On n-pseudo valuation domains (n-PVD)

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Thanks for coming,(**History and References)**

INTRODUCTION

(1) J. R. Hedstrom and E. G. Houston, Pseudo-valuation domains. Pacific J. Math. 4(1978), 551–567.

(2) J. R. Hedstrom and E. G. Houston, Pseudo-valuation domains, II. Houston J. Math.4(1978), 199–207.

Pseudo valuation domains were introduced in 1978 (Hedtrom-Houston) : R is an integral domain with quotient field K. If every prime ideal, say Q, of R satisfies the condition : whenever $xy \in Q$ for some $x, y \in K$, then $x \in Q$ or $y \in Q$, then R is called a pseudo-valuation domain. (Many authors studied this class of domains including myself).

The concept of pseudo valuation domains is a generalization of the concept of valuation domains: R is called an integral domain with quotient field K, then R is called a valuation domain if $x \in R$ or $x^{-1} \in R$ for every nonzero $x \in K$.

INTRODUCTION

(3) D. D. Anderson and M. Zafrullah, Almost Bezout domains, J. Algebra, 142(1991), 285–309.

Almost valuation domains were born in 1991 (D.D. Anderson and Muhammad Zafrulla): R is an integral domain with quotient field K, then R is called an almost valuation domain if for every nonzero $x \in K$, there exists n (n depends on x) such that $x^n \in R$ or $x^{-n} \in R$. So every valuation domain is an almost valuation domain.

For a recent article on almost valuation domains see

[4] N. Mahdou, A. Mimouni and M. Moutui, On almost valuation and almost Bezout rings, Commun. Algebra, 43(2015), 297–308.

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A. Badawi, On pseudo-almost valuation domains, Commun. Algebra 35(2007), 1167–1181.

Pseudo-almost valuation domain was introduced in 2007 (Ayman Badawi): Let R be an integral domain with quotient field K. If every prime ideal, say Q, of R satisfies the condition: For every $x \in K$, there exists a positive integer n such that $x^n \in Q$ or $x^{-n}a \in Q$, then R is called a pseudo-almost valuation domain (It turns out that every almost valuation domain and pseudo-valuation domain is a pseudo-almost valuation domain but not vice-versa).

Definition

Let R be an integral domain with quotient field K, I be a proper ideal of R and $n \geq 1$ be a positive integer. We say that I is a n-powerful ideal of R if whenever $x^n y^n \in I$ for some $x, y \in K$, then $x^n \in R$ or $y^n \in R$. We say I is a n-powerful semiprimary ideal of R if whenever $x^n y^n \in I$ for some $x, y \in K$, then $x^n \in I$ or $y^n \in I$. The concept of powerful ideals was studied by Ayman Badawi and Evan Houston (Powerful ideals, strongly primary ideals, almost pseudo-valuation domains, and conducive domains. Communications in Algebra, 30(4) (2002))

Definition

Let R be an integral domain with quotient field K and $n \geq 1$ be a positive integer. We say that R is an n-pseudo valuation domain (n-PVD) if every prime ideal of R is a n-powerful semiprimary ideal of R. Note that if $n = 1$, then a pseudo-valuation domain in the sense of Houston-Hedstrom is a 1-PVD.

Example

Let $R = Q + X^2C + X^4C[[X]]$, *where* Q *is the field of rational numbers and* C *is the field of complex numbers. Then one can see that* R *is neither a PAVD as in Badawi nor a PVD as in Hedstrom-Houston nor an almost valuation domain as in Anderson-Zafrulla. However, it is easily checked that* R *is a* 4*-PVD with maximal ideal* $M = X^2C + X^4C[[X]]$ *and* $\overline{R} = \overline{Q} + XC[[X]]$ *is a PVD with maximal ideal* $N = \{x \in K \mid x^n \in M\} = XC[[X]]$, *where* $\overline{Q}$ *is the algebraic closure of* Q *inside* C, *and* K *is the quotient field of* R. *Note that* $\overline{R}$ *is not a valuation domain and* R *is not an n-PVD for every* $1 \leq n \leq 3$.

Theorem

Let $n \geq 1$ *and* I *be a prime ideal of an integral domain* R *with quotient field* K. *Then* I *is a n-powerful semiprimary ideal of* R *if and only if* I *is a n-powerful ideal of* R.

Theorem

Assume $P \subseteq Q$ are prime ideals of an integral domain R. If Q is a n-powerful semiprimary ideal of R for some positive integer $n \geq 1$, then P is a n-powerful semiprimary ideal of R.

Theorem

Let $n \geq 1$ and a assume that R is an n-PVD. Then R is a quasilocal domain.

Corollary

Let $n \geq 1$ be a positive integer. Then an integral domain R is an n-PVD if and only if a maximal ideal of R is an n-powerful semiprimary ideal of R if and only if a maximal ideal of R is an n-powerful ideal of R.

Definition

Let R be a commutative ring with $1 \neq 0$ and $n \geq 1$. A proper ideal I of R is called an n-semiprimary ideal of R, if whenever $x^n y^n \in I$ for some $x, y \in R$, then $x^n \in I$ or $y^n \in I$.

Theorem

Let R be a commutative ring with $1 \neq 0$, $n \geq 1$, and I be a proper ideal of R. If I is an n-semiprimary ideal of R, then I is an m-semiprimary ideal of R for every $m \geq n$.

COMMENTS

If I is an n-powerful semiprimary ideal, then I is an n-semiprimary ideal. Thus I is also an m-semiprimary ideal for every integer $m \geq n$, but I need not be an m-poweful semiprimary ideal.

Example

Let $R = F[[X^2, X^5]] = F + FX^2 + X^4F[[X]]$, where F is a field. Then R is quasilocal with maximal ideal $M = (X^2, X^5) = FX^2 + X^4F[[X]]$ and quotient field $K = F[[X]][1/X]$. Clearly M is a 2-semiprimary ideal of R, but not a 3-powerful semiprimary ideal of R since $X^3X^3 = X^6 \in M$, but $X^3 \notin M$. Moreover, M is a 2-powerful semiprimary ideal of R if and only if $char(F) = 2$, and M is an n-powerful semiprimary ideal of R for every integer $n \geq 4$. So, for $R = \mathbb{Z}_2[[X^2, X^5]]$, M is a 2-powerful semiprimary ideal, but not a 3-powerful semiprimary ideal, Thus the "powerful" property fails for M. Let $I = X^4F[[X]]$. Then I is a 2-semiprimary ideal of R, but not a 2-powerful semiprimary ideal of R since $X^2X^2 \in I$, but $X^2 \notin I$. So the "semiprimary" property fails for $I \subseteq J = M$ when $char(F) = 2$.

Definition

Let R be a commutative ring and $n \geq 1$ be a positive integer. A prime ideal P of R is called an n-divided prime ideal of R if $x^n \mid p^n$ (in R) for every $x \in R \setminus P$ and for every $p \in P$. A commutative ring R is called an n-divided ring if every prime ideal of R is an n-divided prime ideal of R. Note that if $n = 1$, then a divided ring in the sense of Dobbs-Badawi is a 1-divided ring.

Corollary

Assume that an integral domain R is an n-PVD for some positive integer $n \geq 1$. Then R is an n-divided domain and the set of all prime ideals of R are linearly ordered by inclusion.

Theorem

Let $n \geq 1$ and R be a root closed integral domain with quotient field K. Then R is a PVD if and only if R is an n-PVD.

Theorem

Let P be a prime ideal of an n-PVD R. Then R/P is an n-PVD.

COMMENTS

We recall from Anderson-Zafrulla that an integral domain R with quotient filed K is called an almost valuation domain if for every nonzero $x \in K$, there is an integer $n \geq 1$ (n depends on x) such that $x^n \in R$ or $x^{-n} \in R$. We have the following definition.

Definition

Let $n \geq 1$ be a positive integer and R be an integral domain with quotient filed K is called an n-valuation domain (n-VD) if for every nonzero $x \in K$, we have $x^n \in R$ or $x^{-n} \in R$.

COMMENTS

It is clear that an n-valuation domain is an almost valuation domain. Also, it is clear that an n-valuation domain (n-VD) is an n-PVD, but an n-PVD need not be an n-VD. Also, an almost valuation domain need not be an n-VD for any positive integer n.

Example

(a) Let $R = \mathbb{Q} + X\mathbb{R}[[X]]$. Then R is a PVD with maximal ideal $X\mathbb{R}[[X]]$ and quotient field $\mathbb{R}[[X]][1/X]$, and thus R is an n-PVD for every positive integer n. However, R is not an n-VD for any positive integer n since $\pi^n, \pi^{-n} \notin R$ for every positive integer n.
(b) Let $R = \mathbb{Z}_p + XF[[X]]$, where p is a positive prime integer and $F = \overline{\mathbb{Z}_p}$ is the algebraic closure of $\mathbb{Z}_p$. Then R is an almost valuation domain with maximal ideal $XF[[X]]$ and quotient field $F[[X]][1/X]$, but not an n-VD for any positive integer n. This follows from the fact that for every $0 \neq a \in F$, there is a positive integer n such that $a^n = 1$; but for every positive integer n, there is a $b \in F$ such that $b^n \notin \mathbb{Z}_p$ and $b^{-n} \notin \mathbb{Z}_p$. Note that R is also a PVD, and thus an n-PVD for every positive integer n.

Theorem

Let R be an n-PVD for some positive integer $n \geq 1$ with maximal ideal M. Suppose that V is an overring of R such that $\frac{1}{s} \in V$ for some $s \in M$. Then V is an n-VD (and hence V is an almost valuation domain).

Theorem

Let R be an n-PVD for some positive integer $n \geq 1$ with maximal ideal M. Suppose that P is a prime ideal of R such that $P \neq M$. Then R_P is an n-VD (and hence R_P is an almost valuation domain). Furthermore, $x^n \in R$ for every $x \in P_P$, and hence $P_P \subset \overline{R}$.

Theorem

Suppose that an integral domain R with quotient field K admits a principal prime ideal P of R that is an n-divided ideal of R for some positive integer $n \geq 1$, then P is a maximal ideal of R. In particular, if P is an n-powerful semiprimary ideal of R for some positive integer $n \geq 1$, then P is a maximal ideal of R and R is an n-VD.

Theorem

Let $n \geq 1$, R be an integral domain with quotient field K and P be a prime ideal of R . Assume that P is an n-powerful semiprimary ideal of R. Then P is an mn-powerful semiprimary ideal of R for every integer $m \geq 1$. Furthermore, if $x^{mn} \in P$ for some integer $m \geq 1$ and $x \in K$, then $x^n \in P$. In particular, if R is an n-PVD, then R is an mn-PVD for every integer $m \geq 1$.

Theorem

Let $n \geq 1$ be an integer and R be an n-PVD with maximal ideal M and with quotient field K. Assume that B is overring of R that is integral over R. Then B is an n-PVD with maximal ideal $\sqrt{MB} = \{x \in B \mid x^n \in M\}$.

Theorem

Let $n \geq 1$ be an integer and R be a quasilocal domain with maximal ideal M and with quotient field K. Then R is an n-PVD if and only if $\overline{R}$ is a PVD with maximal ideal $N = \{x \in K \mid x^n \in M\}$.

Corollary

Let $n \geq 1$ be an integer and R be a quasilocal domain with maximal ideal M and with quotient field K. The following statements are equivalent.

1. *R is an n-PVD.*
2. *$\overline{R}$ is a PVD with maximal ideal $N = \{x \in K \mid x^n \in M\}$.*
3. *$N = \{x \in K \mid x^n \in M\}$ is a maximal ideal of $\overline{R}$ such that $(N : N)$ is a valuation domain with maximal ideal N.*

Corollary

Let P be a nonzero finitely generated prime ideal of an n-PVD R. Then $W = (P : P)$ is an n-PVD with maximal ideal $\sqrt{MW} = \{x \in W \mid x^n \in M\}$. In particular, if R is a Noetherian n-PVD with maximal ideal M, then $(M : M)$ is an n-PVD.

Theorem

Let $n \geq 1$ be an integer and R be an n-PVD with maximal ideal M and with quotient field K. Then every overring of R is an n-PVD if and only if $\overline{R}$ is a valuation domain.

Definition

Let $n \geq 1$ and $A_n(M) = \{x^n \mid x \in K \text{ and } x^n \in M\}$

Theorem

Let $n \geq 1$ and R be a quasilocal integral domain with maximal ideal M, quotient field K, and $I = (A_n(M))$. Then the following statements are equivalent.

1. *R is an n-PVD.*
2. *$V = (I : I)$ is an n-VD with maximal ideal $\sqrt{MV} = \{x \in V \mid x^n \in M\}$, and if $x \in K$ is a nonunit of $\overline{R}$, then $x^n \in M$.*

COMMENTS

We end this talk with several examples.

Example

(a) Let $R = \mathbb{Z}_2[[X^2, X^3]] = \mathbb{Z}_2 + X^2\mathbb{Z}_2[[X]]$. Then R is quasilocal with maximal ideal $M = (X^2, X^3) = X^2\mathbb{Z}_2[[X]]$ and quotient field $K = \mathbb{Z}_2[[X]][1/X]$. It is easily checked that R is an n-PVD if and only if $n \geq 2$ and an n-VD if and only if n is even. First, suppose that n is even. Then
$I = (A_n(M)) = \mathbb{Z}_2X^n + X^{n+2}\mathbb{Z}_2[[X]] \subsetneq M$ and $V = (I : I) = R$ has maximal ideal $M_V = M$. Also,
$M_V = \{x \in V \mid x^n \in M\} \subsetneq \{x \in K \mid x^n \in M\} = X\mathbb{Z}_2[[X]]$. Next, suppose that $n \geq 3$ is odd. Then $I = (A_n(M)) = X^n\mathbb{Z}_2[[X] \subsetneq M$ and $V = (I : I) = \mathbb{Z}_2[[X]]$ has maximal ideal
$M_V = X\mathbb{Z}_2[[X]] = \{x \in K \mid x^n \in M\}$.

(b) Let $R = F[[X^2, X^3]] = F + X^2F[[X]]$, where F is a field. Then R is quasilocal with maximal ideal
$M = (X^2, X^3) = X^2F[[X]]$ and quotient field $F[[X]][1/X]$, and R is an n-PVD if and only if $n \geq 2$. If $char(F) = 2$, then $(A_n(M)) \subsetneq M$ for every integer $n \geq 2$. However, $M = (A_2(M))$ if $char(F) \neq 2$.

Example

(c) Let $R = \mathbb{Z}_p + \mathbb{Z}_pX + X^2F[[X]]$*, where* $F = \overline{\mathbb{Z}_p}$ *is the algebraic closure of* $\mathbb{Z}_p$*. Then* R *is quasilocal with maximal ideal* $M = \mathbb{Z}_pX + X^2F[[X]]$ *and quotient field* $K = F[[X]][1/X]$*. Moreover,* R *is an n-PVD if and only if* $n \geq 2$ *since* $\overline{R} = F[[X]]$ *is a PVD (in fact, a valuation domain). However,* $V = (M : M) = \mathbb{Z}_p + XF[[X]]$ *is an almost valuation domain with maximal ideal* $XF[[X]] = \{x \in K \mid x^n \in M\}$*, but* V *is not an n-VD for any positive integer n. Note that* V *is a PVD, and thus an n-PVD for every positive integer n*

(d) Let F *be a field and* N *a positive integer. Then* $R_N = F + X^NF[[X]]$ *is a quasilocal integral domain with maximal ideal* $M_N = X^NF[[X]]$*, quotient field* $F[[X]][1/X]$*, and integral closure* $\overline{R_N} = F[[X]]$*. Note that* $V_N = (M_N : M_N) = F[[X]]$ *is a valuation domain with maximal ideal* $XF[[X]] = \{x \in V_N \mid x^N \in M_N\} = \sqrt{M_NV_N}$*, and thus* V_N *is an n-VD for every positive integer n. However,* R_N *is an n-PVD if and only if* $n \geq N$*, and* R_N *satisfies condition (if* $x \in K$ *is a nonunit element of* $\overline{R}$*, then* $x^n \in M$*) if and only if* $n \geq N$*.*

Example

(e) Let $R = \mathbb{Z}_3 + \mathbb{Z}_3 X^9 + X^{12}\mathbb{Z}_3[[X]]$*. Then* R *is a quasilocal integral domain with maximal ideal* $M = \mathbb{Z}_3 X^9 + X^{12}\mathbb{Z}_3[[X]]$*, quotient field* $\mathbb{Z}_3[[X]][1/X]$*, and integral closure* $\overline{R} = \mathbb{Z}_3[[X]]$*. Note that* $V = (M : M) = \mathbb{Z}_3 + X^3\mathbb{Z}_3[[X]]$ *is a 3-VD with maximal ideal* $X^3\mathbb{Z}_3[[X]] = \sqrt{MV} = \{x \in V \mid x^3 \in M\}$*. However,* R *is not a 3-PVD since* $(X^2)^3(X^2)^3 \in M$*, but* $X^6 \notin M$*, and* R *does not satisfy condition (if* $x \in K$ *is a nonunit element of* $\overline{R}$*, then* $x^n \in M$*) since* $X^3 \notin M$*.*

COMMENTS

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