TEST NUMBER ONE FOR MATH 221, FALL 2004

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QUESTION 1. *1)* (**10 points) Find the LU-factorization of** $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ -2 & 2 & 7 \end{bmatrix}$.

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ -2 & 2 & 7 \end{bmatrix}}_{A} \xrightarrow{-3R_1+R_2\to R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ -2 & 2 & 7 \end{bmatrix} \xrightarrow{2R_1+R_3\to R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 4 & 9 \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_3\to R_3} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}}_{U}$$

$$E_3E_2E_1A = U$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-3R_1+R_2\to R_2} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1}$$

$$I_3 \xrightarrow{2R_1+R_3\to R_3} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{E_2}$$

$$I_3 \xrightarrow{-2R_2+R_3\to R_3} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{E_3}$$

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$\Longrightarrow$

2) Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -4 & -5 & -7 \\ -1 & -2 & -4 & -5 \end{bmatrix}$

a)(10 points) Solve $AX = \begin{bmatrix} 4 \\ -6 \\ -6 \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 4 \\ -2 & -4 & -5 & -7 & -6 \\ -1 & -2 & -4 & -5 & -6 \end{array}\right] \xrightarrow[1R_1+R_3 \to R_3]{2R_1+R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 4 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & -1 & -2 \end{array}\right] \xrightarrow[1R_2+R_3 \to R_3]{-3R_2+R_1 \to R_1}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + 2x_2 + x_4 = -2$

$x_3 + x_4 = 2$

$0 = 0$

Leading variables :- x_1, x_3

Free variables :- x_2, x_4

Solution

$x_1 = -2 - 2x_2 - x_4$

$x_3 = 2 - x_4$

x_2, x_4 any real #'s.

b) (5 points) Solve $AX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ -2 & -4 & -5 & -7 & 0 \\ -1 & -2 & -4 & -5 & 0 \end{array}\right] \xrightarrow[1R_1+R_2 \to R_3]{2R_1+R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{array}\right] \xrightarrow[1R_2+R_3 \to R_3]{-3R_2+R_1 \to R_1}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Free variables.

$x_1 + 2x_2 + x_4 = 0$

leading $x_3 + x_4 = 0$

$0 = 0$

Solution

$x_1 = -2x_2 - x_4$

$x_3 = -x_4$, x_2, x_4 are any real #!

QUESTION 2. *a)* **(5 points) Find** x **so that the matrix** $\begin{bmatrix} 2 & x \\ x+3 & 2 \end{bmatrix}$ **is singular.**

matrix is singular

$$\begin{vmatrix} 2 & x \\ x+3 & 2 \end{vmatrix} = 0$$

$4 - x(x+3) = 0$

$(4 - x^2 - 3x = 0) \cdot -1$

$x^2 + 3x - 4 = 0$

$(x+4)(x-1) = 0$

$x = -4$

$x = 1$

b) (5 points) Find c **so that the matrix** $\begin{bmatrix} 1 & -1 & 2 \\ -2 & 3 & 6 \\ 0 & c & -2 \end{bmatrix}$ **is nonsingular.**

matrix is nonsingular

$$\begin{vmatrix} 1 & -1 & 2 \\ -2 & 3 & 6 \\ 0 & c & -2 \end{vmatrix} \neq 0$$

$\cancel{a_{31}}^{0} A_{31} + a_{32} A_{32} + a_{33} A_{33} \neq 0$

$c \cdot (-1)^5 \det\begin{bmatrix} 1 & 2 \\ -2 & 6 \end{bmatrix} + -2 \cdot (-1)^6 \det\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \neq 0$

$-c \ (1 \times 6 - 2 \times -2) - 2 \ (1 \times 3 - (-1 \times -2)) \neq$

$-10c - 2 \neq 0$

$\frac{-10c}{-10} = \frac{2}{-10}$

$c \neq -\frac{1}{5}$

$c \neq -\frac{1}{5}$

So

c) The following is an augmented matrix of a system of linear equations: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ -1 & 0 & 1 & a \\ 0 & 2 & b & 6 \end{array}\right]$

(1) (5 points) Find a, b so that the system has unique solution.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ -1 & 0 & 1 & a \\ 0 & 2 & b & 6 \end{array}\right] \xrightarrow{1R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3+a \\ 0 & 2 & b & 6 \end{array}\right] \xrightarrow[-2R_2+R_3\to R_3]{-1R_2+R_1\to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -a \\ 0 & 1 & 2 & 3+a \\ 0 & 0 & b-4 & -2a \end{array}\right]$$

System has a unique solution:-

$b-4 \neq 0 \Rightarrow b \neq 4$.

$-2a$ is any real #

a is any real #

(2) (5 points) Find a, b so that the system has infinitely many solutions.

System has infinitly many solutions:-

$b-4 = 0 \Rightarrow b=4$

$-2a = 0 \Rightarrow a=0$

(3) (5 points) Find a, b so that the system has no solution.

System has no solution.

$b-4 = 0 \Rightarrow b=4$.

$-2a \neq 0 \Rightarrow a \neq 0$.

QUESTION 3. *a)* (**10 points**) **Let** $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ 2 & 5 & 5 \end{bmatrix}$ **Find** A^{-1} **and** $(A^T)^{-1}$.

$$\left[\begin{array}{ccc|ccc} \multicolumn{3}{c|}{\overbrace{\hphantom{1\;0\;0}}^{A}} & & & \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 & 1 & 0 \\ 2 & 5 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_1+R_3 \to R_3]{-3R_1+R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & -3 & 1 & 0 \\ 0 & 5 & 5 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{3}{4} & \frac{1}{4} & 0 \\ 0 & 5 & 5 & -2 & 0 & 1 \end{array}\right] \xrightarrow[-\frac{5}{4}]{-5R_2+R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{3}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 5 & \frac{7}{4} & -\frac{5}{4} & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{5}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{3}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{7}{20} & -\frac{5}{4} & \frac{1}{5} \\ \multicolumn{3}{c|}{\underbrace{\hphantom{0\;0\;1}}_{I_3}} & \multicolumn{3}{c}{\underbrace{\hphantom{0\;0\;1}}_{A^{-1}}} \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & \frac{1}{4} & 0 \\ \frac{7}{20} & -\frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} \multicolumn{3}{c|}{\overbrace{\hphantom{1\;3\;2}}^{AT}} & & & \\ 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 4 & 5 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{4}R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-3R_2+R_1\to R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{7}{4} & 1 & -\frac{3}{4} & 0 \\ 0 & 1 & \frac{5}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{5}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{7}{4} & 1 & -\frac{3}{4} & 0 \\ 0 & 1 & \frac{5}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{5} \end{array}\right] \xrightarrow[\frac{7}{4}R_3+R_1\to R_1]{-\frac{1}{4}R_3+R_2\to R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{3}{4} & \frac{7}{20} \\ 0 & 1 & 0 & 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{5} \\ \multicolumn{3}{c|}{\underbrace{\hphantom{0\;0\;1}}_{I_3}} & & & \end{array}\right] \qquad (A^T)^{-1} = \begin{bmatrix} 1 & -\frac{3}{4} & \frac{7}{20} \\ 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & \frac{1}{5} \end{bmatrix} = (A^{-1})^T$$

b) (5 points) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix}$ **Find the (2, 3)-entry of** A^{-1}.

the (2,3) entry of $A^{-1} = \dfrac{A_{32}}{\det(A)} = \dfrac{(-1)^5 \det\begin{bmatrix} 1 & 3 \\ 0 & 6 \end{bmatrix}}{1 \cdot 5 \cdot 7}$

$= \dfrac{-6}{35}$

c) (5 points) Given
$3x_1 - 7x_2 = 12$
$8x_1 + 2x_2 = -4$
Solve for x_2 **using Cramer's Rule.**

$A = \begin{bmatrix} 3 & -7 \\ 8 & 2 \end{bmatrix}$ $\quad \det(A) = 3 \times 2 - (-7 \times 8) = 62$

$x_2 = \dfrac{\det\begin{bmatrix} 3 & 12 \\ 8 & -4 \end{bmatrix}}{\det(A)} = \dfrac{3 \times -4 - (12 \times 8)}{62} = \dfrac{-108}{62}$

QUESTION 4. *1)*(**7 points) Given** $A \underset{A_1}{\overset{R_1 \leftrightarrow R_3}{\sim}} \underset{A_2}{\overset{3R_3}{\sim}} \overset{2R_1 + R_2}{\sim}$

$A_4 \leftarrow \begin{bmatrix} 0 & 3 & -2 \\ 4 & 6 & 7 \\ 0 & -6 & 6 \end{bmatrix}$. **Find** $det(A)$.

$\det(A_1) = -\det(A)$ $\quad \det(A_2) = 3\det($ $= -3\det$

$\underbrace{\begin{bmatrix} 0 & 3 & -2 \\ 4 & 6 & 7 \\ 0 & -6 & 6 \end{bmatrix}}_{A_4} \overset{R_2 \leftrightarrow R_1}{\sim} \overset{A_5}{\begin{bmatrix} 4 & 6 & 7 \\ 0 & 3 & -2 \\ 0 & -6 & 6 \end{bmatrix}} \overset{2R_2 + R_3 \to R_3}{\sim} \underbrace{\begin{bmatrix} 4 & 6 & 7 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{bmatrix}}_{A_6}$

$\det(A_6) = \det(A_5)$
$24 = 3\det($
$\boxed{\det(A) = 8}$

$\det(A_5) = -\det(A_4) = 3\det(A)$

$\det(A_4) = \det(A_2)$
$= -3\det(A)$

2)(9 points) Given A, B **are** 4×4 **matrices and** $det(A) = -2$ $det(B) = 3$.
Find

a) $det(-2A)$

$= (-2)^4 \det(A)$
$= 16 \cdot -2$
$= -32$

b) $det(A^{-1}B^T)$

$= \det(A^{-1}) \det(B^T)$
$= \dfrac{1}{\det(A)} \cdot \det(B)$
$= \dfrac{-1}{2} \cdot 3 = -\dfrac{3}{2}$

* c) $det(A^{-1} + 2adj(A))$. $= \det(A^{-1} + 2(A^{-1}\det(A) I_4))$

$= \det(A^{-1}(1 + 2\det(A) I_4)) = \det(A^{-1}) \cdot \det([1 + 2\det(A) I_4) = 4$

$\dfrac{1}{\det A} \cdot (\det A)^4$

$= \dfrac{1}{-2}(-4)$

$= \dfrac{1}{2}$

QUESTION 5. *(*8 **points) a) Let** A **be an** $n \times n$ **nonsingular matrix. Prove that** $det(adj(A)) = det(A)^{n-1}$.

$A\,\mathrm{adj}(A) = \det(A) I_n$

$\det(A\,\mathrm{adj}(A)) = \det[\det(A) I_n]$

$\det(A) \det(\mathrm{adj}(A)) = [\det(A)]^n$

$\det(\mathrm{adj}(A)) = \dfrac{[\det(A)]^n}{\det(A)}$

$\det(\mathrm{adj}(A)) = [\det(A)]^{n-1}$

A is nonsingular $\det(A) \neq 0$

b) (6 points) Write down TRUE or FALSE. If False, then give a counter example:

(1) If A, B are $n \times n$ matrices, then $det(A+B) = det(A) + det(B)$ F

$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}$, $A+B = \begin{bmatrix} 7 & 2 \\ 2 & 5 \end{bmatrix}$

$det(A) = 6$ $det(B) = 10$ $[det(A) + det(B)] = 16$ $det(A+B) = 31$

$det(A) + det(B) = 16 \neq det(A+B)$

(2) If A is a 2×2 matrix, then $det(3A) = 3det(A)$ F

Ex:- $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. $3det(A) = 3 \cdot 2 = 6$ $3A = \begin{bmatrix} 3 & 3 \\ 3 & 6 \end{bmatrix}$

$det(A) = 2$ $det(3A) = 9 \neq 3det(A)$.

(3) If A is a 3×3 matrix and singular, then $AX = 0$ has infinitely many solutions. T.

x6

QUESTION 6. (Bonus = 6 points) Let A be a 9×3 matrix, C_1 be the first column of A, C_2 be the second column of A, and C_3 be the third column of A. Suppose that $C_1 = C_3$ and $B = C_1 + C_2 + C_3$. Prove that $AX = B$ has infinitely many solutions.

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Excellent!!

$\underset{9\times3}{A}\underset{3\times1}{X} = \underset{9\times1}{B} \Rightarrow X_1 C_1 + X_2 C_2 + X_3 C_3 = B$

Solve:-

$X_1 C_1 + X_2 C_2 + X_3 C_3 = X_1 C_1 + X_2 C_2 + X_3 C_1$

$= (X_1 + X_3) C_1 + X_2 C_2 = B = C_1 + C_2 + C_3$ $(C_1 = C_3)$

$(X_1 + X_3) C_1 + X_2 C_2 = 2C_1 + C_2$

$X_1 + X_3 = 2$ $\Rightarrow$ $X_1 = 2 - X_3$.

$X_2 = 1$ $\Rightarrow$ $X_2 = 1$

X_3 is any real #

leading variables :- X_1, X_2.

Free variables :- X_3

has infinitely many solution