ON THE DOT PRODUCT GRAPH OF A COMMUTATIVE RING

Ayman Badawi
Department of Mathematics and Statistics, American University of Sharjah, Sharjah, UAE

Let $A$ be a commutative ring with nonzero identity, $1 \leq n < \infty$ be an integer, and $R = A \times A \times \cdots \times A$ ($n$ times). The total dot product graph of $R$ is the (undirected) graph $TD(R)$ with vertices $R^* = R \setminus \{(0, 0, \ldots, 0)\}$, and two distinct vertices $x$ and $y$ are adjacent if and only if $x \cdot y = 0 \in A$ (where $x \cdot y$ denote the normal dot product of $x$ and $y$). Let $Z(R)$ denote the set of all zero-divisors of $R$. Then the zero-divisor dot product graph of $R$ is the induced subgraph $ZD(R)$ of $TD(R)$ with vertices $Z(R)^* = Z(R) \setminus \{(0, 0, \ldots, 0)\}$. It follows that each edge (path) of the classical zero-divisor graph $\Gamma(R)$ is an edge (path) of $ZD(R)$. We observe that if $n = 1$, then $TD(R)$ is a disconnected graph and $ZD(R)$ is identical to the well-known zero-divisor graph of $R$ in the sense of Beck–Anderson–Livingston, and hence it is connected. In this paper, we study both graphs $TD(R)$ and $ZD(R)$. For a commutative ring $A$ and $n \geq 3$, we show that $TD(R)$ (or $ZD(R)$) is connected with diameter two (at most three) and with girth three. Among other things, for $n \geq 2$, we show that $ZD(R)$ is identical to the zero-divisor graph of $R$ if and only if either $n = 2$ and $A$ is an integral domain or $R$ is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Key Words: Annihilator graph; Total graph; Zero-divisor graph.

2010 Mathematics Subject Classification: Primary: 13A15; Secondary: 13B99; 05C99.

1. INTRODUCTION

Let $R$ be a commutative ring with nonzero identity, and let $Z(R)$ be its set of zero-divisors. Recently, there has been considerable attention in the literature to associating graphs with algebraic structures (see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], and [26]). Probably the most attention has been to the zero-divisor graph $\Gamma(R)$ for a commutative ring $R$. The set of vertices of $\Gamma(R)$ is $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices $x$ and $y$ are adjacent if and only if $xy = 0$. The concept of a zero-divisor graph goes back to I. Beck [13], who let all elements of $R$ be vertices and was mainly interested in colorings. The zero-divisor graph $\Gamma(R)$ was introduced by David F. Anderson and Philip S. Livingston in [9], where it was shown, among other things, that $\Gamma(R)$ is connected with $\text{diam}(\Gamma(R)) \in \{0, 1, 2, 3\}$ and $\text{gr}(\Gamma(R)) \in [3, 4, \infty]$. For a recent survey article on zero-divisor graphs, see [12].

Received May 1, 2013; Revised June 14, 2013. Communicated by E. Houston.

Address correspondence to Ayman Badawi, Department of Mathematics and Statistics, American University of Sharjah, P. O. Box 26666, Sharjah, UAE; E-mail: abadawi@aus.edu
Let $A$ be a commutative ring with nonzero identity, $1 \leq n < \infty$ be an integer, and let $R = A \times A \times \cdots \times A$ ($n$ times). Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in R$. Then the dot product $x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \in A$. In this paper, we introduce the total dot product graph of $R$ to be the (undirected) graph $TD(R)$ with vertices $R^* = R \setminus \{(0, 0, \ldots, 0)\}$, and two distinct vertices $x$ and $y$ are adjacent if and only if $x \cdot y = 0 \in A$. Let $Z(R)$ denote the set of all zero-divisors of $R$. Then the zero-divisor dot product graph of $R$ is the induced subgraph $ZD(R)$ of $TD(R)$ with vertices $Z(R)^* = Z(R) \setminus \{(0, 0, \ldots, 0)\}$. It follows that each edge (path) of the classical zero-divisor graph $\Gamma(R)$ is an edge (path) of $ZD(R)$. We observe that if $n = 1$, then $TD(R)$ is a disconnected graph, where $ZD(R)$ is identical to $\Gamma(R)$ in the sense of Beck–Anderson–Livingston, and hence it is connected.

In the second section, for an $1 \leq n < \infty$ and $R = A \times A \times \cdots \times A$ ($n$ times) for some commutative ring $A$, we show (Theorem 2.2) that $ZD(R) = \Gamma(R)$ if and only if either $n = 2$ and $A$ is an integral domain or $R$ is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. If $n = 2$ and $A$ is not an integral domain or $n = 3$ and $A$ is an integral, we show (Theorem 2.3 and Theorem 2.5(1)) that $ZD(R)$ is connected with diameter three. If $n \geq 4$, we show (Theorem 2.5(3)) that $ZD(R)$ is connected with diameter two. If $n \geq 3$, we show (Theorem 2.4) that $TD(R)$ is connected with diameter two. We show (Corollary 2.8) that $Z(D(R))$ contains no cycles if and only if $n = 2$ and $A$ is ring-isomorphic to $\mathbb{Z}_2$. We show (Theorem 2.6) that if $n \geq 3$, then the girth of $ZD(R)$ is three (and hence the girth of $TD(R)$ is three).

We recall some definitions. Let $\Gamma$ be a (undirected) graph. We say that $\Gamma$ is connected if there is a path between any two distinct vertices. For vertices $x$ and $y$ of $\Gamma$, we define $d(x, y)$ to be the length of a shortest path from $x$ to $y$ ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no path). Then the diameter of $\Gamma$ is $\text{diam}(\Gamma) = \sup \{d(x, y) | x$ and $y$ are vertices of $\Gamma\}$. The girth of $\Gamma$, denoted by $\text{gr}(\Gamma)$, is the length of a shortest cycle in $\Gamma$ ($\text{gr}(\Gamma) = \infty$ if $\Gamma$ contains no cycles). A graph $\Gamma$ is complete if any two distinct vertices are adjacent.

Throughout, all rings are commutative with nonzero identity. Let $R$ be a commutative ring. Then $Z(R)$ denotes the set of zero-divisors of $R$, and the distance between two distinct vertices $a, b$ of $TD(R)$ ($ZD(R)$) is denoted by $d_T(a, b)$ ($d_D(a, b)$). If $ZD(R)$ is identical to $\Gamma(R)$, then we write $ZD(R) = \Gamma(R)$; otherwise, we write $ZD(R) \neq \Gamma(R)$. As usual, $\mathbb{Z}$ and $\mathbb{Z}_n$ will denote the integers and integers modulo $n$, respectively. Any undefined notation or terminology is standard, as in [22] or [16].

2. BASIC PROPERTIES OF $TD(R)$ AND $ZR(D)$

We start this section with the following result.

**Theorem 2.1.** Let $A$ be an integral domain and $R = A \times A$. Then $TD(R)$ is disconnected and $ZD(R) = \Gamma(R)$ is connected. In particular, if $A$ is ring-isomorphic to $\mathbb{Z}_2$, then $ZD(R)$ is complete (i.e., $\text{diam}(ZD(R)) = 1$) and $\text{gr}(ZD(R)) = \infty$. If $A$ is not ring-isomorphic to $\mathbb{Z}_2$, then $\text{diam}(ZD(R)) = 2$ and $\text{gr}(ZD(R)) = 4$.

**Proof.** Let $B = \{(a, a), (-a, a), (a, -a) | a \in A^*\}$, and let $x \in B$. Suppose that $y \in R^*$ and $x \cdot y = 0$. Since $A$ is an integral domain, one can easily see that $y \in B$. Let $M = \{(a, 0), (0, a) | a \in A^*\}$ and let $w \in M$. Suppose that $w \cdot s = 0$ for some
s ∈ R*. Again, since A is an integral domain, we conclude that s ∈ M. Thus the vertices (1, 1) and (0, 1) are not connected by a path in TD(R). Hence TD(R) is disconnected. Since A is an integral domain, Z(R)* = M. Let x, y ∈ M. Then x · y = 0 iff xy = (0, 0). Thus ZD(R) = Γ(R). Suppose that A is ring-isomorphic to Z2. Then it is clear that diam(ZD(R)) = 1 and gr(ZD(R)) = ∞. Suppose A is not ring-isomorphic to Z2. Since ZD(R) = Γ(R) and A is an integral domain, diam(ZD(R)) = 2 by [24, Theorem 2.6] and gr(ZD(R)) = 4 by [10, Theorem 2.2]. □

**Theorem 2.2.** Let 2 ≤ n < ∞, A be a commutative ring with 1 ≠ 0, and R = A × A × · · · × A (n times). Then ZD(R) = Γ(R) if and only if either n = 2 and A is an integral domain or R is ring-isomorphic to Z2 × Z2 × Z2.

**Proof.** If n = 2 and A is an integral domain, then by Theorem 2.1 we have ZD(R) = Γ(R). Suppose that R is ring-isomorphic to Z2 × Z2 × Z2. Then by simple hand-calculations, for every x, y ∈ Z(R)*, we have x · y = 0 iff xy = (0, 0, 0), and hence ZD(R) = Γ(R).

Conversely, suppose that ZD(R) = Γ(R). Assume that A is not an integral domain. Then there is an a ∈ Z(A)*. Hence x = (1, a, 0, 0, . . . , 0), y = (a, −1, 0, 0, . . . , 0) ∈ Z(R)*, and x · y = 0, but xy = (0, 0, 0, . . . , 0). Thus x − y is an edge of ZD(R) that is not an edge of Γ(R), a contradiction. Thus A must be an integral domain. Now assume that n = 3 and A is not ring-isomorphic to Z2. Then there is an a ∈ A\{(0, 1, 1, 1, 0, 0, . . . , 0). Let x = (1, 0, a, 0) and y = (−a, 1, 0, 0, . . . , 0). Then x ≠ y and it is clear that x − y is an edge of ZD(R) that is not an edge of Γ(R), a contradiction again. Hence assume that n ≥ 4. Let x = (1, 1, 0, 1, 0, 0, . . . , 0) and y = (−1, 1, 0, 0, . . . , 0, 0). Then x ≠ y, x · y = 0, and xy = (0, 0, . . . , 0), a contradiction. Thus we conclude that either n = 2 and A is an integral domain or R is ring-isomorphic to Z2 × Z2 × Z2. □

In view of Theorem 2.1, we have the following result.

**Theorem 2.3.** Let A be a commutative ring with 1 ≠ 0 that is not an integral domain, and let R = A × A. Then the following statements hold:

1. TD(R) is connected and diam(TD(R)) = 3;
2. ZD(R) is connected, ZD(R) ≠ Γ(R), and diam(ZD(R)) = 3;
3. gr(ZD(R)) = gr(TD(R)) = 3.

**Proof.** (1). Let x = (a, b), y = (c, d) ∈ R*, where x ≠ y, and assume that x · y ≠ 0. Since A is not an integral domain, there are f, g ∈ A* (not necessarily distinct) such that fg = 0. Let w = (−bf, af) and v = (−dg, cg). Note that w, v ∈ Z(R). Clearly x · w = w · v = v · y = 0. Since x · y ≠ 0, w ≠ y and v ≠ x. First, assume that v, w ∈ Z(R)*. If x · y = 0 or w = 0, then x − v · y or x − w · y is a path of length 2 in TD(R) from x to y. Assume that neither x · y = 0 nor y · w = 0. Then x, w, v, y are distinct, and hence x − w − v − y is a path of length 3 in TD(R) from x to y. Now assume that w = (0, 0) or v = (0, 0). If w = (0, 0), then replace w by (f, −f) ∈ Z(R)*, and hence x · w = (a, b) · (f, −f) = 0. Similarly, if v = (0, 0), then replace v by (g, −g) ∈ Z(R)*. Hence if w = (0, 0) or v = (0, 0), then we are able to redefine w and v so that w, v ∈ Z(R)* and x · w = w · v = v · y = 0. Thus as in the earlier
argument, we conclude that there is a path of length at most 3 in $TD(R)$ from $x$ to $y$. Thus $TD(R)$ is connected and $d_f(x, y) \leq 3$ for every $x, y \in R^*$. Now, let $x = (1, 1)$ and $y = (1, 0)$. We show $d_f(x, y) = 3$, and hence $diam(TD(R)) = 3$. Let $w \in R^*$ such that $x \cdot w = 0$. Then $w = (a, -a)$ for some $a \in A^*$. Since $w \cdot y = a \neq 0$, $d_f(x, y) > 2$. Hence $d_f(x, y) = 3$. In particular, let $k, t \in A^*$ such that $kt = 0$, $w = (k, -k)$, and $v = (0, t)$. Then $x - w - v - y$ is a path of length 3 in $TD(R)$ from $x$ to $y$.

(2) Since $A$ is not an integral domain, $ZD(R) \neq \Gamma(R)$ by Theorem 2.2. Let $x, y \in Z(R)^*$. Assume that $x \cdot y \neq 0$. In view of the proof of (1), we are able to find $w, v \in Z(R)^*$ such that either $x - w - y$ is a path in $ZD(R)$ or $x - v - y$ is a path in $ZD(R)$ or $x - w - v - y$ is a path in $ZD(R)$. Hence $diam(ZD(R)) \leq 3$. Let $a \in Z(A)^*$. Then $x = (1, a), y = (0, 1) \in Z(R)^*$. We show $d_f(x, y) = 3$, and thus $diam(ZD(R)) = 3$. Since $x \cdot y \neq 0$, $d_f(x, y) > 1$. Suppose there is a $v = (g, h) \in Z(R)^*$ such that $x - v - y$ is a path of length 2 in $ZD(R)$ from $x$ to $y$. Since $v \cdot y = 0$, we have $h = 0$, and hence $v = (g, 0)$. Since $x \cdot y = 0$, we have $g = 0$ and, thus $v = (0, 0)$, a contradiction. Thus $d_f(x, y) = 3$, and hence $diam(ZD(R)) = 3$.

(3) Since $A$ is not an integral domain, there are $a, b \in A^*$ (not necessarily distinct) such that $ab = 0$. Then $x = (a, 0), y = (0, b), w = (b, a) \in Z(R)^*$. Hence $x - y - w - x$ is a cycle of length 3 in $ZD(R)$. Thus $gr(TD(R)) = gr(ZD(R)) = 3$.

□

**Theorem 2.4.** Let $A$ be a commutative ring with $1 \neq 0, 3 \leq n < \infty$, and let $R = A \times A \times \cdots \times A$ (n times). Then $TD(R)$ is connected and $diam(TD(R)) = 2$.

**Proof.** Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in R^*$, and suppose that $x \cdot y \neq 0$. Then let $M = \{i | x_i = y_i = 0, 1 \leq i \leq n\}$. Suppose that $A$ is not the empty set. Then choose a $k \in M$, and let $w = (w_1, \ldots, w_n) \in R^*$, where $w_k = 1$ and $w_i = 0$ if $i \neq k$. Then $x - w - y$ is a path of length 2 in $TD(R)$ from $x$ to $y$. Thus suppose that $M$ is the empty set. Then $f(x) = min\{i | x_i \neq 0, 1 \leq i \leq n\}$. Thus suppose that $M$ is the empty set. Then $f(x) = 1$ or $f(y) = 1$. We may assume that $f(x) = 1$. Let $v = (x_2y_1 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1, 0, 0, \ldots, 0) \in R$. Suppose that $v \neq (0, 0, \ldots, 0)$. Then it is easy to check that $x \cdot y = v \cdot y = 0$. Since $x \cdot y \neq 0$, $v \neq x$ and $v \neq y$. Hence $x - v - y$ is a path of length 2 in $TD(R)$ from $x$ to $y$. Suppose that $v = (0, 0, \ldots, 0)$. Then $x_1y_2 - x_2y_1 = 0$. Let $w = (-x_2, x_1, 0, 0, \ldots, 0) \in R$. Since $x_1 \neq 0$, $w \in R^*$. Hence $x \cdot w = -x_1x_2 + x_1x_2 = 0$ and $w \cdot y = x_1y_2 - x_2y_1 = 0$. Since $x \cdot w = w \cdot y = 0$ and $x \cdot y \neq 0$, $x \neq w$ and $y \neq w$. Thus $x - w - y$ is a path of length 2 in $TD(R)$ from $x$ to $y$. Hence $TD(R)$ is connected and $diam(TD(R)) = 2$.

□

**Theorem 2.5.** Let $A$ be a commutative ring with $1 \neq 0$. Then the following statements hold:

(1) If $A$ is an integral domain and $R = A \times A \times A$, then $ZD(R)$ is connected ($ZD(R) \neq \Gamma(R)$ by Theorem 2.2) and $diam(ZD(R)) = 3$.

(2) If $A$ is not an integral domain and $R = A \times A \times A$, then $ZD(R)$ is connected ($ZD(R) \neq \Gamma(R)$ by Theorem 2.2) and $diam(ZD(R)) = 2$.

(3) If $4 \leq n < \infty$ and $R = A \times A \times \cdots \times A$ (n times), then $ZD(R)$ is connected ($ZD(R) \neq \Gamma(R)$ by Theorem 2.2) and $diam(ZD(R)) = 2$. 

Let \( x \) and \( ZD/R \) be a commutative ring.\(^3\) Then \( x \cdot y = 1 \neq 0 \). We show \( d_2(x, y) = 3 \). Let \( w = (w_1, w_2, w_3) \in R \) such that \( x \cdot w = w \cdot y = 0 \). Then a trivial calculation leads to \( w_1 = w_2 = w_3 \). Since \( A \) is an integral domain, \( w \in Z(R) \) if and only if \( w = (0, 0, 0) \). Hence \( \text{diam}(ZD(R)) = 3 \).

(2). (Similar to the proof of Theorem 2.4). Let \( x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in Z(R)^* \), and suppose that \( x \cdot y \neq 0 \). Then let \( M = \{i \mid x_i = y_i = 0, 1 \leq i \leq 3\} \). Suppose that \( M \) is not the empty set. Then choose a \( k \in M \), and let \( w = (w_1, w_2, w_3) \in Z(R)^* \), where \( w_k = 1 \) and \( w_i = 0 \). If \( i \neq k \), then \( x \cdot w - y \) is a path of length 2 in \( ZD(R) \) from \( x \) to \( y \). Thus suppose that \( M \) is the empty set. Then let \( f(x) = \min\{i \mid x_i \neq 0, 1 \leq i \leq n\} \) and \( f(y) = \min\{i \mid y_i \neq 0, 1 \leq i \leq 3\} \). Since \( M \) is the empty set, we conclude that \( f(x) = 1 \) or \( f(y) = 1 \). We may assume that \( f(x) = 1 \). Let \( v = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1) \in R \). Suppose that \( v \in Z(R)^* \). Then it is easy to check that \( x \cdot v = v \cdot y = 0 \). Since \( x \cdot y \neq 0 \), \( v \neq x \) and \( v \neq y \). Hence \( x - v - y \) is a path of length 2 in \( ZD(R) \) from \( x \) to \( y \). Suppose that \( v \in Z(R) \). Then choose an \( a \in Z(A)^* \). Then \( a v \in Z(R)^* \) and \( x - a v - y \) is a path of length 2 in \( ZD(R) \) from \( x \) to \( y \). Suppose that \( v = (0, 0, 0) \). Then \( x_1 y_2 - x_2 y_1 = 0 \). Let \( w = (-x_2, x_1, 0) \in Z(R) \). Since \( x_1 \neq 0 \), \( w \in Z(R)^* \). Hence \( x \cdot w = -x_1 x_2 + x_1 x_2 = 0 \) and \( w \cdot y = x_1 y_2 - x_2 y_1 = 0 \). Since \( x \cdot w = w \cdot y = 0 \) and \( x \neq w \) and \( y \neq w \). Thus \( x \cdot w - y \) is a path of length 2 in \( ZD(R) \) from \( x \) to \( y \). Hence \( ZD(R) \) is connected and \( \text{diam}(ZD(R)) = 2 \).

(3). The proof is similar to the proof of Theorem 2.4. Just observe that if \( n \geq 4 \), then \( v \) as in the proof of Theorem 2.4 is in \( Z(R) \). \( \square \)

**Theorem 2.6.** Let \( A \) be a commutative ring with \( 1 \neq 0 \), \( 3 \leq n < \infty \), and \( R = A \times A \times \cdots \times A \) (\( n \) times). Then \( \text{gr}(ZD(R)) = \text{gr}(TD(R)) = 3 \).

**Proof.** Let \( a = (1, 0, \ldots, 0), b = (0, 1, 0, \ldots, 0) \), and \( c = (0, 0, 1, 0, \ldots, 0) \). Then \( a + b + c = a \) is a cycle of length 3. \( \square \)

**Corollary 2.7.** Let \( A \) be a commutative ring with \( 1 \neq 0 \), \( 2 \leq n < \infty \), and \( R = A \times A \times \cdots \times A \) (\( n \) times). Then the following statements are equivalent:

1. \( \text{gr}(ZD(R)) = 3 \);
2. \( \text{gr}(TD(R)) = 3 \);
3. \( A \) is not an integral domain and \( n = 2 \) or \( n \geq 3 \).

**Proof.** This is clear by Theorem 2.3 and Theorem 2.6. \( \square \)

**Corollary 2.8.** Let \( A \) be a commutative ring with \( 1 \neq 0 \), \( 2 \leq n < \infty \), and \( R = A \times A \times \cdots \times A \) (\( n \) times). Then the following statements are equivalent:

1. \( \text{gr}(ZD(R)) = \infty \);
2. \( A \) is ring-isomorphic to \( \mathbb{Z}_2 \) and \( n = 2 \);
3. \( \text{diam}(ZD(R)) = 1 \).
Proof. (1) $\Rightarrow$ (2). Suppose $\text{gr}(ZD(R)) = \infty$. Then $n = 2$ by Theorem 2.6. Hence $A$ is an integral domain by Corollary 2.7. Hence $ZD(R) = \Gamma(R)$ by Theorem 2.2. Thus $A$ is ring-isomorphic to $\mathbb{Z}_2$ by [10, Theorem 2.4]. (2) $\Rightarrow$ (3). It is clear. (3) $\Rightarrow$ (1). Since $\text{diam}(ZD(R)) = 1$, we conclude that $n = 2$ and $A$ is an integral domain by Theorems 2.3 and 2.5. Thus $A$ is ring-isomorphic to $\mathbb{Z}_2$ by Theorem 2.1. Thus $\text{gr}(ZD(R)) = \infty$. $\square$

Corollary 2.9. Let $A$ be a commutative ring with $1 \neq 0$ such that $A$ is not ring-isomorphic to $\mathbb{Z}_2$, $0 \leq n < \infty$, and $R = A \times A \times \cdots \times A$ ($n$ times). Then the following statements are equivalent:

1. $\text{gr}(ZD(R)) = 4$;
2. $ZD(R) = \Gamma(R)$;
3. $TD(R)$ is disconnected;
4. $n = 2$ and $A$ is an integral domain.

Proof. This is clear by Theorem 2.1, Theorem 2.2, Corollary 2.7, and Corollary 2.8. $\square$

Corollary 2.10. Let $A$ be a commutative ring with $1 \neq 0$, $2 \leq n < \infty$, and $R = A \times A \times \cdots \times A$ ($n$ times). Then the following statements are equivalent:

1. $\text{diam}(ZD(R)) = 3$;
2. Either $A$ is not an integral domain and $n = 2$ or $A$ is an integral domain and $n = 3$.

Proof. This is clear by Theorem 2.1, Theorem 2.3, and Theorem 2.5. $\square$

Corollary 2.11. Let $A$ be a commutative ring with $1 \neq 0$, $2 \leq n < \infty$, and $R = A \times A \times \cdots \times A$ ($n$ times). Then the following statements are equivalent:

1. $\text{diam}(ZD(R)) = 2$;
2. Either $A$ is an integral domain that is not ring-isomorphic to $\mathbb{Z}_2$ and $n = 2$, or $A$ is not an integral domain, and $n = 3$, or $n \geq 4$.

Proof. This is clear by Theorem 2.1, Theorem 2.5, and Corollary 2.10. $\square$

Corollary 2.12. Let $A$ be a commutative ring with $1 \neq 0$, $2 \leq n < \infty$, and $R = A \times A \times \cdots \times A$ ($n$ times). Then $\text{diam}(TD(R)) = 3$ if and only if $A$ is not an integral domain and $n = 2$.

Proof. This is clear by Theorem 2.1, Theorem 2.3, and Theorem 2.4. $\square$

Corollary 2.13. Let $A$ be a commutative ring with $1 \neq 0$, $2 \leq n < \infty$, and $R = A \times A \times \cdots \times A$ ($n$ times). Then the following statements are equivalent:

1. $\text{diam}(TD(R)) = 2$;
2. $TD(R)$ is connected and $n \geq 3$;
3. $n \geq 3$.

Proof. The proof is clear by Theorem 2.3 and Theorem 2.4. $\square$
Corollary 2.14. Let $A$ be a commutative ring with $1 \neq 0$, $2 \leq n < \infty$, and $R = A \times A \times \cdots \times A$ ($n$ times). Then $diam(TD(R)) = diam(ZD(R)) = 3$ if and only if $A$ is not an integral domain and $n = 2$.

Proof. This is clear by Corollary 2.10 and Corollary 2.12. □

ACKNOWLEDGMENT

I would like to thank the referee for several helpful suggestions.

REFERENCES


