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TEST # TWO FOR MTH221, SPRING 005

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QUESTION 1. Write down True or False, each = 2 points, total = 16 points

(1) *If A is row equivalent to B, then rank A = rank B.* True

(2) *If A is a 2×2 matrix and $b = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, then the set of all solutions to the linear system $AX = b$ is a subspace of R^2.* False

(3) *Let V be an n-dimensional vector space. If a set of m vectors spans V, then $m = n$.* False

(4) *It is possible to have a matrix A, 4×6 such that the nullity of A is 1.* False

(5) *The interval $(-\infty \ 9)$ is a subspace of R.* False

(6) *If A is a nonzero 6×2 matrix and $AX = 0$ has infinitely many solutions, then $Rank(A) = 1$.* True

(7) *If A is a 3×6 matrix and $Nullity(A) = 3$, then for every b, 3×1, $AX = b$ has infinitely many solutions.* True

(8) $L : R^{2\times2} \longrightarrow R^{2\times2}$ *such that* $L(A) = A + A^T$ *is a linear transformation.* True

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$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

2x2 2x1

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

3x6 6x1

$4 + x =$

~~2+x~~

$L(A+B) = (A+B) + (A+B)^T = A + A^T + B + B^T = L(A) + L(B)$

$L(\alpha A) = \alpha A + (\alpha A)^T = \alpha A + \alpha A^T = \alpha (A + A^T) = \alpha L(A)$

QUESTION 2. *Let* $A = \begin{bmatrix} 1 & -2 & 1 & 1 & 3 \\ 2 & -4 & 1 & 0 & 4 \\ -3 & 6 & 2 & 1 & 7 \end{bmatrix}$

(1) *Find Rank(A) (**6 points**)*
(2) *Find bases for its row space and column space. (**12 points**)*

$$\begin{bmatrix} 1 & -2 & 1 & 1 & 3 \\ 2 & -4 & 1 & 0 & 4 \\ -3 & 6 & 2 & 1 & 7 \end{bmatrix}$$

$-2R_1 + R_2 \to R_2$
$3R_1 + R_3 \to R_3$

$$\begin{bmatrix} 1 & -2 & 1 & 1 & 3 \\ 0 & 0 & -1 & -2 & -2 \\ 0 & 0 & 5 & 4 & 16 \end{bmatrix}$$

$5R_2 + R_3 \to R_3$

$$\begin{bmatrix} 1 & -2 & 1 & 1 & 3 \\ 0 & 0 & -1 & -2 & -2 \\ 0 & 0 & 0 & -6 & 6 \end{bmatrix}$$

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1.) Rank(A) = $\underline{\underline{3}}$

2.) Bases for row space = $\{[1\ -2\ 1\ 1\ 3],\ [2\ -4\ 1\ 0\ 4],\ [-3\ 6\ 2\ 1\ 7]\}$

Bases for column space = $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

QUESTION 3. *Let* $L : R^2 \longrightarrow R^3$ *be a linear transformation such that* $L(-2,1) = (-1,0,3)$ *and* $L(0,3) = (4,1,-1)$. *Find* $L(-4,11)$. *(**10 points**)*

$(-4, 11) = \alpha_1(-2,1) + \alpha_2(0,3)$

$-2\alpha_1 + 0\alpha_2 = -4$ $\quad \boxed{\therefore \alpha_1 = 2}$

$\alpha_1 + 3\alpha_2 = 11$ $\quad \boxed{\therefore \alpha_2 = 3}$

$(-4, 11) = 2(-2,1) + 3(0,3)$

$L(-4,11) = L[2(-2,1)] + L[3(0,3)] = 2L(-2,1) + 3L(0,3)$

$= 2(-1,0,3) + 3(4,1,-1) = \underline{\underline{(10, 3, 3)}}$

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QUESTION 4. *Let $L : R^3 \to R^3$ be defined by*

$$L\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 + x_3 \\ -x_1 + 2x_2 \\ -2x_1 + 3x_2 - x_3 \end{bmatrix}$$

(1) *Find the standard matrix representation of L.*(**6 points**)
(2) *Find Ker(L), then find a basis for ker L.*(**8 points**)
(3) *Find Range(L), then find a basis for range L.*(**8 points**)

$V = (V_1, V_2, V_3)$

$U = (U_1, U_2, U_3)$

$$L(V+U) = L\begin{bmatrix} V_1+U_1 \\ V_2+U_2 \\ V_3+U_3 \end{bmatrix}$$

$$= \begin{bmatrix} V_1+U_1 - V_2 - U_2 + V_3 + U_3 \\ -V_1 - U_1 + 2V_2 + 2U_2 \\ -2V_1 - 2U_1 + 3V_2 + 3U_2 - V_3 - U_3 \end{bmatrix}$$

$$= \begin{bmatrix} V_1 - V_2 + V_3 \\ -V_1 + 2V_2 \\ -2V_1 + 3V_2 - V_3 \end{bmatrix} + \begin{bmatrix} U_1 - U_2 + U_3 \\ -U_1 + 2U_2 \\ -2U_1 + 3U_2 - U_3 \end{bmatrix}$$

$$= L(V) + L(U)$$

$$L(\alpha U) = L\begin{bmatrix} \alpha U_1 \\ \alpha U_2 \\ \alpha U_3 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha U_1 - \alpha U_2 + \alpha U_3 \\ -\alpha U_1 + 2\alpha U_2 \\ -2\alpha U_1 + 3\alpha U_2 - \alpha U_3 \end{bmatrix}$$

$$= \alpha \begin{bmatrix} U_1 - U_2 + U_3 \\ -U_1 + 2U_2 \\ -2U_1 + 3U_2 - U_3 \end{bmatrix}$$

$$= \alpha L(T)$$

$\therefore L : \mathbb{R}^3 \to R^3$ is a linear transformation.

It is already stated that it is lin transfo

① standard matrix for $\mathbb{R}^3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$L\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \qquad L\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \qquad L\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

standard matrix ~~representation of~~ $= \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 0 \\ -2 & 3 & -1 \end{bmatrix}$

$\therefore$ standard matrix representation of $L =$

$$\underbrace{\begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 0 \\ -2 & 3 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 + x_3 \\ -x_1 + 2x_2 \\ -2x_1 + 3x_2 - x_3 \end{bmatrix}$$

② $AX = 0$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 0 \\ -2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 0 \\ -2 & 3 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 + R_2 \to R_2$ | $-R_2 + R_3 \to R_3$

$2R_1 + R_3 \to R_3$ | $R_2 + R_1 \to R_1$

$\boxed{x_2 = -x_3}$

$\boxed{x_1 = -2x_3}$

$$Null(A) = \left\{ \begin{bmatrix} -2x_3 \\ -x_3 \\ x_3 \end{bmatrix} \middle| x_3 \in \mathbb{R} \right\}$$

$$\therefore Ker(L) = \left\{ \begin{bmatrix} -2x_3 \\ -x_3 \\ x_3 \end{bmatrix} \middle| x_3 \in \mathbb{R} \right\}$$

Basis for $Ker(T) = \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$

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(3) $Range(L) = x_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$= Span \left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

Basis for $range(L) = \left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \right\}$

QUESTION 5. (1) *Does the set of vectors $\{x-1, x^2+2x+1, x^2+x-2\}$ form a basis for P_3? Explain (**8 points**).*

$P_3 \approx \mathbb{R}^3$.

$\{\overset{v_1}{(x-1)}, \overset{v_2}{(x^2+2x+1)}, \overset{v_3}{(x^2+x-2)}\} \approx \{\overset{v_1}{(-1,1,0)}, \overset{v_2}{(1,2,1)}, \overset{v_3}{(-2,1,1)}\}$

First, v_1, v_2, v_3 have to be linearly independant. $\boxed{\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0}$

$$\left[\begin{array}{ccc|c} -1 & 1 & -2 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad R_1 + R_2 \to R_2$$

$$\left[\begin{array}{ccc|c} -1 & 1 & -2 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad \begin{array}{c} -3R_2 + R_3 \to R_3 \\ -R_3 + R_1 \to R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} -1 & 0 & -3 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

$4\alpha_3 = 0$
$3\alpha_2 = 0$
$-1\alpha_1 = 0$

$\boxed{\begin{array}{l} \therefore \alpha_3 = 0 \\ \therefore \alpha_2 = 0 \\ \therefore \alpha_1 = 0 \end{array}}$

Since ~~they~~ v_1, v_2, v_3 are linearly independant & they span P_3, the set of vectors $\{(x-1), (x^2+2x+1), (x^2+x-2)\}$ form a basis for P_3.

(2) *Is the span$\{(-2,1,2),(2,1,-1),(2,3,0)\} = R^3$. Explain (**8 points**)*

$\alpha_1(-2,1,2) + \alpha_2(2,1,-1) + \alpha_3(2,3,0) = (a,b,c)$

$$\underbrace{\begin{bmatrix} -2 & 2 & 2 \\ 1 & 1 & 3 \\ 2 & -1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\det(A) = -2 \det\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} - 1 \det\begin{vmatrix} 2 & 2 \\ -1 & 0 \end{vmatrix} + 2 \det\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = -2(3) - 1(2) + 2(4) = \underline{0}$$

Since $\det(A) = 0$, the elements $\{(-2,1,2), (2,1,-1), (2,3,0)\}$ donot span $\mathbb{R}^3$.

(3) *Given that* $S = \{A \in R^{2\times 2} \mid a_{11} + a_{21} = 0\}$ *is a subspace of* $R^{2\times 2}$. *Find a basis for* S.(**8 points**)

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $a_{11} = -a_{21} + 0a_{12} + 0a_{22}$

Free variables = 3 $\boxed{\therefore \text{Dim}(S) = 3}$

Let $a_{12} = 1$, $a_{21} = a_{22} = 0$,

$V_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Let $a_{22} = 1$, $a_{21} = a_{12} = 0$

$V_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Let $a_{21} = 1$, $a_{12} = a_{22} = 0$

$\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$

Basis for S = $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

(4) *Show that* $S = \{f(x) \in P_3 \mid \int_{-1}^{0} f(x)dx = 0\}$ *is a subspace of* P_3. *Find a basis for* S.(**10 points**)

$f(x) = a_0 + a_1 x + a_2 x^2$

$\int_{-1}^{0} f(x) \cdot dx = \int_{-1}^{0} a_0 + a_1 x + a_2 x^2 \cdot dx = 0$

$\left[a_0 x + \frac{1}{2} a_1 x^2 + \frac{1}{3} a_2 x^3\right]_{-1}^{0} = 0$

$0 - \left[-1a_0 + \frac{1}{2} a_1 - \frac{1}{3} a_2\right] = 0$

$\therefore a_0 - \frac{1}{2} a_1 + \frac{1}{3} a_3 = 0$

$\boxed{\therefore a_0 = \frac{1}{2} a_1 - \frac{1}{3} a_2}$

Free variable = 2 $\therefore$ ~~Basis~~ Dim(S) = 2

Let $a_1 = 1$, $a_2 = 0$

$\boxed{V_1 = \frac{1}{2} + x}$

Let $a_2 = 1$, $a_1 = 0$

$\boxed{V_2 = \frac{-1}{3} + x^2}$

$\therefore$ Basis for $S = \left\{ \left(\frac{1}{2} + x\right), \left(\frac{-1}{3} + x^2\right) \right\}$

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QUESTION 6. (**BONUS = 6 points**) *Let* $T : W \longrightarrow V$ *be a linear transformation and* $T(w_1) = v_1$ *for some* $w_1 \in W$ *and* $v_1 \in V$. *Set* $S = \{w \in W \mid T(w) = T(w_1) = v_1\}$. *Prove that* $S = w_1 + Ker(T) = \{w_1 + a \mid a \in Ker(T)\}$.

Set $(S) = \{ \omega \in W \mid T(\omega) = T(\omega_1) = V_1 \}$

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If $T(\omega) = T(\omega_1)$, then $\boxed{\omega = \omega_1}$ & $\boxed{T(\omega) = T(\omega_1) = V_1}$

$\therefore$ ω is a constant

~~Ker(T)~~ Ker$(T) = 0$

since 'ω' is a constant & $AX = 0$!!

$\therefore S = \omega_1 + \text{Ker}(T)$

~~a ∈ Ker(T)~~ Let ~~a~~ a = Ker(T)

$\therefore S = \omega_1 + \text{Ker}(T) = \{ \omega_1 + a \mid$ ~~Ker T~~ $a \in \text{Ker}(T) \}$