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~~Quiz~~ Exam Two, MTH ~~221~~ 205, Spring 2022

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Score = 45/~~50~~ 47

QUESTION 1. (6 points) Solve for $y(t)$.

$$\sin(t)y' + \cos(t)y = (t+1)\sin(t)$$

divide by $\sin(t)$

$y' + \cot(t)\,y = t+1$ — 1st order linear ($\cot(t)$: $P(t)$, $t+1$: $f(t)$)

$I = e^{\int \frac{\cos t}{\sin t}\,dt}$ — $u = \sin t$, $du = \cos t\,dt$, $\frac{du}{\cos t} = dt$

$I = e^{\int \frac{du}{u}} = e^{\ln(\sin t)} = \sin(t)$

$y(t) = \dfrac{\int \sin t\,(t+1)}{\sin(t)}$

$t+1$	+	$\sin t$
1	−	$-\cos t$
0	+	$-\sin t$

$y(t) = \dfrac{-\cos(t)\,(t+1) + \sin(t) + C}{\sin t}$

$y(t) = -(t+1)\cot(t) + 1 + \dfrac{C}{\sin(t)}$

6 ✓

QUESTION 2. (6 points) Solve the following DE.

$$\frac{dy}{dx} = \frac{1}{-x + e^{-y}}$$

$\frac{dx}{dy} = -x + e^{-y}$

$x' = -x + e^{-y}$

$x' + 1x = e^{-y}$ — $f(y)$ — 1st order linear

$I = e^{\int dy} = e^{y}$

$x = \dfrac{\int e^{y}(e^{-y})\,dy}{e^{y}} = \dfrac{y + C}{e^{y}} = ye^{-y} + Ce^{-y}$

10 ✓

QUESTION 3. **(6 points)** Solve for $y(t)$

$$y' + \cos(2t)y = \frac{\cos(2t)}{y}$$

$y' + \cos(2t)y = y^{-1}\cos(2t)$ bernoulli

$n=-1 \quad 1-n=2 \quad w=y^2 \quad y=\sqrt{w}$

$w' + 2\cos(2t)w = 2\cos(2t)$ 1st order linear

$I = e^{\int 2\cos(2t)\,dt} = e^{\sin(2t)}$

$y(t) = \dfrac{\int e^{\sin(2t)} 2\cos(2t)\,dt}{e^{\sin(2t)}}$ → $u=\sin 2t$, $du = 2\cos 2t\,dt$

$y(t) = \dfrac{\int e^u\,du}{e^{\sin(2t)}} = \dfrac{e^{\sin(2t)}+C}{e^{\sin(2t)}} = 1 + \dfrac{C}{e^{\sin(2t)}}$

$y = \sqrt{1 + Ce^{-\sin(2t)}}$

QUESTION 4. **(6 points)** Solve for $y(t)$

$$\frac{dy}{dt} = \frac{y^2-6y+8}{t}$$

$\int \dfrac{dy}{y^2-6y+8} = \int \dfrac{dt}{t}$

$\dfrac{1}{(y-2)(y-4)} = \dfrac{a}{(y-2)} + \dfrac{b}{(y-4)}$

$a = \dfrac{1}{-2}$

$b = 1/2$

$\int \dfrac{dy}{(y-2)(y-4)} = \ln(t) + C$

$\int \left(-\frac{1}{2}\frac{1}{y-2} + \frac{1}{2}\frac{1}{y-4}\right) dy = \ln(t) + C$

$-\frac{1}{2}\int \frac{dy}{y-2} + \frac{1}{2}\int \frac{dy}{y-4} = \ln(t) + C$

$-\frac{1}{2}\ln(y-2) + \frac{1}{2}\ln(y-4) = \ln(t) + C$

$-\frac{1}{2}\left(\ln(y-2) - \ln(y-4)\right) = \ln(t) + C$

$-\frac{1}{2}\ln\left(\frac{y-2}{y-4}\right) = \ln(t) + C$

$\ln\left(\frac{y-2}{y-4}\right) = -2\ln(t) + C$

$\frac{y-2}{y-4} = e^{-2\ln(t)+C}$

$\frac{y-2}{y-4} = t^{-2}e^C$

$\frac{y-2}{y-4} = \frac{e^C}{t^2}$

QUESTION 5. (6 points) Solve the following DE

$$(e^x + y + 2xy)dx + (x^2 + y^2 + x + 4)dy = 0$$

F_x F_y

$(F_x)_y = 1 + 2x$ $(F_y)_x = 2x + 1$

exact

$$\int F_x\,dx = \int e^x + y + 2xy\,dx$$

$$= e^x + xy + x^2y + k(y)$$

$$F_y = x + x^2 + k'(y)$$

$$x + x^2 + k'(y) = x^2 + x + y^2 + 4$$

$$k'(y) = y^2 + 4$$

$$k(y) = \int y^2 + 4 = \frac{y^3}{3} + 4y + C$$

Solution

$$e^x + xy + x^2y + \frac{1}{3}y^3 + 4y + C = 0$$

QUESTION 6. Imagine that we have a metal bar at temperature of 120C is placed in a room with constant temperature of 24c. After 1 minute the temperature of the bar is 60C.

(i) Find the time it will take the bar to reach a temperature of 30C.(Give your answer to the nearest one decimal)

$T_c = 24°C \qquad T(0) = 120°C \qquad T(1) = 60°C$

$T' = k(T-24)$

$T' = kT - 24k$

$T' - kT = -24k$

$I = e^{\int -k\,dt} = e^{-kt}$

$T(t) = \dfrac{\int e^{-kt}(-24k)\,dt}{e^{-kt}}$

$T(t) = \dfrac{24e^{-kt} + C}{e^{-kt}} = 24 + \dfrac{C}{e^{-kt}}$

$= 24 + Ce^{kt}$

$120 = 24 + C$

$C = 96$

$60 = 24 + 96e^{k}$

$\frac{3}{8} = e^{k}$

$k = -0.98$

$T(t) = 24 + 96e^{-0.98t}$

$30 = 24 + 96e^{-0.98t}$

$\frac{1}{16} = e^{-0.98t}$

$t = 2.8$ min ✓

(ii) Find the temperature of the bar after 4 minutes. (Give your answer to the nearest one decimal)

$T(4) = 24 + 96e^{-0.98(4)} = 25.9°C$ ✓

QUESTION 7. (SHOW THE WORK)(8 points) Imagine a 100-gallons tank initially contains 20 gallons of fresh water (i.e., at t = 0, amount of salt is zero). A brine solution containing one pound of salt per gallon is poured into the tank at the rate of 4 gal/min, while the well-stirred mixture leaves the tank at the rate of 2 gal/min.

(i) Find the amount of salt in the tank after 4 minutes.

$$C(t) = \frac{A(t)}{20+2t}$$

$$A'(t) = \text{In - out} = 4 - \frac{2A(t)}{2(10+t)}$$

$$A'(t) = 4 - \frac{A(t)}{10+t}$$

$$A'(t) + \frac{1}{10+t}A(t) = 4$$

$$I = e^{\int \frac{1}{10+t}dt} = 10+t$$

$$A(t) = \frac{4\int(10+t)dt}{10+t} = 4\frac{(10t+t^2/2)+C}{10+t} = \frac{40t+2t^2+C}{10+t}$$

$$A(0)=0 \quad \text{so} \quad C=0$$

$$A(t) = \frac{40t+2t^2}{10+t}$$

$$A(4) = \frac{40(4)+2(4)^2}{14} = 13.7 \text{ pounds}$$

Q5//

(ii) Find the concentration of the salt in the tank after 4 minutes.

$$C(4) = \frac{13.7}{20+8} = 0.5 \text{ pounds/gal}$$

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