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Final Exam, Spring 2025

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Score = $\frac{56}{58}$ / 52

QUESTION 1. (i) **(3 points)** Let $A = \{1, 2, 6, 10, 7, 14, 15\}$. Define "<=" on A such that a" $\leq$ "b iff $b \mid a$. Then " <= " is a partial order relation on A.

a " ≤ " b
b | a

Write down T or F

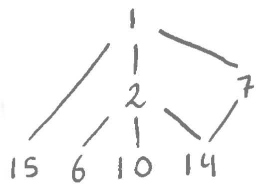
a. $6 \leq 12$ F

b. $14 \leq 7$ T

c. $2 \leq 1$ T

Draw the Hesse diagram of the above relation **(3 points)**

2, 6, 10, 7, 14, 15 " ≤ " 1
6, 10, 14 " ≤ " 2
14 " ≤ " 7



QUESTION 2. (i) **(8 points)** Let $a_n = 7a_{n-1} - 12a_{n-2} + 60n$. Find a general formula for a_n. Do not find C_1, C_2.

$a_n - 7a_{n-1} + 12a_{n-2} = 60n$

• solve for homogeneous a_n.

$a_n - 7a_{n-1} + 12a_{n-2} = 0 \rightarrow \alpha^2 - 7\alpha + 12 = 0 \rightarrow \alpha_1 = 4, \alpha_2 = 3$

$a_h(n) = C_1(4)^n + C_2(3)^n$

• solve for particular a_n:

$a_p(n) = b_1 n + b_0$

$a_p(n) - 7a_p(n-1) + 12a_p(n-2) = 60n$

$b_1 n + b_0 - 7(b_1 n - b_1 + b_0) + 12(b_1 n - 2b_1 + b_0) = 60n$

$b_1 n + b_0 - 7b_1 n + 7b_1 - 7b_0 + 12b_1 n - 24b_1 + 12b_0 = 60n$

$6b_1 n + (-17b_1 + 6b_0) = 60n$

$6b_1 = 60 \rightarrow b_1 = 10$

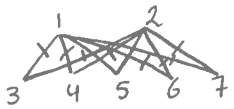
$-17(10) + 6b_0 = 0 \rightarrow b_0 = \frac{85}{3}$

$a_p(n) = 10n + \frac{85}{3}$

$a_n = C_1(4)^n + C_2(3)^n + 10n + \frac{85}{3}$

(ii) **(3 points)** Is $K_{2,5}$ an Eulerian trail (path)? If yes, construct such trail (path).

-yes , because exactly 2 verticies have odd degrees

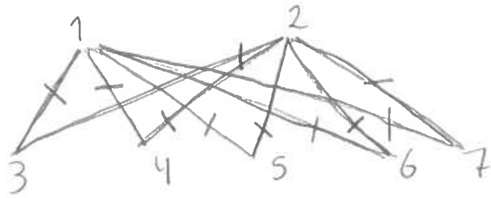


1 - 3 - 2 - 6 - 1 - 4 - 2 - 7 - 1 - 5 - 2

QUESTION 3. (9 points)

(i) **(3 points)** The digits 1, 2, ..., 9 will be used to construct 6-digits cars plates. How many car-plates can be constructed if the second digit must be three, the 4th digit must be 5, and the last digit must be even ? Note that each plate consists of 6 digits and there is no repetition.

4 even 5 odd



$\overline{6} \times \overline{1} \times \overline{5} \times \overline{1} \times \overline{4} \times \overline{4}$

(ii) **(3 points)** There are 1020 positive integers, where each is of the form $12k$ for some integer k. Then there are at least m integers out of the given 1020 integers say $a_1, \ldots, a_m$ such that $a_1 \pmod{15} = a_2 \pmod{15} = \cdots = a_m \pmod{15}$. What is the best value of m? show the work

$$|\text{co-domain}| = \frac{15}{\gcd(15,12)} = \frac{15}{3} = 5$$

$$m = \left\lceil \frac{1020}{5} \right\rceil = 204$$

(iii) **(3 points)** The digits 1, 2, ..., 9 will be used to construct car plates, where each plate has 5 digits. If the first and the forth digits must be even, and the 5th digit must be odd. Assume no repetition. How many car plates can be constructed ?

$\overline{4} \times \overline{6} \times \overline{5} \times \overline{3} \times \overline{5}$

QUESTION 4. Let $V = \{2, 3, 4, 8, 9, 27\}$. Two vertices $v_1, v_2 \in V$ are connected by an edge if and only if $ab \pmod 6 = 0$.

(i) **(4 points)** By drawing the graph, convince me that the graph is a $K_{m,n}$ for some integers m, n. Find m, and n

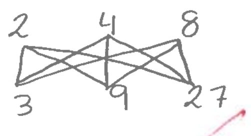
(2x3) mod 6 = 0 (4x9) mod 6 = 0

(2x9) mod 6 = 0 (4x27) mod 6 = 0

(2x27) mod 6 = 0 (8x9) mod 6 = 0

(3x4) mod 6 = 0 (8x27) mod 6 = 0

(3x8) mod 6 = 0



m = 3

n = 3

(ii) **(3 points)** Is the graph an Eulerian circuit? If yes, construct such circuit.

-no because the degrees of the verticies are not even

(iii) **(3 points)** Is the graph a Hamiltonian? if yes, then convince me

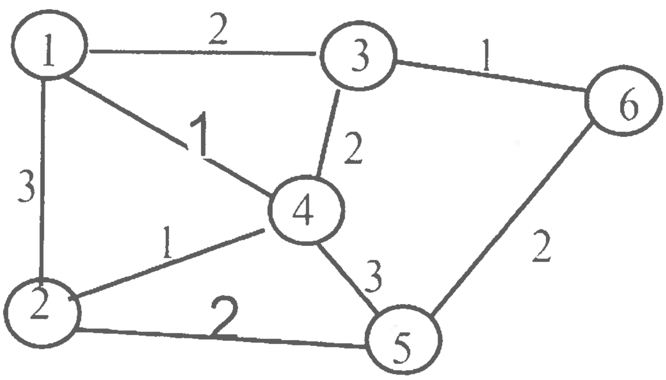
-yes, because C6 is a subgraph of it

27 - 8 - 9 - 4 - 3 - 2 - 27

QUESTION 5. (8 points)

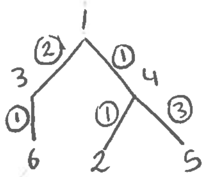
Stare at the below picture.

Consider the network (nodes, links and their weights) in the figure below.



Use Dijkstra's Algorithm to construct the minimum weight spanning tree from vertex 1 to every other vertex.

	1	2	3	4	5	6
1	$\boxed{0}$	3^1	2^1	1^1	∞	∞
4	X	2^4	2^1	$\boxed{1^1}$	4^4	∞
2	X	$\boxed{2^4}$	2^1	X	4^4	∞
3	X	X	$\boxed{2^1}$	X	4^4	3^3
6	X	X	X	X	4^4	$\boxed{3^3}$
5	X	X	X	X	$\boxed{4^4}$	X



QUESTION 6. (**8 points**) Use Math Induction and prove that $64 \mid [(n(n+2)(n+4)(n+6)]$ for every even integer $n \geq 2$.

1- prove it for $n = 2$:

$2 \times (2+2) \times (2+4) \times (2+6) = 2 \times 4 \times 6 \times 8 = 384$

$\frac{384}{64} = 6 \to$ true

2- assume $64 \mid [n(n+2)(n+4)(n+6)]$ is true for some even integer $n \geq 2$

3- prove it for the next even integer $n+2 \to 64 \mid (n+2)(n+4)(n+6)(n+8)$

$$(n+2)(n+4)(n+6)(n+8) = \underbrace{n(n+2)(n+4)(n+6)}_{\text{proved by step 2}} + 8\underbrace{(n+2)(n+4)(n+6)}_{\substack{\text{even numbers} \\ \to \text{can be written as } 2m,\ m \in Z}}$$

$8 \times 2m \times 2m \times 2m$

$= 64m \to$ multiple of 64

$\to$ therefore, $64 \mid (n+2)(n+4)(n+6)(n+8)$ is true

- conclusion : $64 \mid [n(n+2)(n+4)(n+6]$ is true for every even integer $n \geq 2$

QUESTION 7. (3 points) write True or False

(i) $\forall x \in Q^*, \exists y \in Z^*$ such that $xy \in Z$ T

$\frac{2}{3} \times 3 = 2$

(ii) $\exists! y \in Q$ such that $\forall x \in Z^*,\ xy - 4x = 0$ T

$x(y-4) \to y = 4$ T

(iii) If $\exists x \in Q^*$ such that $x^3 - 4x = 0$, then $\overbrace{2^2 = 20}^{F}$ F

$x = \pm 2 \in Q$

QUESTION 8. (3 points) Let $A = \{3, 1, 5, 7\}$ and $B = \{3, 2, 9\}$ Write True or False

(i) $\{(3, 5), (9, 1)\} \subseteq P(B \times A)$ F

(ii) $\{\varphi, \{2, 3\}\} \in P(B)$ F

$P(B) = \{B, \emptyset, \{3\}, \{2\}, \{9\}, \{3, 2\}, \{3, 9\}, \{2, 9\}$

(iii) $\{(5, 3)\} \in P(A \times B)$ T