Name Nourah Safeer, ID g00106051

Exam I: MTH 111, Spring 2025

Ayman Badawi

Points = $\frac{59}{60}$ Excellent !!

QUESTION 1. (6 points) Given $y = 8$ is the directrix of a parabola that has the point $(6, 5)$ as its vertex point.

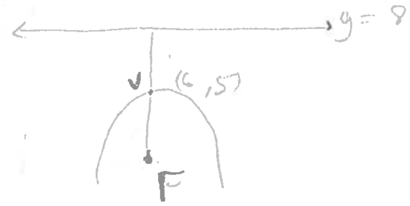
a) Find the equation of the parabola

distance btw directrix & vertex, $|d| = 8 - 5 = 3$

$\therefore$ equation of parabola $\Rightarrow -4d(x - x_0) = (y - y_0)^2$

$\Rightarrow -4(3)(x - 6) = (y - 5)^2$

$\Rightarrow \boxed{-12(x-6) = (y-5)^2}$



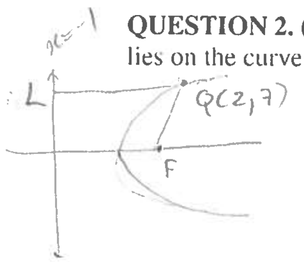
b) Find the focus of the parabola.

Focus $F_d = (6, 5 - d)$

$= (6, 5-3)$

$= \boxed{(6, 2)}$

QUESTION 2. (3 points) Given that $x = -1$ is the directrix of a parabola that has focus F. If the point $Q = (2, 7)$ lies on the curve of the parabola, find $|QF|$ (i.e., find the distance between F and Q).



From properties of parabola, we know: $|QL| = |QF|$

$|QL| = \sqrt{(2+1)^2 - (7-7)^2}$

$\boxed{|QF| = 3 \text{ units}}$

QUESTION 3. (8 points) Given $(-3, -1), (-3, 9)$ are the vertices of the major axis of an ellipse (recall major axis is the longer axis) and $(-3, 7)$ is one of the foci.

(i) Find the vertices of the minor axis (shorter axis). (you may want to draw such ellipse (roughly) on the right-side).

$|V_1F_1| = |V_2F_2| \therefore |V_2F_2| = 9 - 7 = 2$

centre $= \left(\frac{-3-3}{2}, \frac{7+1}{2}\right)$ $\because |CF_1| = |CF_2|$

$= (-3, 4)$

$\therefore |CF_1| = 3$

From (ii) $\Rightarrow K/2 = 5$

$|CF_1|^2 = \left(\frac{K}{2}\right)^2 - b^2$

$9 = 25 - b^2$

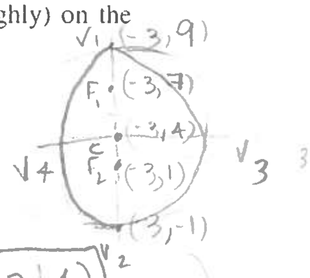
$b = \sqrt{25 - 9}$

$b = \sqrt{16}$

$b = 4$ units

$\therefore V_3 = (-3-4, 4) = \boxed{(-7, 4)}$

$V_4 = (-3+4, 4) = \boxed{(1, 4)}$



(ii) Find the ellipse-constant K.

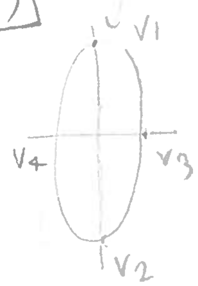
K = length of major axis $= |V_1V_2|$

$\therefore \boxed{K = 10}$

(iii) Find the second foci of the ellipse.

$V_2F_2 = 2 \quad \therefore F_2 = (-3, -1+2)$

$= \boxed{(-3, 1)}$



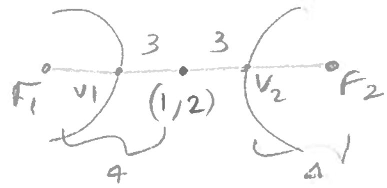
(iv) Find the equation of the ellipse.

general equation $\Rightarrow \frac{(x - x_0)^2}{(b)^2} + \frac{(y - y_0)^2}{(K/2)^2} = 1$

$\frac{(x+3)^2}{16} + \frac{(y-4)^2}{25} = 1$

QUESTION 4. (8 points)

Draw roughly the hyperbola $\frac{(x-1)^2}{9} - \frac{(y-2)^2}{7} = 1$. Then find



a) The hyperbola-constant K.

$(K/2)^2 = 9$ $\boxed{K = 6}$

$\frac{K}{2} = 3$

b) The two vertices of the hyperbola.

$v_1 = (1-3, 2)$ $v_2 = (1+3, 2)$

$\boxed{v_1 = (-2, 2)}$ $\boxed{V_2 = (4, 2)}$

c) The foci of the hyperbola.

$CF_1 = CF_2 = \sqrt{\left(\frac{K}{2}\right)^2 + b^2}$

$= \sqrt{9+7}$

$= 4$ units

$\therefore F_1 = (1-4, 2)$

$\boxed{F_1 = (-3, 2)}$

$F_2 = (1+4, 2)$

$\boxed{F_2 = (5, 2)}$

QUESTION 5. (8 points) Given $y = x^2 + 6x + 11$ is an equation of a parabola. Write the equation of the parabola in the standard form. Then

a) Draw roughly the parabola.

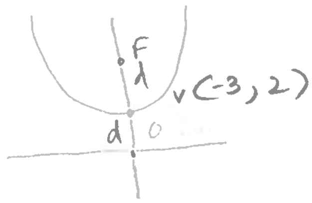
$y = x^2 + 6x + 11$

$y = (x)^2 + 2(3)x + (3)^2 - (3)^2 + 11$

$y = (x+3)^2 - 9 + 11$

$y - 2 = (x+3)^2$

$4\left(\frac{1}{4}\right)(y-2) = (x+3)^2$



b) Find the equation of the directrix line.

$d = 1/4 = 0.25$

$\therefore$ eqn of directrix $\Rightarrow y = 2 - 0.25$

$\boxed{y = 1.75}$

c) Find the focus of the parabola.

$F = (-3, 2 + d)$

$F = (-3, 2 + 0.25)$

$\boxed{F = (-3, 2.25)}$

QUESTION 6. (8 points) a) Given two lines $L_1: x = 3t+1, y = 2t+3, z = 10t+2$ ($t \in R$) and $L_2: x = 6w-5, y = 6w-3, z = -3w+5$ ($w \in R$). Convince me that $L_1 \perp L_2$. Given that L_1 intersects L_2 in a point Q. Find Q.

If $L_1 \perp L_2 \Rightarrow N_1 \cdot N_2 = 0$

$N_1 = \langle 3, 2, 10 \rangle$, $N_2 = \langle 6, 6, -3 \rangle$

$N_1 \cdot N_2 = 18 + 12 - 30$

$N_1 \cdot N_2 = 0$

$\boxed{\therefore L_1 \perp L_2}$

Q satisfies eqns of L_1 & L_2

$\therefore$ we have $3t+1 = 6w-5$ —①

$2t+3 = 6w-3$ —②

$10t+2 = -3w+5$ —③

① − ② $\Rightarrow 3t+1-2t-3 = \cancel{6w}-5-\cancel{6w}+3$

$t - 2 = -2$

$t = 0$

$(\therefore w = 21)$

$\because$ It is give L_1 intersect L_2

$Q = (x, y, z)$

$Q = (3t+1, 2t+3, 10t+2)$

$\boxed{Q = (1, 3, 2)}$ ✓

QUESTION 7. (15 points)

Given that $q_1 = (0, 4, 2)$, $q_2 = (2, 1, -1)$, and $q_3 = (2, 3, 5)$ are not co-linear.

a) Find a vector F that is perpendicular to both vectors $\overrightarrow{q_1q_2}$ and $\overrightarrow{q_1q_3}$.

$\overrightarrow{q_1q_2} = \langle 2, -3, -3 \rangle$

$\overrightarrow{q_1q_3} = \langle 2, -1, 3 \rangle$

If $\vec{F} \perp \overrightarrow{q_1q_2}$ & $\vec{F} \perp \overrightarrow{q_1q_3}$

then $\vec{F} = \overrightarrow{q_1q_2} \times \overrightarrow{q_1q_3}$

$$F = \begin{vmatrix} i & j & k \\ 2 & -3 & -3 \\ 2 & -1 & 3 \end{vmatrix}$$

$= i(-9-3) - j(6+6) + k(-2+6)$

$= -12i - 12j + 4k$

$\therefore \boxed{\vec{F} = \langle -12, -12, 4 \rangle}$

b) Find the area of the triangle with vertices q_1, q_2, q_3.

Area of $\triangle = \dfrac{|\overrightarrow{q_1q_2} \times \overrightarrow{q_1q_3}|}{2}$

$= \dfrac{|\vec{F}|}{2}$

$= \dfrac{\sqrt{(12)^2 + (12)^2 + (4)^2}}{2}$

$= \dfrac{\sqrt{304}}{2}$ ✓

$\Rightarrow \dfrac{17.43}{2}$

$\Rightarrow 8.715$

$\therefore$ The area of the $\triangle$ is $\boxed{8.715 \text{ sq. units}}$

c) Find the equation of the plane, say P, that passes through q_1, q_2, q_3.

$\vec{F}$ is a vector $\perp$ to P

$\Rightarrow P: -12x - 12y + 4z = c$

consider $q_1\ (0, 4, 2)$

$-12(0) - 12(4) + 4(2) = c$

$c = -48 + 8$

$c = -40$

$\therefore$ equation of plane P

$\boxed{-12x - 12y + 4z = -40}$ ✓

d) The point $Q = (1, 1, 10)$ does not lie in the plane, P, as in (c). Find $|QP|$ (the distance between Q and the plane, P)

eqn of $P \Rightarrow -12x - 12y + 4z + 40 = 0$

$|QP| = \dfrac{|\text{sub. } Q \text{ in } P|}{\sqrt{304}}$ $= \dfrac{56}{\sqrt{304}}$ $= \boxed{\cancel{0.184} \text{ units}}$

0.5

e) Find a vector F such that $|F| = 2025$ that is perpendicular to both vectors $\overrightarrow{q_1q_2}$ and $\overrightarrow{q_1q_3}$.

F is $\perp$ to $\overrightarrow{q_1q_2}$ & $\overrightarrow{q_1q_3}$

$\therefore \boxed{F = 2025 \langle -12, -12, 4 \rangle}$

D

$|\vec{F}|$ from (b) $= \sqrt{304}$

$2025 = \sqrt{304}\, a$

$|a| = \dfrac{2025}{17.43}$

$a = 116.17$

$|D|$

Let $D = \langle -12, -12, 4 \rangle$

$\therefore \vec{F} = 116.17 \langle -12, -12, 4 \rangle$

QUESTION 8. (4 points) The line $L_1: x = 2t+1, y = 3t+1, z = at+5$ ($t \in R$) lies entirely in the plane $P: 3x + 2y + 4z = c$. Find the values of a and c.

At $t = 0$

$x = 1, y = 1, z = 5$

substituting in P

$3(1) + 2(1) + 4(5) = c$

$3 + 2 + 20 = c$

$\boxed{c = 25}$ ✓

substituting L_1 in P $\because$ it lies entirely in P

$3(2t+1) + 2(3t+1) + 4(at+5) = 25$

$6t + 3 + 6t + 2 + 4at + 20 = 25$

$(12 + 4a)t + \cancel{25} = \cancel{25}$

$12 + 4a = 0$

$4a = -12$

$\boxed{a = -3}$ ✓

$\dfrac{2025}{\sqrt{304}}$

$-12 - 12 + 40 + 40$