

Exam Three, MTH 213 , Summer 2021

Ayman Badawi

(Stop working at 13:00 pm/ submit your solution by 13:12 pm, DO NOT SUBMIT BY EMAIL) ———— 42

QUESTION 1. (**10 points**)(**SHOW THE WORK**)

(i) Use Math. Induction and prove that $\sum_{i=1}^{i=n} i^2 = 1 + 4 + 9 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all integers $n \geq 1$.

(ii) Use Math. Induction and prove that $2^{(4n+1)} + 13$ is divisible by 15 for all integers $n \geq 1$.

QUESTION 2. (**11 points**)(**SHOW THE WORK**)

(i) Consider the linear recurrence $a_n = 9a_{n-2} + 2^n$ such that $a_0 = 2.20$ and $a_1 = 1.40$. Find the general formula for a_n.

(ii) Use the formula $a_n = 9a_{n-2} + 2^n$ and calculate a_2. Then use the formula that you discovered in (i) and find a_2 and a_4

(iii) Consider the linear recurrence $a_n = a_{n-1} + 2n + 3$ such that $a_0 = 3$. Find the general formula for a_n.

QUESTION 3. (**SHOW THE WORK**)(**6 points**)
Let $A = \{-8, -4, -3, -2, -1, 0, 1, 2, 3, 4, 8\}$. Define "=" on A such that for all $a, b \in A$ we have $a" = "b$ if and only if $a \ (mod\ 7) = b \ (mod\ 7)$. Then "=" is an equivalence relation (DO NOT SHOW THAT).

(i) Find all equivalence classes of $" = "$.

(ii) As in the class notes, we can view "=" as a subset of $A \times A$. How many elements does $" = "$ have? (you do not need to find the set "=")

QUESTION 4. (**SHOW THE WORK**)(**5 points**) Given "=" is a relation on $A = \{0, 2, 3, 5, 8\}$ such that $" = " = \{(0,0), (2,2), (3,3), (5,5), (8,8), (2,5), (5,2), (3,5), (3,2), (2,3)\}$. Stare at $" = "$ and answer the following:

(i) Is "=" a reflexive? Explain briefly

(ii) Is "=" symmetric? Explain briefly

(iii) Is "=" transitive? Explain briefly

(iv) Is "=" an equivalence relation? Explain briefly

(v) Is "=" a partial order relation ? Explain briefly

QUESTION 5. (**SHOW THE WORK**)(**4 points**)
Define "$\leq$" on $A = \{2, 8, 9, 11, 13\}$ such that for all $a, b \in A$ we have $a" \leq "b$ if and only if $b - a \in \{-4, -1, 1, 0\}$. Then "$\leq$" is not a partial order relation on A. By example, explain which **axioms** fail (i.e., check all AXIOMS and tell me which one is valid and which one is invalid).

QUESTION 6. (**SHOW THE WORK**)(**6 points**) Consider the following code

```
For i = 2 to (n^2 + 1)
S = i^4 + B * C - 3 * i
    For k = 1 to i
      L = k^5 + i^3 - 2
    next k
next i
```

(i) Find the exact number of arithmetic operations that the code will execute.

(ii) What is the complexity of the code? i.e., find O(code).

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1. i) At $n=1$

$$\frac{n(n+1)(2n+1)}{6} = \frac{1(2)(3)}{6} = \underline{\underline{1}}$$

Assume that $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$ for some integer $k=n$.

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= (k+1)\left[\frac{2k^2+k+6k+6}{6}\right]$$

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$$=(k+1)\left[\frac{2k^2+7k+6}{6}\right]$$

$$= \underline{\underline{\frac{(k+1)(k+2)(2k+3)}{6}}}$$

$\therefore \sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$ and by math. induction,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

ii) At $n=1$,

$2^{4n+1}+13 = 2^{4+1}+13 = 32+13 = 45$ which is divisible by 15.

Assume that $2^{4k+1}+13$ is divisible by 15 for some integer $k=n$.

We prove that $2^{4(k+1)+1}+13$ is also divisible then.

$$2^{4(k+1)+1}+13 = 2^{4k+5}+13 = 2^{4k+1}\cdot 2^4 + 13$$

$$= 2^{4k+1}\cdot 2^4 + 13 + 13(2^4) - 13(2^4)$$

$$= 2^4\left(2^{4k+1}+13 - 13 + \frac{13}{2^4}\right)$$

$$= 2^4\left(2^{4k+1}+13\right) - 13\cdot 2^4 + 13$$ Typo

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$= 2^4\underbrace{(2^{4k+1}+13)}_{\text{divisible by 15}} - \underbrace{195}_{\text{divisible by 15}}$

5

$\therefore\ 2^{4(k+1)+1}+13$ is divisible by 15.

Hence, by math. induction, $2^{4n+1}+13$ is divisible by 15 $\forall n \geq 1$.

$$\begin{array}{r} 16 \\ \times 13 \\ \hline 48 \\ 160 \\ \hline 208 \\ -13 \\ \hline 195 \end{array}$$

150

2. i) $a_n = 9a_{n-2} + 2^n$; $a_0 = 2.2$, $a_1 = 1.4$

$a_n - 9a_{n-2} = 2^n$

H : $\alpha^n - 9\alpha^{n-2} = 0$

$\alpha^2 - 9 = 0$

$\alpha = \pm 3$

$H = c_1 3^n + c_2(-3)^n$

$P = \cancel{A5}\ A2^n$

$a_n - 9a_{n-2} = 2^n$

$A2^n - 9A2^{n-2} = 2^n$

$A2^n - 9A2^n \cdot 2^{-2} = 2^n$

$2^n\left(A - \frac{9A}{4}\right) = 2^n$

$A - \frac{9A}{4} = 1$

$4A - 9A = 4$

$-5A = 4$

$A = -4/5$

$\therefore\ a_n = c_1 3^n + c_2(-3)^n - \frac{4}{5}2^n$

$a_0 = c_1 + c_2 - \frac{4}{5} = 2.2$ —— (1)

$a_1 = 3c_1 - 3c_2 - \frac{8}{5} = 1.4$ —— (2)

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① × 3 + ②

$3c_1 + 3c_2 - \frac{12}{5} = 6.6$

$3c_1 - 3c_2 \quad \frac{8}{5} = 1.4$

$6c_1 - 4 = 8$

$6c_1 = 12$

$c_1 = 2$

$c_2 = 2.2 + \frac{4}{5} - 2$

$= \frac{11 + 4 - 10}{5} = 1$

$\therefore \; a_n = 2 \cdot 3^n + (-3)^n - \frac{4}{5} 2^n$

ii) $a_n = 9 a_{n-2} + 2^n$

$a_2 = 9 a_0 + 2^2 = 9(2.2) + 4 = 23.8$

$a_2 = 2 \cdot 3^2 + (-3)^2 - \frac{4}{5} 2^2$

$= 18 + 9 - \frac{16}{5} = 23.8$

$a_4 = 2 \cdot 3^4 + (-3)^4 - \frac{4}{5} 2^4$

$= 162 + 81 - \frac{64}{5}$

$= 230.2$

iii) $a_n - a_{n-1} = 2n + 3$

H: $\alpha - 1 = 0$

$\alpha = 1$

$H = c_1 \cdot 1^n = c_1$

$P = An^2 + Bn$

$a_n - a_{n-1} = 2n + 3$

$An^2 + Bn - A(n-1)^2 - B(n-1) = 2n + 3$

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① 2.2 × 5 = 11.0

① 2.2 × 9 = 19.8

③ 27 × 5 = 135; 135 − 16 = 23.8 (~~119~~ / 5)

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$$\cancel{An^2} + \cancel{Bn} - \cancel{An^2} - A + 2An - \cancel{Bn} + B = 2n+3$$

$-2ab$
$-2n1$
$-2n$

$$2An + B - A = 2n + 3$$

$$2A = 2 \Rightarrow A = \underline{\underline{1}}$$

$$B - A = 3$$

$$B = 3 + A = \underline{\underline{4}}$$

$$\therefore \ a_n = c_1 + n^2 + 4n$$

$$a_0 = c_1 = 3$$

$$\therefore \ a_n = 3 + n^2 + 4n$$

5

3. a "=" b iff $a \pmod 7 = b \pmod 7$.

i) $[0] = \{0\}$
$[1] = \{1, 8\}$
$[2] = \{2\}$
$[3] = \{3, -4\}$
$[4] = \{4, -3\}$
$[-1] = \{-1\}$
$[-2] = \{-2\}$
$[-8] = \{-1, -8\}$

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$\therefore$ equivalence classes are $\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \overline{-1}, \overline{-2}$,

ii) It will have $1 + 2^2 + 1 + 2^2 + 2^2 + 2^2 + 2^2 + 1 = 19$

4. i) Yes, since each element in A "equals" itself.
ii) No, since $3 " = " 5$ but $5 " \neq " 3$
iii) Yes, since $3 " = " 5$, $5 " = " 2$ and $3 " = " 2$.
iv) Since "=" is not symmetric, it is not an equivalence relation.
v) Since "=" is not anti symmetric (e.g. $2 " = " 3$ and $3 " = " 2$) it cannot be a partial order relation.

$5 " \neq " 3$

NO $5 " = " 2$, $2 " = " 3$

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5. Reflexive:

"$\leq$" is reflexive since $a-a=0 \in \{-4,-1,1,0\}$ $\forall a \in A$.

~~Sym~~ Anti-symmetric:

choose $a=8$, $b=9$

we can see $a-b=8-9=-1 \in \{-4,-1,1,0\}$. $\therefore b \leq a$
but, $b-a=9-8=1 \in \{-4,-1,10\}$. $\therefore a \leq b$.

$\therefore$ This relation is not anti symmetric

Transitive:

~~There are no three~~
let $a=8$, $b=9$, $c=13$

$b \leq a$ ~~a≤b~~ from above

~~b−c = 13 9~~
$b-c=9-13=-4 \in \{-4,-1,1,0\}$

$\therefore c \leq b$

But $c \not\leq a$ $\therefore$ $a-c=8-13=-5 \notin \{-4,-1,1,0\}$

$\therefore$ "$\leq$" is not transitive.

6. i)

Outer loop	Outer loop operations	Inner loop operations
2	7	2×8
n^2+1	7	$(n^2+1) \times 8$

~~No. of times outer loop will run~~

$= n^2+1-2+1 = \underline{n^2}$

Typo

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Total no. of operations $= n^2 \cdot 6 + n^2 \left(\frac{16 + 8(n^2+1)}{2} \right)$

ii) $O(\text{code}) = n^4$