Name Alaa Elkouni , ID 84191

Definitely, a beautiful piece of art!

Exam One, MTH 221 , Spring 2022

Ayman Badawi

SCORE = 52 / 52

QUESTION 1. (10 points) Let $T : R^3 \to R^3$ such that $T(x_1, x_2, x_3) = (3x_1,\ 3x_2,\ -x_1 + 4x_2 + 3x_3)$

3 0 0   0 3 0   −1 4 3

(i) **(4 points)** Find all eigenvalues of T.

$$A = \begin{array}{ccc} x_1 & x_2 & x_3 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \\ -1 & 4 & 3 \end{array}$$

$C_A(\alpha) = |\alpha I_3 - A|$

$$\begin{matrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{matrix} - \begin{matrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ -1 & 4 & 3 \end{matrix} = \begin{vmatrix} \alpha-3 & 0 & 0 \\ 0 & \alpha-3 & 0 \\ 1 & -4 & \alpha-3 \end{vmatrix} = (\alpha-3)(\alpha-3)(\alpha-3) = 0$$

lower triangle

$\therefore \alpha = 3$

4

(ii) **(4 points)** For each eigenvalue a of T find the corresponding eigenspace E_a and write E_a as span..

$$E_3 = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -4 & 0 & 0 \end{array}\right]$$

READ!

$x_1 = 4x_2$

$0 = 0$

solution $E_3 = \{(4x_2, x_2, x_3) \mid x_2, x_3 \in R\}$

$E_3 = \{x_2(4,1,0),\ x_3(0,0,1)\}$

$E_3 = \text{span}\{(4,1,0), (0,0,1)\}$

4

(iii) **(2 points)** Find all points in the domain (R^3) such that $T(x_1, x_2, x_3) = -3(x_1, x_2, x_3)$

because −3 is not an eigen value of T, there does not exist a non-zero point where $T(x_1, x_2, x_3) = -3(x_1, x_2, x_3)$.

$\therefore$ solution set $= \{(0,0,0)\}$

2

QUESTION 2. (16 points) Let $T: R^4 \to R^3$ such that

1 -1 2 2 -1 1 -2 -1 -3 3 -6 -5

$$T(x_1, x_2, x_3, x_4) = (x_1 - x_2 + 2x_3 + 2x_4,\ -x_1 + x_2 - 2x_3 - x_4,\ -3x_1 + 3x_2 - 6x_3 - 5x_4)$$

(i) **(2 points)** Find the standard matrix presentation of T, say M.

$$M = \begin{bmatrix} 1 & -1 & 2 & 2 \\ -1 & 1 & -2 & -1 \\ -3 & 3 & -6 & -5 \end{bmatrix}$$

(ii) **(6 points)** Find all points in the domain of T (i.e., points in R^4) such that $T(x_1, x_2, x_3, x_4) = (2, 2, -2)$

Form Augmented Matrix:

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 2 & 2 \\ -1 & 1 & -2 & -1 & 2 \\ -3 & 3 & -6 & -5 & -2 \end{array}\right]$$

$R_1 + R_2 \to R_2 \qquad 3R_1 + R_3 \to R_3$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

$-2R_2 + R_1 \to R_1 \qquad -R_2 + R_3 \to R_3$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 0 & -6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

READ!

$x_1 = x_2 - 2x_3 - 6$

$x_4 = 4$

$0 = 0$

x_1, x_4 are leading variables

x_2, x_3 are free variables

solution set $\{(x_2 - 2x_3 - 6, x_2, x_3, 4) \mid x_2, x_3 \in R\}$

6

(iii) **(4 points)** Find $Z(T) = Ker(T) = Null(T)$ (i.e., find all points in the domain of T such that $T(x_1, x_2, x_3, x_4) = (0,0,0)$).[be careful, in view of (ii), it should be clear] and write $Z(T)$ as span of independent points.

using (ii)

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

READ!

$x_1 = x_2 - 2x_3$

$x_4 = 0$

$0 = 0$

$Z(T) = \{(x_2 - 2x_3, x_2, x_3, 0) \mid x_2, x_3 \in R\}$

$= \{x_2(1,1,0,0), x_3(-2,0,1,0)\} = \text{span}\{(1,1,0,0), (-2,0,1,0)\}$

(iv) **(4 points)** Find the $rank(M)$, the $dim(Range(T))$, and write $Range(T)$ as span of independent points. [Maybe the calculations that you did in (ii) are helpful!]

Rank (M) = dim (Range(T)) = # of independent rows = 2

Range (T) = columnspace (T) = span$\{(1,-1,-3), (2,-1,-5)\}$

1 2 1 1 -2 -4 -2 -1 -1 -2 -1 0 0000

QUESTION 3. (6 points) Given $D = \{(x_1 + 2x_2 + x_3 + x_4,\ -2x_1 - 4x_2 - 2x_3 - x_4,\ -x_1 - 2x_2 - x_3,\ 0) \mid x_1, x_2, x_3, x_4 \in R\}$ is a subspace of R^4. Find a basis of D and write D as a span of independent points.

$D = \{ x_1(1,-2,-1,0),\ x_2(2,-4,-2,0),\ x_3(1,-2,-1,0),\ x_4(1,-1,0,0) \}$

$D = \text{span}\{(1,-2,-1,0), (2,-4,-2,0), (1,-2,-1,0), (1,-1,0,0)\}$

(a) basis of D = span $\{(1,-2,-1,0), (2,-4,-2,0), (1,-2,-1,0), (1,-1,0,0)\}$

(b)

$$D = \begin{matrix} 1 & -2 & -1 & 0 \\ 2 & -4 & -2 & 0 \\ 1 & -2 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{matrix} \rightarrow \begin{matrix} 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{matrix}$$

independent points

$\therefore D = \text{span}\{(1,-2,-1,0), (0,1,1,0)\}$

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$-2R_1 + R_2 \rightarrow R_2$ $\quad -R_1 + R_3 \rightarrow R_3$ $\quad -R_1 + R_4 \rightarrow R_4$

QUESTION 4. (6 points) Consider the following system of Linear Equations.

$$x_1 + 5x_2 - 8x_3 = 0$$
$$-2x_1 + ax_2 + 7x_3 = c$$
$$-3x_1 - 15x_2 + bx_3 = d$$

For what values of a, b, c, d will the system have unique solution?

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & c \\ 1 & 5 & -8 & 0 \\ -2 & a & 7 & c \\ -3 & -15 & b & d \end{array}$$

unique solution $\rightarrow$ $|A| \neq 0$; no free variables

$|A| = (1)(10+a)(b-24)$

$|A| = (10+a)(b-24)$

$a \neq -10 \quad b \neq 24 \quad c \in R \quad d \in R$

$2R_1 + R_2 \rightarrow R_2$ $\quad 3R_1 + R_3 \rightarrow R_3$

$$\begin{array}{ccc|c} 1 & 5 & -8 & 0 \\ 0 & 10+a & -9 & c \\ 0 & 0 & b-24 & d \end{array}$$

upper triangle

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$x_1 = -5x_2 + 8x_3$

$x_2 = \dfrac{9x_3 + c}{(10+a)}$

$x_3 = \dfrac{d}{(b-24)}$

QUESTION 5. (8 points) Let A be a 4×4 matrix. Given

$$A \xrightarrow{-3R_2} B \xrightarrow{R_1 \leftrightarrow R_4} C \xrightarrow{-4R_1+R_3 \to R_3} D = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix} \xrightarrow{2R_1 + R_3 \to R_3} E = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 4 & 4 & 6 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$$E \xrightarrow{R_2 \leftrightarrow R_4} F = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 4 & 4 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-4R_2 + R_3 \to R_3} G = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$|G| = (1)(1)(-4)(1) = -4$

$|G| = |F|$

$|E| = -|F|$

$|D| = |E| = -|F| = -|G|$

$|C| = |D| = -|G| = -(-4) = 4$

Find

(a) **(4 points)** $|A|$

$|B| = -|C| = |G|$

$|B| = |G|$

$|A| = -\frac{1}{3}|B| = -\frac{1}{3}|G| = -\frac{1}{3} \times -4 = \frac{4}{3}$ ✓

9

(b) **(4 points)** $|2AC^T| = 2^4 |A||C| = 16 \times \frac{4}{3} \times 4 = \frac{256}{3}$ ✓

$|C^T| = |C|$

QUESTION 6. (6 points) Let $T : R^2 \to R^2$ be a linear transformation such that $T(1,0) = (2,1)$ and $T(4,1) = (12,6)$

(i) **(4 points)** Find the standard matrix presentation of T [Note $(4,1) = (4,0) + (0,1)$]

$T(4,1) = T(4,0) + T(0,1)$

$(12,6) = 4(2,1) + T(0,1)$

$T(0,1) = (12,6) - (8,4)$

$T(0,1) = (4,2) \longleftarrow$ 2nd column

$T(1,0) = (2,1) \longleftarrow$ 1st column

$T = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$

6

(ii) **(2 points)** Find $T(-5,7)$

$T(-5,7) = -5\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 7\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -10 \\ -5 \end{bmatrix} + \begin{bmatrix} 28 \\ 14 \end{bmatrix} = \begin{bmatrix} 18 \\ 9 \end{bmatrix}$

Faculty information

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates.
E-mail: abadawi@aus.edu, www.ayman-badawi.com