - \cdots - a_{m-1} - (u + s)$ be a shortest path from $s$ to $u + s$ in $T_0(\Gamma(R))$ of length $m.$ If $m = n,$ then we are done; so assume that $m < n.$ Since $-u = 2s - (s + a_1) + (a_1 + a_2) - \cdots - (a_{m-2} + a_{m-1}) - (a_{m-1} + (u + s)),$ we have $R = (s, s + a_1, a_1 + a_2, a_2 + a_3, \ldots, a_{m-1} + (u - s))$ is generated by $m + 1$ elements of $Z(R).$ Hence $n \leq m + 1$ by [5, Theorem 3.4]. Thus $n = m + 1,$ which is a contradiction since $n$ is an odd integer and $m + 1$ is an even integer. Hence $d_{T_0}(u + s, s) = n = d_T(u + s, s).$

**Theorem 4.5.** Let $R$ be a commutative ring.

1. diam$(T_0(\Gamma(Z_2))) = 0 < \infty = \text{diam}(T(\Gamma(Z_2))).$
2. diam$(T_0(\Gamma(Z_3))) = 1 < \infty = \text{diam}(T(\Gamma(Z_3))).$
3. If $|R| \geq 4,$ then diam$(T_0(\Gamma(R))) = \text{diam}(T(\Gamma(R))).$

**Proof.** Parts (1) and (2) are easily verified; so we may assume that $|R| \geq 4.$ Then $T(\Gamma(R))$ is connected if and only if $T_0(\Gamma(R))$ is connected by Theorem 4.2, and diam$(T_0(\Gamma(R))) \leq \text{diam}(T(\Gamma(R)))$ by Lemma 4.1. Thus diam$(T(\Gamma(R))) = \infty$ if and only if diam$(T_0(\Gamma(R))) = \infty$ by Corollary 4.3 and the remarks before Theorem 4.2. Hence we may assume that diam$(T(\Gamma(R))) = n < \infty,$ and thus $R$ is not an integral domain. Let $z \in Z(R)^*.$ If $n$ is an odd integer, then $d_T(1 + z, z) = n = d_{T_0}(1 + z, z)$ by Lemma 4.4(5), and hence diam$(T(\Gamma(R))) = \text{diam}(T_0(\Gamma(R))) = n$ by Lemma 4.1. If $n$ is an even integer, then $d_T(1 - z, z) = d_{T_0}(1 - z, z) = n$ by Lemma 4.4(4), and thus diam$(T(\Gamma(R))) = \text{diam}(T_0(\Gamma(R))) = n$ by Lemma 4.1. Hence diam$(T_0(\Gamma(R))) = \text{diam}(T(\Gamma(R)))$ for all rings $R$ with $|R| \geq 4.$

The next example follows directly from Theorem 4.5 and the discussion preceding Theorem 4.2.

**Example 4.6.**

(a) diam$(T_0(\Gamma(Z_2)))$ We have observed that diam$(T_0(\Gamma(Z_2))) = 0,$ diam$(T_0(\Gamma(Z_3))) = 1,$ and diam$(T_0(\Gamma(Z_p))) = \infty$ when $p \geq 5$ is prime. Let $n = p_1^{m_1} \cdots p_k^{m_k}$ for distinct primes $p_i$ and $m_i \geq 1.$ If $k = 1$ and $m_1 \geq 2,$ then diam$(T_0(\Gamma(R))) = \infty.$ If $k \geq 2,$ then diam$(T_0(\Gamma(R))) = 2.$

(b) diam$(T_0(\Gamma(Z_{n_1} \times \cdots \times Z_{n_k})))$ Let $R = Z_{n_1} \times \cdots \times Z_{n_k}$ with $2 \leq n_1 \leq \cdots \leq n_k$ and $k \geq 2.$ Then diam$(T_0(\Gamma(R))) = 2.$

The girth of $T_0(\Gamma(R))$ is also easily determined. Recall from [5, Theorem 2.6(3)] that if $Z(R)$ is an ideal of $R,$ then gr$(T(\Gamma(R))) = 3$ if and only if $|Z(R)| \geq 3,$
\[ \text{gr}(T(\Gamma(R))) = 4 \text{ if and only if } 2 \notin Z(R) \text{ and } |Z(R)| = 2, \text{ and } \text{gr}(T(\Gamma(R))) = \infty \text{ otherwise.} \] (Note that if \(|Z(R)| = 2\), then \(R\) is isomorphic to \(\mathbb{Z}_4\) or \(\mathbb{Z}_2[X]/(X^2)\), and \(2 \in Z(R)\) in either case. So, “the \(\text{gr}(T(\Gamma(R))) = 4 \text{ case}” \text{ cannot actually happen when } Z(R) \text{ is an ideal of } R. \) If \(Z(R)\) is not an ideal of \(R\), then \(\text{gr}(T(\Gamma(R))) = 4 \text{ if and only if } R \cong \mathbb{Z}_2 \times \mathbb{Z}_2, \text{ and } \text{gr}(T(\Gamma(R))) = 3 \text{ otherwise} [5, \text{ Theorem 3.14}]. \) Thus \(\text{gr}(T(\Gamma(R))) \in \{3, 4, \infty\} \). \text{ Note that } \text{gr}(T(\Gamma(R))) \leq \text{gr}(T_0(\Gamma(R))) \text{ since } T_0(\Gamma(R)) \text{ is a (induced) subgraph of } T(\Gamma(R)). \]

We next give explicit calculations for \(\text{gr}(T(\Gamma(R)))\) and \(\text{gr}(T_0(\Gamma(R)))\). These calculations show that \(\text{gr}(T_0(\Gamma(R))) = \text{gr}(T(\Gamma(R)))\) unless \(R\) is isomorphic to \(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_9, \text{ or } \mathbb{Z}_3[X]/(X^2)\).

**Theorem 4.7.** Let \(R\) be a commutative ring. Then \(\text{gr}(T(\Gamma(R))) \in \{3, 4, \infty\} \). Moreover,

1. \(\text{gr}(T(\Gamma(R))) = \infty \text{ if and only if either } R \text{ is an integral domain or } R \text{ is isomorphic to } \mathbb{Z}_4 \text{ or } \mathbb{Z}_2[X]/(X^2),\)
2. \(\text{gr}(T(\Gamma(R))) = 4 \text{ if and only if } R \text{ is isomorphic to } \mathbb{Z}_2 \times \mathbb{Z}_2, \text{ and}\)
3. \(\text{gr}(T(\Gamma(R))) = 3 \text{ otherwise.}\)

**Proof.** By [5, Theorem 2.6(3); 5, Theorem 3.14], \(\text{gr}(T(\Gamma(R))) = 3 \text{ unless } R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ or } |Z(R)| \leq 2. \) If \(R \cong \mathbb{Z}_2 \times \mathbb{Z}_2, \text{ then } \text{gr}(T(\Gamma(R))) = 4. \) If \(|Z(R)| \leq 2, \text{ then } R \text{ is either an integral domain or isomorphic to } \mathbb{Z}_4 \text{ or } \mathbb{Z}_2[X]/(X^2). \) In each of these cases, \(\text{gr}(T(\Gamma(R))) = \infty. \) The result now follows. \(\square\)

**Theorem 4.8.** Let \(R\) be a commutative ring. Then \(\text{gr}(T_0(\Gamma(R))) \in \{3, 4, \infty\} \). Moreover,

1. \(\text{gr}(T_0(\Gamma(R))) = \infty \text{ if and only if either } R \text{ is an integral domain or } R \text{ is isomorphic to } \mathbb{Z}_4, \mathbb{Z}_2[X]/(X^2), \text{ or } \mathbb{Z}_2 \times \mathbb{Z}_2,\)
2. \(\text{gr}(T_0(\Gamma(R))) = 4 \text{ if and only if } R \text{ is isomorphic to } \mathbb{Z}_9 \text{ or } \mathbb{Z}_3[X]/(X^2), \text{ and}\)
3. \(\text{gr}(T_0(\Gamma(R))) = 3 \text{ otherwise.}\)

**Proof.** Note that \(\text{gr}(T_0(\Gamma(R))) \leq \text{gr}(Z_0(\Gamma(R))) \text{ since } Z_0(\Gamma(R)) \text{ is a (induced) subgraph of } T_0(\Gamma(R)). \) Thus Theorem 4.8 follows directly from Theorem 3.3(2) since one can easily verify that the rings \(\mathbb{Z}_4, \mathbb{Z}_2[X]/(X^2), \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_9, \mathbb{Z}_3 \times \mathbb{Z}_3, \text{ and } \mathbb{Z}_3[X]/(X^2)\) have \(\text{gr}(T_0(\Gamma(R))) \) equal to \(\infty, \infty, 3, 4, 3, \) and \(4, \) respectively. \(\square\)

We close this section with the analog of Example 2.9 for \(\text{gr}(T_0(\Gamma(R)))\). It follows directly from Theorem 4.8.

**Example 4.9.** (a) \(\text{gr}(T_0(\Gamma(Z_n))))\) Let \(R = \mathbb{Z}_n\) with \(n \geq 2.\) Then \(\text{gr}(T_0(\Gamma(Z_n))) = \infty \text{ if } n \text{ is prime, } \text{gr}(T_0(\Gamma(Z_4))) = \infty, \text{gr}(T_0(\Gamma(Z_9))) = 4, \text{ and } \text{gr}(T_0(\Gamma(R))) = 3 \text{ otherwise.}\)

(b) \(\text{gr}(T_0(\Gamma(Z_{n_1} \times \cdots \times Z_{n_k}))))\) Let \(R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}\) with \(2 \leq n_1 \leq \cdots \leq n_k \text{ and } k \geq 2. \) Then \(\text{gr}(T_0(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2))) = \infty, \text{ and } \text{gr}(T_0(\Gamma(R))) = 3 \text{ otherwise.}\)
5. Zero-Divisor Paths and Regular Paths in $T_0(\Gamma(R))$

Let $R$ be a commutative ring and $x, y \in R^*$ be distinct. We say that $x-a_1-\cdots-a_n-y$ is a zero-divisor path from $x$ to $y$ if $a_1, \ldots, a_n \in Z(R)^*$ and $a_i + a_{i+1} \in Z(R)$ for every $0 \leq i \leq n$ (let $a_0 = x$ and $a_{n+1} = y$). We define $d_Z(x, y)$ to be the length of a shortest zero-divisor path from $x$ to $y$ if $d_Z(x, x) = 0$ and $d_Z(x, y) = \infty$ if there is no such path) and $\text{diam}_Z(R) = \sup\{d_Z(x, y) \mid x, y \in R^*\}$. Thus $d_T(x, y) = d_{T_0}(x, y) \leq d_Z(x, y)$, for every $x, y \in R^*$. In particular, if $x, y \in R^*$ are distinct and $x+y \in Z(R)$, then $x-y$ is a zero-divisor path from $x$ to $y$ with $d_Z(x, y) = 1$. For any commutative ring $R$, we have $\max\{\text{diam}(Z_0(\Gamma(R))), \text{diam}(T_0(\Gamma(R)))\} \leq \text{diam}_Z(R)$. However, if $R$ is a quasilocal reduced ring with $|\text{Min}(R)| \geq 3$, then $\text{diam}(Z_0(\Gamma(R))) \leq 2$ by Theorem 2.4, but $\text{diam}_Z(R) = \infty$ since there is no zero-divisor path from 1 to any $x \in Z(R)^*$ (cf. Theorem 5.1(1)). Also, $\text{diam}(T_0(\Gamma(Z_{1225}))) = 2 < 3 = \text{diam}_Z(Z_{1225})$ by Examples 4.6 and 5.5. Note that $\text{diam}_Z(Z_2) = 0$, $\text{diam}_Z(Z_3) = 1$, and $\text{diam}_Z(R) = \infty$ for any other integral domain $R$.

We first determine when there is a zero-divisor path between every two distinct elements of $R^*$.

Theorem 5.1. Let $R$ be a commutative ring that is not an integral domain. Then there is a zero-divisor path from $x$ to $y$ for every $x, y \in R^*$ if and only if one of the following two statements holds.

1. $R$ is reduced, $|\text{Min}(R)| \geq 3$, and $R = (z_1, z_2)$ for some $z_1, z_2 \in Z(R)^*$.

2. $R$ is not reduced and $R = (z_1, z_2)$ for some $z_1, z_2 \in Z(R)^*$.

Moreover, if there is a zero-divisor path from $x$ to $y$ for every $x, y \in R^*$, then $R$ is not quasilocal and $\text{diam}_Z(R) \in \{2, 3\}$.

Proof. Suppose that there is a zero-divisor path from $x$ to $y$ for every $x, y \in R^*$. First, assume that $R$ is reduced and not an integral domain. Since $Z_0(\Gamma(R))$ is connected if and only if $|\text{Min}(R)| \geq 3$ by Theorem 2.4, we have $|\text{Min}(R)| \geq 3$. Let $y \in Z(R)^*$. Then there is a zero-divisor path $1-a_1-\cdots-a_n-y$ from 1 to $y$ for some $a_1, \ldots, a_n \in Z(R)^*$. Thus $z = 1 + a_1 \in Z(R)^*$, and hence $R = (a_1, z)$. If $R$ is not reduced, then a similar argument, as in the reduced case, shows that $R = (z_1, z_2)$ for some $z_1, z_2 \in Z(R)^*$.

Conversely, assume that (1) holds. Thus $1 = w + z$ for some $w, z \in Z(R)^*$. Let $x, y \in R^*$ be distinct. Then $x = xw + xz$ and $y = yw + yz$. We consider two cases. Case one: assume that $x, y \in Z(R)^*$. Then we are done by Theorem 2.4. Case two: assume that $x \notin Z(R)$. Hence $xw, xz \in Z(R)^*$. Suppose that $x + y \notin Z(R)$. Then assume that either $xw = yw$ or $y = \pm yw$. Then $x-(-xw) - y$ is the desired zero-divisor path of length two from $x$ to $y$. Next, assume that $xw \neq yw, yw \neq 0$ and $y \neq \pm yw$. Then $x-(-xw)-(yw)-y$ is the desired zero-divisor path of length three from $x$ to $y$. Finally, assume that $yw = 0$. Since $y \neq 0$ and $y = yw + yz$, we have $yz = y \neq 0$. Thus $x-(-xz) - y$ is the desired zero-divisor path of length two from

1250074-12
Theorem 5.4. (1) Let \( R = R_1 \times R_2 \) for commutative quasilocal rings \( R_1, R_2 \) with maximal ideals \( M_1, M_2 \), respectively. If there are \( a_1 \in U(R_1) \) and \( a_2 \in U(R_2) \) with \((2a_1, 2a_2) \in U(R)\) and \((3a_1, 3a_2) \not\in Z(R)\), then \( \text{diam}_Z(R) \in \{3, \infty\} \). Moreover, \( \text{diam}_Z(R) = 3 \) if either \( R_1 \) or \( R_2 \) is not reduced.

(2) Let \( R = R_1 \times \cdots \times R_n \) for commutative rings \( R_1, \ldots, R_n \) with \( n \geq 3 \). Then \( \text{diam}_Z(R) = 2 \).

**Proof.** (1) Let \( a = (a_1, a_2), b = (2a_1, 2a_2) \in U(R) \). Then \( a \neq b \) and \( d_Z(a, b) \neq 1 \) since \( a + b = (3a_1, 3a_2) \not\in Z(R) \). Assume that there is an \( f = (m_1, m_2) \in R^* \) such that \( a - f - b \) is a zero-divisor path from \( a \) to \( b \). Thus \( f \in Z(R)^* \); so either \( m_1 \in M_1 \) or \( m_2 \in M_2 \). If \( m_1 \in M_1 \), then \( m_1 + a_1, m_1 + 2a_1 \in U(R_1) \). Hence \( m_2 + a_2, m_2 + 2a_2 \in M_2 \), since \( a + f, b + f \in Z(R) \). But then \( a_2 = (m_2 + 2a_2) - (m_2 + a_2) \in M_2 \), a contradiction. In a similar manner, \( m_2 \in M_2 \) also leads to a contradiction; so no such \( f \) exists. Thus \( d_Z(a, b) \geq 3 \); so \( \text{diam}_Z(R) \in \{3, \infty\} \).

The “moreover” statement now follows from Theorem 5.1.

(2) We have \( \text{diam}_Z(R) \in \{2, 3\} \) by Theorem 5.1 since \(|\text{Min}(R)| \geq n \geq 3 \). Let \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in R^* \) with \( x + y \not\in Z(R) \). We may assume that \( x_1 \neq 0 \). Let \( z = (-x_1, -y_2, 1, \ldots, 1, 0) \in Z(R)^* \). Then \( x + z, y + z \in Z(R) \); so \( x - z - y \) is the desired zero-divisor path from \( x \) to \( y \) of length 2. Hence \( \text{diam}_Z(R) = 2 \).

The following example shows that all possible values for \( \text{diam}_Z(R) \) given in Corollary 5.2 and Theorem 5.4 may be realized. The details are left to the reader.

**Example 5.5.** (a) \( \text{diam}_Z(Z_n) \) We have already observed that \( \text{diam}_Z(Z_2) = 0 \), \( \text{diam}_Z(Z_3) = 1 \), and \( \text{diam}_Z(Z_p) = \infty \) when \( p \geq 5 \) is prime. Let \( R = Z_n \) with
Let \( n \geq 2 \) and \( n \) not prime. Let \( n = p_1^{m_1} \cdots p_k^{m_k} \) for distinct primes \( p_i \) and \( m_i \geq 1 \). If either \( k = 1 \), or \( k = 2 \) and \( m_1 = m_2 = 1 \), then \( \text{diam}_Z(R) = \infty \). If \( k = 2 \), \( p_1, p_2 \geq 5 \), and \( m_1 + m_2 \geq 3 \), then \( \text{diam}_Z(R) = 3 \). Otherwise, \( \text{diam}_Z(R) = 2 \).

(b) \( \text{diam}_Z(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}) \) Let \( R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k} \) with \( 2 \leq n_1 \leq \cdots \leq n_k \) and \( k \geq 2 \). If \( k = 2 \) and \( n_1, n_2 \) are prime, then \( \text{diam}_Z(R) = \infty \). If \( k = 2 \) and \( n_1 = p_1^{m_1}, n_2 = p_2^{m_2} \) for primes \( p_1, p_2 \geq 5 \) and \( m_1 + m_2 \geq 3 \), then \( \text{diam}_Z(R) = 3 \). Otherwise, \( \text{diam}_Z(R) = 2 \).

Let \( x, y \in R^* \) be distinct. We say that \( x - a_1 - \cdots - a_n - y \) is a regular path from \( x \) to \( y \) if \( a_1, \ldots, a_n \in \text{Reg}(R) \) and \( a_i + a_{i+1} \in Z(R) \) for every \( 0 \leq i \leq n \) (let \( a_0 = x \) and \( a_{n+1} = y \)). We define \( d_{\text{reg}}(x, y) \) to be the length of a shortest regular path from \( x \) to \( y \) \( (d_{\text{reg}}(x, x) = 0 \) and \( d_{\text{reg}}(x, y) = \infty \) if there is no such path), and \( \text{diam}_{\text{reg}}(R) = \sup \{d_{\text{reg}}(x, y) \mid x, y \in R^* \} \). Thus \( d_R(x, y) = d_{T_0}(x, y) \leq d_{\text{reg}}(x, y) \) for every \( x, y \in R^* \). In particular, if \( x, y \in R^* \) are distinct and \( x + y \in Z(R) \), then \( x - y \) is a regular path from \( x \) to \( y \) with \( d_{\text{reg}}(x, y) = 1 \). For any commutative ring \( R \), we have \( \max \{\text{diam}(T_0(\Gamma(R))), \text{diam}(\text{Reg}(\Gamma(R)))\} \leq \text{diam}_{\text{reg}}(R) \). Note that \( \text{diam}(T_0(\Gamma(\mathbb{Z}_{60}))) = 2 < \infty = \text{diam}_{\text{reg}}(\mathbb{Z}_{60}) \) and \( \text{diam}(\text{Reg}(\Gamma(\mathbb{Z}_{60}))) = 1 < 2 = \text{diam}_{\text{reg}}(\mathbb{Z}_{6}) \). However, if \( R \) is an integral domain, then \( T_0(\Gamma(R)) = \text{Reg}(\Gamma(R)) \); so all three diameters are equal. Moreover, \( \text{diam}_{\text{reg}}(\mathbb{Z}_2) = 0, \text{diam}_{\text{reg}}(\mathbb{Z}_3) = 1 \) and \( \text{diam}_{\text{reg}}(R) = \infty \) for any other integral domain \( R \). Hence \( \text{diam}_Z(R) = \text{diam}_{\text{reg}}(R) \) for any integral domain \( R \).

**Theorem 5.6.** Let \( R \) be a commutative ring with \( \text{diam}(T_0(\Gamma(R))) = n < \infty \).

1. Let \( u \in U(R), s \in R^* \), and \( P \) be a shortest path from \( s \) to \( u \) of length \( n - 1 \) in \( T_0(\Gamma(R)) \). Then \( P \) is a regular path from \( s \) to \( u \).
2. Let \( u \in U(R), s \in R^* \), and \( P : s - a_1 - \cdots - a_n = u \) be a shortest path from \( s \) to \( u \) of length \( n \) in \( T_0(\Gamma(R)) \). Then either \( P \) is a regular path from \( s \) to \( u \), or \( a_1 \in Z(R)^* \) and \( a_1 - \cdots - a_n = u \) is a regular path from \( a_1 \) to \( u \) of length \( n - 1 = d_{T_0}(a_1, u) \).

**Proof.**

1. If \( n = 2 \), then \( P \) is a regular path from \( s \) to \( u \) by definition. Thus we may assume that \( n > 2 \). Since \( d_{T_0}(z, u) \) is either \( n - 1 \) or \( n \) for every \( z \in Z(R)^* \) by Lemma 4.4(1) and \( d_{T_0}(s, u) = n - 1 \), we conclude that \( P \) must be a regular path.

2. Suppose that \( P \) is not a regular path; so \( a_i \in Z(R)^* \) for some \( 1 \leq i \leq n - 1 \). Since \( d_{T_0}(z, u) \) is either \( n - 1 \) or \( n \) for every \( z \in Z(R)^* \) by Lemma 4.4(1) and \( d_{T_0}(s, u) = n \), we must have \( a_1 \in Z(R)^* \) and \( a_i \in \text{Reg}(R) \) for every \( 2 \leq i \leq n - 1 \). Thus \( a_1 - \cdots - a_n - u \) is a regular path of length \( n - 1 = d_{T_0}(a_1, u) \). \( \square \)

We next determine when there is a regular path between every two distinct elements of \( R^* \).
Theorem 5.7. Let $R$ be a commutative ring.

(1) If $s \in \text{Reg}(R)$ and $w \in \text{Nil}(R)^*$, then there is no regular path from $s$ to $w$.

(2) If $R$ is reduced and quasilocal, then there is no regular path from any unit to any nonzero nonunit in $R$.

In particular, if there is a regular path from $x$ to $y$ for every $x, y \in R^*$, then either $R$ is reduced and not quasilocal or $R$ is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

Proof. (1) Let $s \in \text{Reg}(R)$ and $w \in \text{Nil}(R)^*$. Since $a + w \in \text{Reg}(R)$ for every $a \in \text{Reg}(R)$ by Lemma 2.1(4), there is no regular path from $s$ to $w$.

(2) Let $M$ be the maximal ideal of $R$, $x \in U(R)$, and $0 \neq y \in M$. Suppose that there is a regular path $x - x_1 - \cdots - x_n - y$. Then $x + x_1 = z_1 \in Z(R) \subseteq M$; so $x_1 = -x + z_1 \in U(R)$. In a similar manner, each $x_i \in U(R)$. But then $x_n + y \in U(R)$, a contradiction.

The “in particular” statement is clear by parts (1) and (2) above and the remarks preceding Theorem 5.6.

\[ \square \]

Theorem 5.8. Let $R$ be a commutative ring. Then there is a regular path from $x$ to $y$ for every $x, y \in R^*$ if and only if $R$ is reduced, $\text{Reg}(\Gamma(R))$ is connected, and for every $z \in Z(R)^*$ there is a $w \in Z(R)^*$ such that $d_Z(z, w) > 1$ (possibly with $d_Z(z, w) = \infty$).

Proof. Suppose that there is a regular path from $x$ to $y$ for every $x, y \in R^*$. Then $R$ is reduced by Theorem 5.7, and it is clear that $\text{Reg}(\Gamma(R))$ is connected. Let $z \in Z(R)^*$, and let $z - a_1 - \cdots - 1$ be a regular path from $z$ to 1. Then $a_1 \in \text{Reg}(R)$ and $w = -(z + a_1) \in Z(R)^*$. Thus $z \neq w$ and $z + w \notin Z(R)$; so $d_Z(z, w) > 1$.

Conversely, suppose that $R$ is reduced, $\text{Reg}(\Gamma(R))$ is connected, and for every $z \in Z(R)^*$ there is a $w \in Z(R)^*$ such that $d_Z(z, w) > 1$ (possibly with $d_Z(z, w) = \infty$). Let $x, y \in R^*$. If $x, y \in \text{Reg}(R)$, then there is nothing to prove. First, assume that $x \in Z(R)^*$ and $y \in \text{Reg}(R)$. Since $x \in Z(R)^*$, there is a $w \in Z(R)^*$ such that $d_Z(x, w) > 1$. Then $x + w \notin Z(R)$; so $x + w = -w \in Z(R)$ for some $u \in \text{Reg}(R)$.

Since $\text{Reg}(\Gamma(R))$ is connected, let $u - u_1 - \cdots - y$ be a regular path from $u$ to $y$. Then $x - u - u_1 - \cdots - y$ is a regular path from $x$ to $y$. Next, assume that $x, y \in Z(R)^*$. Then again as above, there are $u, v \in \text{Reg}(R)$ such that $x + u \in Z(R)$ and $y + v \in Z(R)$. If $u = v$, then $x - u - y$ is a regular path from $x$ to $y$. So assume that $u \neq v$. Since $\text{Reg}(\Gamma(R))$ is connected, let $u - \cdots - v$ be a regular path from $u$ to $v$. Then $x - u - \cdots - v - y$ is a regular path from $x$ to $y$.

\[ \square \]

In view of Theorems 2.3 and 5.8, we have the following result.

Corollary 5.9. Let $R$ be a reduced commutative ring with $|\text{Min}(R)| = 2$. Then there is a regular path from $x$ to $y$ for every $x, y \in R^*$ if and only if $\text{Reg}(\Gamma(R))$ is connected.
Recall from [9] that a commutative ring $R$ is a p.p. ring if every principal ideal of $R$ is projective. For example, a commutative von Neumann regular ring is a p.p. ring, and $\mathbb{Z} \times \mathbb{Z}$ is a p.p. ring that is not von Neumann regular. It was shown in [15, Proposition 15] that a commutative ring $R$ is a p.p. ring if and only if every element of $R$ is the product of an idempotent element and a regular element of $R$ (thus a commutative p.p. ring that is not an integral domain has non-trivial idempotents). We show that a commutative p.p. ring $R$ that is not an integral domain has $\text{diam}_{\text{reg}}(R) = 2$, but first a lemma.

**Lemma 5.10.** Let $R$ be commutative ring, $u,v \in \text{Reg}(R)$, and $e \in \text{Idem}(R)$. Then $eu + (1 - e)v \in \text{Reg}(R)$.

**Proof.** Let $eu + (1 - e)v = w \in R$, and suppose that $cw = 0$ for some $c \in R$. Then $ew = e[eu + (1 - e)v] = eu$ and $(1 - e)w = (1 - e)[eu + (1 - e)v] = (1 - e)v$. Thus $ceu = cewe = 0$ and $(1 - e)v = c(1 - e)w = 0$, and hence $ce = (1 - e) = 0$ since $u, v \in \text{Reg}(R)$. Thus $c = ce + c(1 - e) = 0$; so $eu + (1 - e)v = w \in \text{Reg}(R)$.

**Theorem 5.11.** Let $R$ be a commutative p.p. ring that is not an integral domain. Then there is a regular path from $x$ to $y$ for every $x, y \in R^*$. Moreover, $\text{diam}_{\text{reg}}(R) = \text{diam}(T_0(\Gamma(R))) = \text{diam}(T(\Gamma(R))) = 2$.

**Proof.** Let $x, y \in R^*$ be distinct, and suppose that $x + y \notin Z(R)$. We consider three cases. Case one: assume that $x, y \in Z(R)^*$. Since $x + y \notin Z(R)$, necessarily $x + y \in \text{Reg}(R)$, and thus $x - (x + y)$ is the desired regular path of length two from $x$ to $y$. Case two: assume that $x, y \in \text{Reg}(R)$. Since $R$ is a p.p. ring and not an integral domain, there is an $e \in \text{Idem}(R) \setminus \{0, 1\}$. Hence $w = -(1 - e)x + ey \in \text{Reg}(R)$ by Lemma 5.10. Since $e(1 - e) = 0$ and $e \notin \{0, 1\}$, we have $x + w = ex - ey = e(x - y) \in Z(R)$ and $y + w = (e - 1)x - (e - 1)y = (e - 1)(x - y) \in Z(R)$. Thus $x - w - y$ is the desired regular path of length two from $x$ to $y$. Case three: assume that $x \in \text{Reg}(R)$ and $y \in Z(R)^*$. Hence $y = fu$ for some $f \in \text{Idem}(R) \setminus \{0, 1\}$ and $u \in \text{Reg}(R)$. Then $h = -(1 - f)x + fu \in \text{Reg}(R)$ by Lemma 5.10. Since $f(1 - f) = 0$ and $f \notin \{0, 1\}$, we have $x + h = fx - fu = f(x - u) \in Z(R)$ and $y + h = (f - 1)x \in Z(R)$. Thus $x - h - y$ is the desired regular path of length two from $x$ to $y$; so $\text{diam}_{\text{reg}}(R) \leq 2$.

For the “moreover” statement, we first note that $T(\Gamma(R))$ is connected with $\text{diam}(T(\Gamma(R))) = 2$ by [5, Corollary 3.6] since $R$ has a non-trivial idempotent. Thus $2 = \text{diam}(T(\Gamma(R))) = \text{diam}(T_0(\Gamma(R))) \leq \text{diam}_{\text{reg}}(R) \leq 2$ by Theorem 4.7, since $|R| \geq 4$; so we have the desired equality.

**Corollary 5.12.** Let $R$ be a commutative von Neumann regular ring that is not a field. Then there is a regular path from $x$ to $y$ for every $x, y \in R^*$. Moreover, $\text{diam}_{\text{reg}}(R) = 2$.

**Corollary 5.13.** Let $R$ be a commutative ring. If there is an $e \in \text{Idem}(R) \setminus \{0, 1\}$, then $\text{Reg}(\Gamma(R))$ is connected with $\text{diam}(\text{Reg}(\Gamma(R))) \in \{0, 1, 2\}$. 

1250074-16
**Proof.** Let \( u, v \in \text{Reg}(R) \) be distinct, \( u + v \notin Z(R) \), and \( e \in \text{Idem}(R) \setminus \{0, 1\} \). Then \( w = -eu + (1 - e)v \in \text{Reg}(R) \) by Lemma 5.10; so \( u - w - v \) is the desired path from \( u \) to \( v \) in \( \text{Reg}(\Gamma(R)) \) of length two. Thus \( \text{Reg}(\Gamma(R)) \) is connected and \( \text{diam}(\text{Reg}(\Gamma(R))) \leq 2 \).

One easily verifies that \( \text{diam}(\text{Reg}(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2))) = 0 \), \( \text{diam}(\text{Reg}(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3))) = 1 \), and \( \text{diam}(\text{Reg}(\Gamma(\mathbb{Z}_5 \times \mathbb{Z}_5))) = 2 \). Thus all possible values for \( \text{diam}(\text{Reg}(\Gamma(R))) \) in Corollary 5.13 may be realized.

We next determine \( \text{diam}_{\text{reg}}(R) \) for \( R = \mathbb{Z}_n \) and \( R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k} \). The details are left to the reader; they follow directly from Theorem 5.7 and Corollary 5.12.

**Example 5.14.** (a) \( \text{diam}_{\text{reg}}(\mathbb{Z}_n) \) We have already observed that \( \text{diam}_{\text{reg}}(\mathbb{Z}_2) = 0 \), \( \text{diam}_{\text{reg}}(\mathbb{Z}_3) = 1 \), and \( \text{diam}_{\text{reg}}(\mathbb{Z}_p) = \infty \) when \( p \geq 5 \) is prime. Let \( R = \mathbb{Z}_n \) with \( n \geq 2 \) and \( n \) not prime. Then \( \text{diam}_{\text{reg}}(R) = 2 \) if \( n \) is the product of (at least 2) distinct primes. Otherwise, \( \text{diam}_{\text{reg}}(R) = \infty \).

(b) \( \text{diam}_{\text{reg}}(\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}) \) Let \( R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k} \) with \( 2 \leq n_1 \leq \cdots \leq n_k \) and \( k \geq 2 \). Then \( \text{diam}_{\text{reg}}(R) = 2 \) if every \( n_i \) is prime. Otherwise, \( \text{diam}_{\text{reg}}(R) = \infty \).

The rings in Theorem 5.11 and Corollary 5.12 are reduced and not quasilocal. We next give an example of a reduced non-quasilocal ring \( R \) that is not an integral domain such that there is no regular path from \( x \) to \( y \) for some \( x, y \in R^* \).

**Example 5.15.** Let \( I = 2X\mathbb{Z}[X] \) be an ideal of \( \mathbb{Z}[X] \), and let \( R = \mathbb{Z}[X]/I \). Then \( R \) is reduced, not quasilocal, and \( Z(R) = X\mathbb{Z}[X]/I \cup 2\mathbb{Z}[X]/I \). Note that \( R \neq (Z(R)) \); so \( T_0(\Gamma(R)) \) is not connected by Corollary 4.3. Thus there is no regular path from \( x \) to \( y \) for some \( x, y \in R^* \). It may easily be shown that there is no regular path from \( x = 1 + I \) to \( y = X + I \).

We have \( \text{diam}(T_0(\Gamma(R))) \leq \min\{\text{diam}_Z(R), \text{diam}_{\text{reg}}(R)\} \) for any commutative ring \( R \). Examples 4.6, 5.5 and 5.14 show that all three diameters may be different. For \( n = 5^2 \cdot 7^2 = 1225 \), we have \( \text{diam}(T_0(\Gamma(\mathbb{Z}_n))) = 2 < 3 = \text{diam}_Z(\mathbb{Z}_n) < \infty = \text{diam}_{\text{reg}}(\mathbb{Z}_n) \). For \( n = 2^2 \cdot 3 \cdot 5 = 60 \), we have \( \text{diam}(T_0(\Gamma(\mathbb{Z}_n))) = \text{diam}_Z(\mathbb{Z}_n) = 2 < \infty = \text{diam}_{\text{reg}}(\mathbb{Z}_n) \). Also, \( \text{diam}(T_0(\Gamma(\mathbb{Z}_{35}))) = \text{diam}_{\text{reg}}(\mathbb{Z}_{35}) = 2 < \infty = \text{diam}_Z(\mathbb{Z}_{35}) \).

We could also define \( \text{gr}_Z(R) \) and \( \text{gr}_{\text{reg}}(R) \) by only using cycles in \( Z(R)^* \) and \( \text{Reg}(R) \), respectively. However, this gives nothing new since \( \text{gr}_Z(R) = \text{gr}(Z_0(\Gamma(R))) \) and \( \text{gr}_{\text{reg}}(R) = \text{gr}(\text{Reg}(\Gamma(R))) \). We have already determined \( \text{gr}(Z_0(\Gamma(R))) \) in Theorem 3.3(2), and \( \text{gr}(\text{Reg}(\Gamma(R))) \) has been studied in [5, Theorems 2.6 and 3.14].

We end this paper by giving \( \text{gr}(\text{Reg}(\Gamma(R))) \) for \( R = \mathbb{Z}_n \) and \( R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k} \); details are left to the reader.

**Example 5.16.** (a) \( \text{gr}(\text{Reg}(\Gamma(\mathbb{Z}_n))) \) Let \( R = \mathbb{Z}_n \) with \( n \geq 2 \). Then \( \text{gr}(\text{Reg}(\Gamma(R))) = \infty \) if \( n = 4, n = 6 \), or \( n \) is prime; \( \text{gr}(\text{Reg}(\Gamma(R))) = 4 \) if \( n = p^m \) with \( p \geq 3 \) prime and \( m \geq 2 \); and \( \text{gr}(\text{Reg}(\Gamma(R))) = 3 \) otherwise.
(b) \((\text{gr}(\text{Reg}(\Gamma(Z_{n_1} \times \cdots \times Z_{n_k}))))\) Let \(R = Z_{n_1} \times \cdots \times Z_{n_k}\) with \(2 \leq n_1 \leq \cdots \leq n_k\) and \(k \geq 2\). Then \(\text{gr}(\text{Reg}(\Gamma(R))) = \infty\) if \(n_{k-1} = 2\) and \(n_k = 2, 3, 4,\) or 6. Otherwise, \(\text{gr}(\text{Reg}(\Gamma(R))) = 3\).

References


