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Exam One, MTH 221 , Spring 2022

41 Excellent

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SCORE = $\frac{41}{42}$

QUESTION 1. (8 points) Let $T : P^3 \to P^2$ such that $T(a_2x^2 + a_1x + a_0) = a_0x + (a_0 + a_2)$. Then T is an R-homomorphism (i.e., linear transformation).

(i) **(3 points).** Find the co-matrix presentation of T. (Hint: the co-linear transformation of T is $L : R^3 \to R^2$)

$\mathbb{R}^3 \to \mathbb{R}^2$

$(a_2, a_1, a_0) \to (a_0, a_0 + a_2)$

$\boxed{m = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}}$ ✓

(ii) **(3 points)** Find all polynomials in P_3 (the domain of T) such that $T(a_2x^2 + a_1x + a_0) = 3x + 4$.

$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 3 \\ 1 & 0 & 1 & 4 \end{array}\right] \underset{R_2 - R_1 \to R_2}{\sim} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 3 \\ 1 & 0 & 0 & 1 \end{array}\right] \to \begin{matrix} a_0 = 3 \\ a_2 = 1 \end{matrix}$ $\quad a_1 \in \mathbb{R}$

$\boxed{T = \{(x^2 + a_1x + 3) \mid a_1 \in \mathbb{R}\}}$ ✓

(iii) **(2 points)** Find all polynomials in P_3 such that $T(a_2x^2 + a_1x + a_0) = 0 = 0x + 0$ (i.e., find $Z(T) = Ker(T) = Null(T)$ (hint: (ii) might be useful.)

$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$ $\begin{matrix} a_0 = 0 \\ a_2 = 0 \end{matrix}$ $\quad a_1 \in \mathbb{R}$

$Z(T) = \{(\cancel{0}\ a_1x\ \cancel{0}) \mid a_1 \in \mathbb{R}\}$ $= \cancel{T}\ Z(T) = \{(a_1x) \mid a_1 \in \mathbb{R}\}$

~~$Z(T) = \text{span}(0, 1, 0)$~~

$= \text{span}\{x\}$ ✓

QUESTION 2. (10 points) Let $T : R^3 \to R^3$ such that

$$T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, -x_1 - x_3, 2x_1 + 2x_2 + 3x_3)$$

(i) **(2 points)** Find the standard matrix presentation of T, say M.

$$M = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ 2 & 2 & 3 \end{bmatrix}$$

(ii) **(5 points)** Find M^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - 2R_1 \to R_3]{R_2 + R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - R_3 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\boxed{M^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}}$$

(iii) **(2 points)** If T^{-1} exists, then find a formula for T^{-1}, i.e., $T^{-1}(a_1, a_2, a_3) =$

M^{-1} exists so T^{-1} exists

$$\boxed{T^{-1}(a_1, a_2, a_3) = (2a_1 - a_2 - a_3, a_1 + a_2, -2a_1 + a_3)}$$

(iv) **(1 points)** Find $T^{-1}(1, 2, 4)$.

$$T^{-1}(1, 2, 4) = (2(1) - 2 - 4, 1 + 2, -2(1) + 4)$$

$$\boxed{= (-4, 3, 2)}$$

QUESTION 3. (6 points) Given $D = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R^{2\times2} \mid a+d=0$, where $a,b,c,d \in R\}$. Convince me that D is a subspace of $R^{2\times2}$. Write D as a span of independent matrices. Find $dim(D)$

$a+d=0 \to a=-d \to \boxed{d=-a}$

$D = \{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mid a,b,c \in \mathbb{R}\}$

$= \{ a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mid a,b,c \in \mathbb{R}\}$

$D = \text{span}\{(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix})\}$

$\dim(D) = 3$

because we can write D as a span of independant points, we know that D is a subspace of $\mathbb{R}^{2\times2}$

QUESTION 4. (4 points) (i) Convince me that $D = \{x^3 + a_2x^2 + a_2x + a_2 \mid a_2 \in R\}$ is not a subspace of P_4.

$D(0) = x^3$

hence the image of $D(0)$ is not the origin and so it cannot be a subspace of P_4

you mean $0 \notin D$, if $a_2 = 0$, then we get $x^3 \neq 0$

ii) Convince me that $D = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R^{2\times2} \mid a=b$ OR $c=d$, where $a,b,c,d \in R\}$ is not a subspace of $R^{2\times2}$.
(Hint: in this question, it is easier to show that D is not closed under addition, i.e., find two elements a, b in D such that $a+b$ is not in D)

$A = \begin{bmatrix} 1 & 5 \\ 6 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 \\ 8 & 9 \end{bmatrix}$ are 2 matrices that exist in D

$\begin{bmatrix} 1 & 5 \\ 6 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 14 & 15 \end{bmatrix} = C$

matrix C is not in D as it does not satisfy the conditions and so D is not closed under addition and so cannot be a subspace

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc} 1 & 1 & -1 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{array}\right] \begin{array}{ccc} 1 & 0 & 0 \\ +R_1 & 0 & 1 \\ 0 & \phi & 0 \end{array}$$

$$\left[\begin{array}{ccc} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right] \begin{array}{c} +R_3 \\ -R_3 \\ \end{array} \begin{array}{ccc} 1 & 0 & 0 \\ 1 & 0 & \phi \\ 0 & 1 & 0 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & \phi & 0 & 1 & \end{array}\right.$$

QUESTION 5. (10 points) Let A be a 3×3 matrix such that $C_A(\alpha) = (\alpha - 2)^2(\alpha - 4)$. Let $D = A^2 + A^{-1} + I_3$. Given $E_2 = span\{(1,1,-1),(0,0,1)\}$ and $E_4 = span\{(-1,0,2)\}$

(i) (4 points) Find $|D|$.

eigenvalues of $A = 2, 2, 4$

$$D = (2)^2 + \frac{1}{2} + 1 = 5.5 \, (\times 2)$$
$$= 4^2 + \frac{1}{4} + 1 = 17.25$$
} eigenvalues of D

$|D| = 5.5 \times 5.5 \times 17.25$

$\boxed{= 521.8125}$

(ii) (2 points) Find Trace(D)

$5.5 + 5.5 + 17.25 = \boxed{28.25}$

(iii) (4 points) Is A diagnoliazable? If yes, then find a diagonal matrix D and invertible matrix Q such that $Q^{-1}AQ = D$.

A is diagnolizable as → $\dim(E_2) = 2$ and $\alpha = 2$ is repeated twice
→ $\dim(E_4) = 1$ and $\alpha = 4$ is repeated once

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad \& \quad Q = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ -1 & 0 & 2 \end{bmatrix}$$

column

$$Q = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ -1 & 1 & 2 \end{bmatrix}$$

QUESTION 6. (4 points) Find a matrix A, 3×2 such that

$$\left(A \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}\right)^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

transpose both sides → $A \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$

subtract → $A \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \\ 0 & -2 \end{bmatrix}$

multiply inverse from right → $A \cancel{\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}} = \underset{3\times 2}{\begin{bmatrix} 0 & -1 \\ 2 & 0 \\ 0 & -2 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}}$

first collumn of A

$5 \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + -2 \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \\ 4 \end{bmatrix}$

second collumn of A

$-2 \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + -1 \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$

$$\boxed{A = \begin{bmatrix} 2 & 1 \\ 10 & -4 \\ 4 & 2 \end{bmatrix}}$$