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MTH 532, Exam One

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ALL RINGS ARE COMMUTATIVE with $1 \neq 0$

QUESTION 1. 1) Let I be a prime ideal of R and suppose that $I_1I_2 \subseteq I$ for some proper ideals I_1, I_2 of R. Prove that $I_1 \subseteq I$ or $I_2 \subseteq I$. [Note if $i_1 \in I_1$ and $i_2 \in I_2$, then $i_1i_2 \in I_1I_2$].

Solution: Assume that neither $I_1 \subseteq I$ **nor** $I_2 \subseteq I$**. Then there exist** $a \in I_1 \setminus I$ **and** $b \in I_2 \setminus I$**. Since** $I_1I_2 \subseteq I$**, we conclude that** $ab \in I$**. Since** I **is prime and** $ab \in I$**, we conclude that** $a \in I$ **or** $b \in I$**, a contradiction. Thus** $I_1 \subseteq I$ **or** $I_2 \subseteq I$**.**

2) Let I be a proper ideal of R such that whenever $xy \in I$ for some $x, y \in R$, then $x^n \in I$ or $y^n \in I$ for some integer $n \geq 1$. Prove $\sqrt{I}$ is a prime ideal of R.

Solution: Assume $ab \in \sqrt{I}$**. Hence** $(ab)^m = a^mb^m \in I$ **for some positive integer** $m \geq 1$**. Thus, by hypothesis, there is an integer** n **such that** $(a^m)^n = a^{mn} \in I$ **or** $(b^m)^n = b^{mn} \in I$**. Hence, by the definition of the** $\sqrt{I}$**, we conclude that** $a \in \sqrt{I}$ **or** $b \in \sqrt{I}$**. Thus** $\sqrt{I}$ **is a prime ideal of** R**.**

QUESTION 2. 1) Let e be an idempotent of R. Prove that $1 - e$ is an idempotent of R.

Solution: Since $(1-e)^2 = 1 - 2e + e^2 = 1 - 2e + e = 1 - e$**, we conclude that** $(1 - e)$ **is an idempotent of** R**.**

2) Let e be an idempotent of R such that $e \notin \{0, 1\}$. Prove that $\frac{R}{(1-e)R}$ is ring-isomorphic to eR [note that $e(1 - e) = 0$ and $x - ex = (1 - e)x$; also we conclude that $\frac{R}{eR}$ is ring-isomorphic to $(1 - e)R$, do not prove this, it is clear]

Solution: Let $f : R \to eR$ **such that** $f(a) = ea$ **for every** $a \in R$**. We show** f **is a ring homomorphism**
i)Let $a, b \in R$ **. Then f(a + b) = e(a + b) = ea + eb = f(a) + f(b); also** $f(ab) = e(ab) = e^2(ab) = (ea)(eb) = f(a)f(b)$**. Thus** f **is a ring homomorphism.**
ii) We show f **is onto. Let** $y \in eR$**. Then** $y = ed$ **for some** $d \in R$**. Thus** $f(d) = ed = y$**. Hence** f **is ONTO.**
iii) We show $Ker(f) = (1 - e)R$**. Let** $k \in Ker(f)$**. Then** $f(k) = ek = 0$**. Since** $ek = 0$**, we conclude that** $k = k - ek = (1 - e)k$**. Thus** $k \in (1 - e)R$**. Hence** $Ker(f) \subseteq (1 - e)R$**. Now, let** $w \in (1 - e)T$**. Then** $w = (1 - e)g$ **for some** $g \in R$**. Thus** $f(w) = f((1 - e)g) = e((1 - e)g) = 0$**. Hence** $w \in Ker(f)$**. Thus** $Ker(f) = (1 - e)R$**. By the 1st isomorphism Theorem, we conclude that** $\frac{R}{(1-e)R}$ **is ring-isomorphic to** eR**.**

3) Prove that R is ring-isomorphic to $eR \times (1 - e)R$. [Maybe the concept of co-prime and the CRT are helpful here.]

Solution: Since $e + (1 - e) = 1 \in R$**, we conclude that** eR **and** $(1 - e)R$ **are coprime. It is clear that** $eR \cap (1 - e)R = \{0\}$**. Hence, by the CRT,** $R = R/(eR \cap (1 - e)R) \cong R/eR \times R/(1 - e)R \cong (1 - e)R \times eR$ **(by ii).**

4) Assume that R has exactly one maximal ideal of R. Prove that $0, 1$ are the only idempotent of R.

Solution: Some of you used (iii) to prove it, which is a correct proof. Here is another proof. Let M **be the maximal ideal of** R**. Assume there is an idempotent** e **of** R **such that** $e \notin \{0, 1\}\}$ **. Since every proper ideal of** R **is contained in a maximal ideal of** R**, we conclude that** $eR \subseteq M$ **and** $(1-e)R \subseteq M$**. Thus** $e+1-e = 1 \in M$**, a contradiction. Hence** 0 **and** 1 **are the only idempotents of** R**.**

QUESTION 3. Given $T : Z_5 \to Z_{10}$ be a ring-homomorphism such that $T(1) \neq 0$.

a) Find the range of T and the Kernel of T

Solution: By staring, since $T(1) \neq 0$, we conclude that $Ker(T) \neq Z_5$. Since Z_5 is a field, $\{0\}$ is the only proper ideal of R. Hence $Ker(T) = \{0\}$. Since $Z_5 = Z_5/Ker(T) \cong Range(T)$, we conclude that $Range(T)$ is a subring of Z_{10} with 5 elements. Since $T(1)$ is an idempotent of Z_{10} and $T(1) \neq 0$, we conclude that $T(1) \in \{1, 5, 6\}$

i) Assume $T(1) = 1$. Then $0 = T(0) = T(1+1+\cdots+1)(5\, times) = T(1)+\cdots T(1) = 5 \in Z_{10}$, impossible since $0 \neq 5$ in Z_{10}.

ii) Assume $T(1) = 5$. Then $T(2) = T(1+1) = T(1) + T(1) = 0 \in Z_{10}$, impossible since $Ker(T) = \{0\}$.

iii) Assume $T(1) = 6$. Then Range(T) = $\{6, 2, 8, 4, 0\}$ is a subring of Z_{10} with 5 element. NOTE, since $Z_5 \cong Range(T)$ and Z_5 is a field , we conclude that $Range(T) = \{6, 2, 8, 4, 0\}$ is a field with 6 as the multiplicative identity.

b) I claim that Z_{10} has a a subring, say D, that is a field. If I am wrong, then explain. If I am right, then find D, and find the multiplicative inverse of each nonzero element of D.

Solution: From a(iii), $D = Range(T) = \{6, 2, 8, 4, 0\}$ is is a field with 6 as the multiplicative identity. Now 2^{-1} (in D) = 8 and hence 8^{-1} (in D) = 2; 4^{-1} (in D) = 4; 6^{-1} (in D) = 6

c) Let $T : Z_3 \to Z_9$ be a ring-homomorphism. Find the range of T and the Kernel of T. [May be (4) in Question 2 is helpful]

Solution: Since Z_9 has one maximal ideal $M = \{0, 3, 6\} = 3Z_9$, by question 2(4) we conclude that $0, 1$ are the only idempotent of Z_9.

i) Assume $T(1) = 1$. Then $0 = T(0) = T(1 + 1 + 1) = T(1) + T(1) + T(1) = 3$, impossible since $3 \neq 0$ in Z_9.

ii) Assume $T(1) = 0$. Then $Ker(T) = Z_3$ and $Range(T) = \{0\}$, so $T(1) = 0$ is the correct answer.

QUESTION 4. a) Let $f(x) = 15x^7 + 10x^4 + 12x^2 + 18x + 22 \in Z[x]$. Is $f(x)$ irreducible in Z[x]? explain.

Solution: Since $2 \nmid 15, 2 \mid 10, 2 \mid 12, 2 \mid 18, 2 \mid 22$, but $4 \nmid 22$, we conclude that $f(x)$ is irreducible by Eisenstein's result.

b) Let $R = Z_{24}$ and $D = \{f(x) \in R[x] \mid f(x) \in U(R[x])$ and degree of $f(x) = 2\}$. Find $|D|$(i.e., find the size of D, but do not write down all elements of D)

Solution: From a HW problem, $ax^2 + bx + c$ is a unit in Z_{24} if and only if $a, b \in Nil(Z_{24})$ and $c \in U(Z_{24})$. Since $ax^2 + bx + c$ must be of degree 2, $a \neq 0$. By a HW problem $Nil(Z_{24}) = 6Z_{24} = \{0, 6, 12, 18\}$. Thus $|Nil(Z_{24})| = 4$. We know $|U(Z_{24})| = \phi(24) = (2^3 - 2^2)(3^1 - 3^0) = 8$. Hence a has 3 choices, b has 4 choices, and c has 8 choices. Thus $|D| = 3 \times 4 = 96$.

c) Let $R = Z_{10}$. Is $f(x) = 4x^7 + 6x^2 + 8 \in Z(R[x])$? explain

Solution: Since $5 \in Z(Z_{10})$ and $5(f(x)) = 0$, we conclude that $f(x) \in Z(Z_{10}[x])$.

d) We know if R is an integral domain, then $char(R) = 0$ or $char(R) = p$ for some positive prime integer p. Give me an example of a ring R such that R is not an integral domain, but $char(R) = p$ for some positive prime integer p.

Solution: Let $R = Z_5 \times Z_5$. Since the order of $(1, 1)$ under addition is 5, we conclude that $Char(R) = 5$. Since $(1, 0)(0, 1) = (0, 0)$, R is not an integral domain.

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