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Department of Mathematics and Statistics
American University of Sharjah

Final Exam – Spring 2022
MTH 205 – Diff. Equations

Date: Thursday March 19, 2022 Time: 5:00-7:00 pm.

Student Name	Student ID Number
Fahad Alzara	88243

1. This exam has 6 page plus this cover page.
2. *No communication of any kind!*
3. *Do not open this exam until you are told to begin.*
4. *No Questions are allowed during the examination.*
5. *Do not separate the pages of the exam.*
6. *Scientific calculators are allowed but cannot be shared. Graphing Calculators are not allowed.*
7. *Turn off all cell phones and remove all headphones.*

Failing to abide by any of the above exam rules may result in a disciplinary action taken against you

Student signature: ____________________

Name Fahad Alzara, ID 88243

Final Exam, MTH 205, Spring 2022

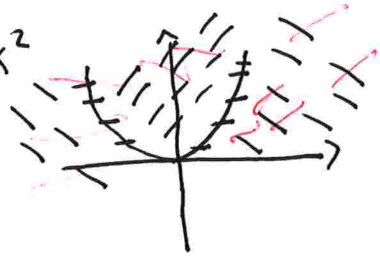
Ayman Badawi

Score = $\frac{\quad}{64}$

QUESTION 1. (6 points) (SHOW THE WORK)

(i) (2 points) Sketch the direction field of $y' = x - \sqrt{y}$, where $y \geq 0$.

Let $y' = 0 \Rightarrow 0 = x - \sqrt{y} \Rightarrow y = x^2$

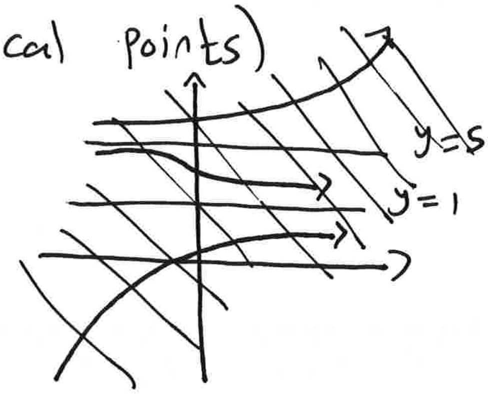


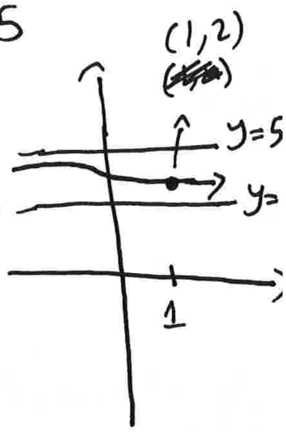
(ii) (3 points) Assume that $y(x)$ is passing through $(1, 2)$ and it is the solution to the autonomous differential equation $y' = y^2 - 6y + 5$. Roughly, sketch $y(x)$.

$y' = y^2 - 6y + 5 \Rightarrow$ Let $y' = 0 \Rightarrow 0 = y^2 - 6y + 5$

$y = 5,\ y = 1$ (Critical points)

sign of y': +++ (1) --- (5) +++





(iii) (1 point)Let $y(x)$ as in (ii), find $Lim_{x\to\infty}\ y(x)$

$\lim_{x\to\infty} y(x) = 1$

QUESTION 2. (6 points) (SHOW THE WORK) Solve the following differential equation.

$$\frac{dy}{dx} = \frac{1}{2x + x^2y^2e^{(-y)}}$$

$\frac{dy}{dx} = \frac{1}{2x + x^2y^2e^{(-y)}}$ Flip $\rightarrow \frac{dx}{dy} = 2x + x^2y^2e^{-y}$

$\Rightarrow x' - 2x = x^2y^2e^{-y}$ (1st order non-linear)

Let $w = x^{1-n}$ where $n = 2 \Rightarrow w = x^{1-2} = x^{-1}$

$\therefore$ ~~$w' - 2(2)w = 2y^2e^{-y} \Rightarrow w' - 4w = 2y^2e^{-y}$~~

$\therefore w' - 2(-1)w = -1(y^2e^{-y}) \Rightarrow w' + 2w = -y^2e^{-y}$

~~$I = e^{\int 2dt}$~~ $I = e^{\int 2\,dy} = e^{2y}$ $\Rightarrow$

$y'x$ and y'

x^{n-1} and x^n

QUESTION 3. (6 points) (SHOW THE WORK) Consider the differential equation $(x+1)y' + y = 0$. It is clear that 0 is an ordinary value. Imagine we need to solve for y(x) by a power series around 0, i.e., $y(x) = \sum_{i=0}^{\infty} a_i x^i$.

(i) (1 point) Which of the values $x = 1, -1, 2$ are ordinary ?

$$(1+x)y' + y = 0 \Rightarrow y' + \frac{1}{1+x} y = 0$$

$x=2$ and $x=1$ are ordinary but $x=-1$ is not ordinary because it is not continuous and hence its derivatives are not differentiable

* (ii) (5 points) Assume that $y(0) = 4$, find the recurrence formula for the series $y(x) = \sum_{i=0}^{\infty} a_i x^i$, and calculate the exact values of a_0, a_1, a_2, a_3.

$$\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots = y$$

$$y' = a_1 + 2a_2 x^2 + \ldots + n a_n x^{n-1} + (n+1) a_{n+1} x^n + \ldots$$

$$(x+1)y' + y = 0 \Rightarrow xy' + y' + y = 0$$

$$x(\ldots + n a_n x^{n-1} + \ldots) + (\ldots + (n+1) a_{n+1} x^n + \ldots) + (\ldots + a_n x^n + \ldots) = 0 + \ldots + 0$$

$$\therefore x^n (n a_n + (n+1) a_{n+1} + a_n) = 0 x^n$$

$$n a_n + (n+1) a_{n+1} + a_n = 0 \Rightarrow a_{n+1} = \frac{(\cancel{1+n}) a_n}{(\cancel{1+n})} = a_n$$

$$\boxed{\text{The recurrence formula is } a_{n+1} = a_n}$$

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QUESTION 4. (6 points) (SHOW THE WORK)

(i) (4 points) Find the general solution for $y(t)$, where $(3t^2+1)y'' - 6ty' = 0$.

$$(3t^2+1)y'' - 6ty' = 0$$

Let $w = y'$, $w' = y''$

$$\therefore (3t^2+1)w' - 6tw = 0 \quad (1^{st} \text{ order linear})$$

$$w' - \frac{6t}{3t^2+1} w = 0$$

$$I = e^{-\int \frac{6t}{3t^2+1} dt} = e^{-\ln|3t^2+1|} = \frac{1}{3t^2+1}$$

$$w = \frac{\int \cancel{\left(\frac{1}{3t^2+1}\right)}(0)\, dt}{\left(\frac{1}{3t^2+1}\right)}$$

$$w = \frac{0 + c}{\left(\frac{1}{3t^2+1}\right)}$$

$$\boxed{w = c(3t^2+1), \quad y = \int c(3t^2+1)\, dt}$$ NEXT PAGE

(ii) (2 points) Find the largest interval I around the x-axis so that the differential equation $\underbrace{\sqrt{x-1}}_{}y'' + \underbrace{\frac{3}{x-3}}_{}y' + \underbrace{\frac{2}{x-4}}_{}y = 0$, where $y(2) = 4$ and $y'(2) = 6$, has unique solution.

cont. except at $x = 4$

cont. at $x > 1$

cont. except at $x = 3$

$\therefore$ Interval $= (1, 3)$

QUESTION 5. (12 points, this question is about Laplace) (SHOW THE WORK)

(i) (2 points) Find $\ell^{-1}\left\{\frac{s-1}{(s-2)^3}\right\}$

$$L^{-1}\left\{\frac{s-1-1+1}{(s-2)^3}\right\} = L^{-1}\left\{\frac{s-2}{(s-2)^3} + \frac{1}{(s-2)^3}\right\} = e^{2t}t + \frac{e^{2t}}{2}t^2$$

(ii) (4 points) solve for $y(t)$, where $y' + 5y = e^{-t} - 6\int_0^t y(r)\,dr$, where $y(0) = 0$.

$$L\{\} \Rightarrow sY(s) + 5Y(s) = \frac{1}{s+1} - 6\frac{Y(s)}{s}$$

$$Y(s)\left(s + 5 + \frac{6}{s}\right) = \frac{1}{s+1} \Rightarrow Y(s)\left(\frac{s^2+5s+6}{s}\right) = \frac{1}{s+1}$$

$$\Rightarrow Y(s) = \frac{s}{(s+1)(s+2)(s+3)}$$

$$\frac{s}{(s+1)(s+2)(s+3)} = \frac{a}{s+1} + \frac{b}{s+2} + \frac{c}{s+3}$$

$a = -\frac{1}{2}$, $b = 2$, $c = -\frac{3}{2}$

$$y(t) = L^{-1}\{Y(s)\}$$

$$y(t) = L^{-1}\left\{-\frac{1}{2}\cdot\frac{1}{s+1} + \frac{2}{s+2} - \frac{3}{2}\cdot\frac{1}{s+3}\right\} = -\frac{1}{2}e^{-t} + 2e^{-2t} - \frac{3}{2}e^{-3t}$$

(iii) (3 points) Solve for $x(t)$ only, where $x(t) + y'(t) = 15$ and $x'(t) + y(t) = 5t$, $x(0) = 10$ and $y(0) = 0$.

$$x(t) + y'(t) = 15$$

$$x'(t) + y(t) = 5t$$

apply laplace to both

$$X(s) + sY(s) = \frac{15}{s} \quad \text{— (1)}$$

$$sX(s) - 10 + Y(s) = \frac{5}{s^2}$$

$$sX(s) + Y(s) = \frac{5}{s^2} + 10 \quad \text{— (2)}$$

$$X(s) = \frac{\begin{bmatrix} \frac{15}{s} & s \\ \frac{5}{s^2}+10 & 1 \end{bmatrix}}{\begin{bmatrix} 1 & s \\ s & 1 \end{bmatrix}}$$

$$X(s) = \frac{\frac{15}{s} - s\left(\frac{5}{s^2}+10\right)}{1-s^2}$$

$$X(s) = \frac{\frac{15}{s} - \frac{5}{s} - 10s}{1-s^2} \Rightarrow$$

(iv) (3 points) Use the undetermined method with Laplace as explained in the class to find the general form of y_p, but do not find the exact y_p.

$$y^{(3)} - y'' = xe^x$$

Let $y^{(3)} - y'' = 0$ and Let $y = e^{mt}$

$\Rightarrow$ Char (H.D.E) $\Rightarrow m^3 - m^2 = 0 \Rightarrow m^2(m-1) = 0$

$m = 0 \rightarrow y_1 = e^{0t} = 1$

$m = 0 \rightarrow y_2 = te^{0t} = t \qquad \therefore y_h = c_1 + c_2 t + c_3 e^t$

$m = 1 \rightarrow y_3 = e^t$

$Y(s) = \frac{\quad}{s^2(s-1)} \cdot L\{xe^x\} = \frac{\quad}{s^2(s-1)} \cdot \frac{\quad}{(s-1)^2} = \frac{\quad}{s^2(s-1)^3}$

$Y(s) = \frac{c_1}{s} + \frac{c_2}{s^2} + \frac{c_3}{(s-1)} + \frac{c_4}{(s-1)^2} + \frac{c_5}{(s-1)^3}$ Continued back of the page

QUESTION 6. (6 points) (SHOW THE WORK) Use the variation method to find the general solution ($y_g(t)$) to

$$t^2y'' - 2y = \frac{3}{t^2}, \quad where \ t > 0$$

Let $t^2y'' - 2y = 0$ and let $y = t^n$, $y' = nt^{n-1}$, $y'' = (n^2 - n)t^{n-2}$

$\Rightarrow (n^2 - n)t^n - 2t^n = 0$

$t^n(n^2 - n - 2) = 0$

Since $t^n \neq 0$ because $t > 0$, $n^2 - n - 2 = 0$

$(n-2)(n+1) = 0 \Rightarrow \begin{array}{l} n = 2 \rightarrow y_1 = t^2 \\ n = -1 \rightarrow y_2 = t^{-1} = \frac{1}{t} \end{array}$

$y_p = v_1 y_1 + v_2 y_2 \qquad \therefore y_h = c_1 t^2 + c_2 t^{-1}$

two conditions:

(1) $v_1' y_1 + v_2' y_2 = 0 \Rightarrow \left(t^2 v_1' + \frac{1}{t} v_2' = 0\right) \frac{-2}{t}$

(2) $v_1' y_1' + v_2' y_2' = \frac{3}{t^4} \Rightarrow 2t v_1' - t^{-2} v_2' = \frac{3}{t^4}$

(1) $-2t v_1' - \frac{2}{t^2} v_2' = 0$, add (1) and (2)

$\Rightarrow -\frac{2}{t^2} v_2' - \frac{1}{t^2} v_2' = \frac{3}{t^4} \Rightarrow -\frac{3}{t^2} v_2' = \frac{3}{t^4} \Rightarrow v_2' = -\frac{1}{t^2}$

QUESTION 7. (6 points) (SHOW THE WORK) Imagine a steal ball weighing 128 lb (note that mass, m = w/g = 128/32) is suspended from a spring, the spring is stretched 2 ft. The ball started in motion with no initial velocity by displacing it 6 ft above the equilibrium position. Assume no air resistance and no external force.

(i) (5 points) Find an expression, $x(t)$, for the position of the ball at any time t.

$X = 2$ ft

weight $= 128$ lb

$m = 4$ kg

$k = \frac{\text{weight}}{X} = \frac{128}{2}$

$k = 64$ N/m

$x(0) = -6$

$x'(0) = 0$

$mx''(t) + \cancel{a_1 x'(t)} + kx(t) = \cancel{F(t)}$

$\Rightarrow 4x'' + 64x = 0$

Let $y = e^{mt}$, Char(L.D.E) $\downarrow$ $4m^2 + 64 = 0$

$\Rightarrow m = \pm 4i$

$\therefore x(t) = C_1 \cos(4t) + C_2 \sin(4t)$

$x(0) = -6 = C_1 \cos(0) + \cancel{C_2 \sin(0)} \Rightarrow C_1 = -6$

$x'(t) = -4C_1 \sin(4t) + 4C_2 \cos(4t)$

$x'(0) = 0 = 4C_2 \Rightarrow C_2 = 0$

$\boxed{\therefore x(t) = -6\cos(4t)}$

(ii) (1 point) The position of the ball at $t = \pi/12$.

at $t = \frac{\pi}{12} \rightarrow x(\frac{\pi}{12}) = -6\cos(4 \cdot \frac{\pi}{12})$

$\boxed{x = -3 \text{ ft}}$

QUESTION 8. (6 points) Solve the differential equation $(1 + \frac{y}{x})dx + \frac{x}{y}dy = 0$

$\underbrace{(1 + \frac{y}{x})dx + \frac{x}{y}dy = 0}_{0\text{-homogeneous}}$

$\frac{1}{u(u+2)} = \frac{A}{u} + \frac{B}{u+2}$, $A \to \frac{1}{2}$, $B \to -\frac{1}{2}$

Let $y = xu$, $dy = u\,dx + x\,du$

$\Rightarrow F_x(1,u)\,dx + F_y(1,u)[u\,dx + x\,du] = 0$

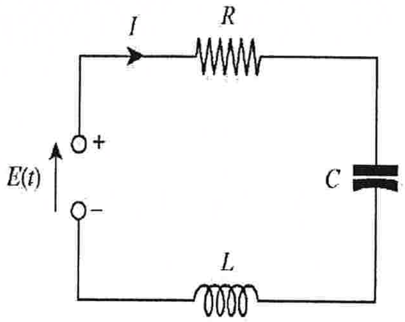
$\Rightarrow (1+u)dx + (\frac{1}{u})[u\,dx + x\,du] = 0$

$(1 + u + 1)dx + (\frac{x}{u})du = 0 \Rightarrow (2+u)dx = -\frac{x}{u}du$

$\Rightarrow \int \frac{1}{x}dx = \int -\frac{1}{u(2+u)}du \Rightarrow \ln|x| = \int \frac{1}{2} \cdot \frac{1}{u}du - \frac{1}{2}\int \frac{1}{u+2}du$

$\Rightarrow \ln|x| = \frac{1}{2}\ln|u| - \frac{1}{2}\ln|u+2| + c \Rightarrow \ln|x| - \frac{1}{2}\ln|\frac{y}{x}| + \frac{1}{2}\ln|\frac{y}{x} + 2| = c$

QUESTION 9. **(6 points)** **(SHOW THE WORK)** imagine the following electrical circuit. Stare at it.



$$i = \frac{dq}{dt} = q'$$

Given that $R = 10\,ohms$, $C = 0.001\,farad$, $L = 0.5\,henry$, and E is constant, where E = 12 volts. Assume no initial current and no initial charge at $t = 0$ when the voltage is first applied. Find the amount of charge $q(t)$ on the capacitor at any time t.

$$q(0) = 0$$
$$i(0) = 0 \to q'(0) = 0$$

$$L\frac{di}{dt} + iR + \frac{q}{c} = E$$

$$Lq'' + Rq' + \frac{1}{c}q = E$$

$$\Rightarrow 0.5q'' + 10q' + 1000q = 12$$

OK

Let $\frac{1}{2}q'' + 10q' + 1000q = 0$ and let $y = e^{mt}$

Char(H.D.E) $\Rightarrow \frac{1}{2}m^2 + 10m + 1000 = 0$

$$m = -10 \pm 10\sqrt{19}\,i$$

$$q(t) = e^{-10t}\left[C_1\cos(10\sqrt{19}\,t) + C_2\sin(10\sqrt{19}\,t)\right]$$

$q_p = A = \text{Const.}$, $q' = 0$, $q'' = 0$ $\therefore 1000A = 12 \Rightarrow A = 0.012$

$$\therefore q(t) = e^{-10t}\left[C_1\cos(10\sqrt{19}\,t) + C_2\sin(10\sqrt{19}\,t)\right] + 0.012$$

QUESTION 10. **(4 points)** solve the differential equation $\frac{dy}{dx} = \frac{1}{\cos(2x+y)e^{(\sin(2x+y))}} - 2$

$$\frac{dy}{dx} = \frac{1}{\cos(2x+y)e^{\sin(2x+y)}} - 2$$

$\Rightarrow$

Let $u = 2x + y$, $\frac{du}{dx} = 2 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 2$

$$\frac{du}{dx} = \frac{1}{\cos(u)e^{\sin(u)}} \Rightarrow \int \cos(u)e^{\sin(u)}\,du = \int dx$$

$\Rightarrow$