Absorbing Ideals in Commutative Rings: A Survey

Ayman Badawi

In memory of Paul-Jean Cahen

1 Introduction

Let R be a commutative ring with $1 \neq 0$ and I be a proper ideal of R. Then I is called a 2-absorbing ideal of R as in [10] if whenever $abc \in I$ for some $a, b, c \in R$, then $ab \in I$ or $bc \in I$ or $ac \in I$. Over the past 15 years, there has been considerable attention in the literature to 2-absorbing ideals of commutative rings and their generalizations, for example, see [1–5, 9–11, 13–21, 23–26, 30–38, 40–56]. A more general concept than 2-absorbing ideals is the concept of n-absorbing ideals. Let $n \geq 1$ be a positive integer. A proper ideal I of R is called an *n-absorbing ideal* of R as in [2] if $a_1, a_2, \ldots, a_{n+1} \in R$ and $a_1 a_2 \cdots a_{n+1} \in I$, then there are n of the a_i's whose product is in I. In this article, we survey some recent developments on conjectures (see, [2, 9], and [23]) concerning n-absorbing ideals of commutative rings. We survey some classifications of factorization-commutative rings in terms of absorbing ideals. We survey some properties of n-absorbing ideals in ring extensions. We strongly recommend that the reader keeps the first survey article [9] in hand while reading this paper.

2 Conjectures on n-Absorbing Ideals of Commutative Rings

Let I be a proper ideal of a commutative ring R. Then $\sqrt{I}$ denoted the radical ideal of R. A proper ideal of R is called a *strongly n-absorbing ideal* of R as in [2] if whenever $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of R, then the product of some

A. Badawi (✉)
Department of Mathematics & Statistics, The American University of Sharjah, Sharjah, UAE
e-mail: abadawi@aus.edu

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n of the I_j's is contained in I. It is clear that a strongly-n-absorbing ideal of a commutative ring R is an n-absorbing ideal of R.

Anderson and Badawi in [2] made the following conjectures:

Conjecture I If I is an n-absorbing ideal of a commutative ring R, then $(\sqrt{I})^n \subseteq I$.

Conjecture II If I is an n-absorbing ideal of a commutative ring R, then I is a strongly n-absorbing ideal of R.

Conjecture III If I is an n-absorbing ideal of a commutative ring R, then $I[X]$ is an n-absorbing ideal of $R[X]$.

Choi and Walker in [28] gave an affirmative answer for Conjecture I for any positive integer n, and G. Donadze independently in [35] gave an alternative proof of Conjecture I. It was shown in [10] that Conjecture II is correct for $n = 2$. Conjectures II and III were verified in [2] for any positive integer n when R is a Prüfer domain. Also, Conjecture III was verified in [2] when $n = 2$. Laradji in [47] proved that Conjectures II and III are valid for any positive integer n when R is an arithmetical ring (e.g., if R is a Prufer domain). It was shown in [47] that if $I[X]$ is an n-absorbing ideal of $R[X]$, then I is a strongly n-absorbing ideal of R, and hence if Conjecture III is true, then Conjecture II is true.

We recall that a commutative ring R is said to be a *U-ring* provided R has the property that an ideal contained in a finite union of ideals must be contained in one of those ideals. Recall that a Prufer domain is a U-ring. The authors in [53] proved the following result.

Theorem 2.1 ([53, Theorem 2.4]) *If R is a U-ring, then Conjecture II holds.*

We recall from [39] and [6] that an integral domain R is called a *pseudo-valuation domain* (PVD) if R has exactly one maximal ideal M such that $(M : M)$ is a valuation domain. We recall that if $f(x) = a_n x^n + \cdots + a_0 \in R[x]$, then $C(f)$ is the ideal $(a_n, \ldots, a_0)R$. A ring R is called a *Gaussian ring* if $C(fg) = C(f)C(g)$ for every $f, g \in R[x]$. The authors in [53] proved the following result.

Theorem 2.2

(1) *[53, Theorem 2.6]. If R is a U-ring that is a Gaussian ring, then Conjecture III holds.*

(2) *[53, Theorem 2.7]. Let $n \geq 2$. Suppose that R is a PVD with maximal ideal M and I is a proper ideal of R such that $\sqrt{I} \neq M$. Then I is an n-absorbing ideal of R if and only if $I[x]$ is an n-absorbing ideal of $R[x]$.*

Since if Conjecture III holds, then Conjecture II holds by [47, Theorem 2.9(i)], in light of Theorem 2.2 we have the following result.

Corollary 2.3

(1) *If R is a U-ring that is a Gaussian ring, then Conjectures II and III hold.*

(2) *Let $n \geq 2$. Suppose that R is a PVD with maximal ideal M and I is a proper ideal of R such that $\sqrt{I} \neq M$. Then I is an n-absorbing ideal of R if and only if*

$I[x]$ is an n-absorbing ideal of $R[x]$, if and only if I is a strongly n-absorbing ideal of R.

We recall from [33] and [12] that a commutative ring R is called a *divided ring* if $Q \subset xR$ for every prime ideal Q of R, and $x \in R \setminus Q$ and it is called a *locally divided* ring as in [15] if R_P is a divided ring for every prime ideal P of R.

Recently, Choi in [27] proved the following result.

Theorem 2.4 ([27, Corollary 13]) *Let R be a locally divided ring. Then Conjectures II and III hold.*

Since a PVD is a divided ring (and hence locally divided), we conclude that Corollary 2.3(ii) is a particular case of Theorem 2.4.

We recall from [1] that the AF-dimension of a ring R, denoted by AF-dim(R), is the smallest positive integer n such that each proper ideal of R can be written as a finite product of n-absorbing ideals of R; if no such n exists, then AF-dim(R) = ∞. A ring R is an FAF-ring if AF-dim(R) $< \infty$.

The following are examples of FAF-rings.

Example 2.5

(1) [1, Corollary 3.9]. Let $d \in \mathbb{Z}-\{0, 1\}$ be a square-free integer such that $4 \mid (d-1)$ and $8 \mid (d-5)$. Then $R = \mathbb{Z}[\sqrt{d}]$ is an FAF-ring and $AF\text{-}dim(R) = 2$.
(2) [1, Corollary 4.4]. Let R be a finite direct product of fields. Then R and $R[X]$ are FAF-rings.

Choi in [27] proved the following result.

Theorem 2.6 ([27, Theorem 39 (4)]) *Assume that R is an FAF-ring. Then Conjectures II and III hold.*

3 2-AB-Rings and Factorization Rings

We recall from [21] that a commutative ring R is called a 2-AB-ring if every 2-absorbing ideal of R is prime.

The authors in [21] proved the following results.

Theorem 3.1 ([21, Theorem 2.3]) *Let R be a commutative ring with $1 \neq 0$. The following statements are equivalent.*

(1) *R is a 2-AB-ring.*
(2) *R has exactly one maximal ideal, say M, such that the prime ideals of R are linearly ordered (by inclusion) and $IM = P$ for every 2-absorbing ideal I of R and every minimal prime ideal P over I.*
(3) *R has exactly one maximal ideal, say M, such that the prime ideals of R are linearly ordered (by inclusion) and P is the only minimal 2-absorbing ideal over P^2 for every prime ideal P of R.*

Let $n \geq 2$ be a positive integer. The authors in [43] extended the concept of 2-AB-rings to n-AB-rings. We recall from [43] that a commutative ring R is called an n-AB-ring if every n-absorbing ideal of R is a prime ideal of R. They obtained similar results to those in Theorem 3.1.

Theorem 3.2 ([43, Theorem 2.13]) *Let R be a commutative ring with $1 \neq 0$. The following statements are equivalent.*

(1) *R is an n-AB-ring.*
(2) *R has exactly one maximal ideal, say M, such that the prime ideals of R are linearly ordered (by inclusion) and $IM = P$ for every n-absorbing ideal I of R and every minimal prime ideal P over I.*
(3) *R has exactly one maximal ideal, say M, such that the prime ideals of R are linearly ordered (by inclusion) and P is the only minimal n-absorbing ideal over P^n for every prime ideal P of R.*

4 Commutative Rings with 2-Absorbing Factorization

Let R be a commutative ring with $1 \neq 0$. Then R is called a *TAF*-ring if every ideal of R is a finite product of 2-absorbing ideals. The authors in [50] obtained the following results.

Theorem 4.1 ([50, Theorem 3.3]) *Any TAF-ring is a finite direct product of one-dimensional domains and zero-dimensional quasi-local rings having nilpotent maximal ideal. In particular, a TAF-ring of dimension one having a unique height-zero prime ideal is a domain.*

Theorem 4.2 ([50, Corollary 3.4]) *Let R be a commutative ring. The following are equivalent.*

(1) *$R[X]$ is a TAF-ring.*
(2) *R is a von Neumann regular TAF-ring.*
(3) *R is a finite direct product of fields.*

In view of Theorem 4.2, we have the following example.

Example 4.3 Let $R = \mathbb{Z}_5 \times \mathbb{Q} \times \mathbb{R} \times \mathbb{Z}_{11}$. Then R and $R[X]$ are TAF-rings by Theorem 4.2.

The authors in [22] proved the following result.

Theorem 4.4 ([22, Theorem 2.3]) *Let R be a commutative ring. Then $R[X]$ is a principal ideal ring if and only if R is ring-isomorphic to a finite direct product of fields.*

In view of Theorems 4.4 and 4.2, we have the following result.

Corollary 4.5 *Let R be a commutative ring. The following are equivalent.*

(1) *$R[X]$ is a TAF-ring.*
(2) *R is a von Neumann regular TAF-ring.*
(3) *R is a finite direct product of fields.*
(4) *$R[X]$ is a principal ideal ring.*

Let R be an integral domain. We recall the following definitions.

(1) We say R has *finite character* if every $x \in R - \{0\}$ belongs to only finitely many maximal ideals of R.
(2) R is called an *atomic* domain if every nonzero non-unit can be written in at least one way as a finite product of irreducible elements.
(3) R is a *discrete valuation ring (DVR)* if R is a principal ideal domain (PID) with exactly one nonzero maximal ideal.
(4) R is an ACCP-domain if there is no infinite strictly ascending chain of principal ideals.

We recall from [50] that a proper ideal I of R is called a TA-ideal if I is a finite product of 2-absorbing ideals.

Theorem 4.6 ([50, Theorem 4.3]) *Let R be an integral domain that is not a field with exactly one maximal ideal M. The following are equivalent.*

(1) *R is a TAF-domain.*
(2) *R is one-dimensional and every principal ideal of R is a TA-ideal.*
(3) *R is atomic, one-dimensional and every atom of R generates a TA-ideal.*
(4) *R is atomic and M^2 is universal (i.e. $M^2 \subseteq aR$ for each atom $a \in R$).*
(5) *R is an atomic PVD.*
(6) *R is a PVD which satisfies ACCP.*
(7) *$(M : M)$ is a DVR with maximal ideal M.*
Furthermore, if R is Noetherian, then the integral closure R' of R is a DVR with maximal ideal M.

Theorem 4.7 ([50, Theorem 4.4])
Let R be an integral domain. The following are equivalent.

(1) *R is a TAF-domain.*
(2) *R has finite character and R_M is a TAF-domain for each maximal ideal M of R.*
(3) *R has finite character and R_M is an atomic PVD for each maximal ideal M of R.*
(4) *R has finite character and R_M is an ACCP PVD for each maximal ideal M of R.*
(5) *R is a one-dimensional domain which has finite character and every principal ideal of R is a TA-ideal.*
(6) *R is a one-dimensional ACCP-domain that has finite character and every principal ideal generated by an atom is a TA-ideal.*

If R is a Noetherian domain, then we have the following result.

Theorem 4.8 ([50, Corollary 4.5]) *For a Noetherian domain R that is not a field, the following are equivalent.*

(1) *R is a TAF-domain.*
(2) *R_M is a TAF-domain for each maximal ideal M of R.*
(3) *R_M is a PVD for each maximal ideal M of R.*
(4) *R'_M is a DVR with maximal ideal MR_M for each maximal ideal M of R.*
(5) *R is one-dimensional and every principal ideal generated by an atom is a TA-ideal.*

Theorem 4.9

(1) *[50, Corollary 4.7]. Let R be a Noetherian domain. If R is a TAF-domain, then so is every overring of R.*
(2) *[50, Corollary 4.8]. Let $K \subseteq L$ be a field extension. Then $K + XL[X]$ is a TAF-domain.*
(3) *[50, Corollary 4.11]. Let $d \in \mathbb{Z} - \{0, 1\}$ be a square-free integer such that $4 \mid (d-1)$. Then $\mathbb{Z}[\sqrt{d}]$ is a TAF-domain if and only if $8 \mid (d-5)$.*

5 Commutative Rings with Absorbing Factorization

We recall from [1] that the AF-dimension of a ring R, denoted by AF-dim(R), is the smallest positive integer n such that each proper ideal of R can be written as a finite product of n-absorbing ideals of R; if no such n exists, then AF-dim(R) = ∞. A ring R is an FAF-ring if AF-dim(R) $< \infty$. Recall that a *ZPI-ring* is a ring whose proper ideals can be written as a product of prime ideals. Hence, $AF - dim(R)$ measures, in some sense, how far R is from being a ZPI-ring.

The following is a structure theorem for the FAF-rings.

Theorem 5.1 ([1, Theorem 4.2]) *Any FAF-ring is a finite direct product of one-dimensional domains and zero-dimensional local rings with nilpotent maximal ideal. In particular, an FAF-ring of Krull dimension one having unique height-zero prime ideal is a domain.*

Recall that a ring R is said to be *special primary* if R has exactly one maximal ideal M and every proper ideal of R is a power of M. Note that if R is a ZPI ring, then R is a special primary ring.

Recall that R is called a chained ring if $a \mid b$ or $b \mid a$ for every $a, b \in R$.

Theorem 5.2 ([1, Proposition 3.4]) *A chained ring R is an FAF-ring if and only if R is a special primary ring.*

The next result says that the AF-dimension of a factor (resp. fraction) ring is bounded above by the AF-dimension of the ring.

Theorem 5.3 ([1, Proposition 3.5]) *Let R be an FAF-ring and T a factor or a fraction ring of R. Then $AF - dim(T) \leq AF - dim(R)$.*

Theorem 5.4 ([1, Proposition 3.6]) *Let $R_1, \ldots, R_k$ be FAF-rings and $R = R_1 \times \cdots \times R_k$. Then $AF - dim(R) = max\{AF - dim(R_i) \mid 1 \leq i \leq k\}$.*

Denote by $Min(I)$ the set of minimal prime ideals over an ideal I.

Theorem 5.5 ([1, Proposition 3.7]) *Let R be an FAF-ring and I a proper ideal. Then $Min(I)$ is finite.*

Theorem 5.6 ([1, Proposition 3.8]) *Let R be a finite ring of order m such that $p^{n+2} \nmid m$ for each prime p. Then $AF - dim(R) \leq n$. Moreover, $AF - dim(\mathbb{Z}_{p^{n+1}}[X]/(X^2, pX)) = n + 1$.*

Recall that if R is a ring, then $Spec(R) = \{P \mid \text{P is a prime ideal of } R\}$.

Theorem 5.7 ([1, Theorem 5.4]) *Let R be a commutative Noetherian one-dimensional domain with nonzero conductor $(R : R')$, where R' is the integral closure of R. The following are equivalent.*

(1) *R is an FAF-domain.*
(2) *R_M is an FAF-domain for each maximal ideal M of R.*
(3) *The spectral map $Spec(R') \to Spec(R)$ is bijective.*

In view of Theorem 5.7, we have the following example.

Example 5.8 ([1, Example 5.5])

(1) $AF - dim(\mathbb{Z}[2i]) = 3$.
(2) $R = \mathbb{Z}[\sqrt[3]{4}]$ is an FAF-ring. Since $R' = \mathbb{Z}[\sqrt[3]{2}]$ and $R \subseteq R'$ is a root extension (i.e., $z^2 \in R$ for each $z \in R'$), the map $Spec(R') \to Spec(R)$ is bijective. Hence R is an FAF-domain by Theorem 5.7.
(3) $R = \mathbb{Z}[\sqrt[3]{10}]$ is not an FAF-ring. Note that $R' = \mathbb{Z}[t]$ with $t = \frac{1+\sqrt[3]{10}+\sqrt[3]{100}}{3}$. Furthermore, $(3, t)$ and $(3, t-1)$ are two distinct prime ideals lying over $(3, 1-\sqrt[3]{10})$ in $\mathbb{Z}[\sqrt[3]{10}]$. Thus R is not an FAF-ring by Theorem 5.7.
(4) Let K be a field. Consider the Noetherian one-dimensional domains $A = K + X(X-1)K[X]$ and $B = K + X^n K[X]$ for some $n \geq 2$. Their integral closure is $K[X]$. Consider the spectral maps $Spec(K[X]) \to Spec(A)$ and $Spec(K[X]) \to Spec(B)$. Since only the second one is bijective, we get that B is an FAF-domain while A is not.

Theorem 5.9 ([1, Corollary 4.4]) *Let R be a commutative ring. The following are equivalent.*

(1) *$R[X]$ is an FAF-ring.*
(2) *R is a von Neumann regular FAF-ring.*
(3) *R is a finite direct product of fields.*
(4) *$R[X]$ is a ZPI-ring.*

Since $R[X]$ is a TFT-ring if and only if R is a finite direct product of fields by Corollary 4.5 if and only if R is an FAF-ring by Theorem 5.9, we have the following result.

Corollary 5.10 *Let R be a commutative ring. The following are equivalent.*

(1) *$R[X]$ is a TAF-ring.*
(2) *R is a von Neumann regular TAF-ring.*
(3) *R is a finite direct product of fields.*
(4) *$R[X]$ is a principal ideal ring.*
(5) *$R[X]$ is an FAF-ring.*
(6) *R is a von Neumann regular FAF-ring.*
(7) *$R[X]$ is a ZPI-ring.*

For a one-dimensional domain R, we have the following result.

Theorem 5.11 ([1, Theorem 4.3]) *Let R be a one-dimensional domain. The following are equivalent.*

(1) *R is an FAF-domain.*
(2) *R has finite character and there is some positive integer d such that $AF-dim(R_M) \leq d$ for each maximal ideal M of R.*

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