PROBLEM SET 1

HAMID SAGBAN

Exercise 1. *Let S be a set with binary operation $*$, such that*

(1) $(S, *)$ *is a semigroup.*

(2) *There is an $e \in S$ such that $x * e = x$ for every $x \in S$*

(3) *For every $x \in S$, there is a $y \in S$ such that $x * y = e$*

Prove that $(S, *)$ is a group.

Proof. We can rewrite (3) as for every $x \in S$, there is an element $x^{-1} \in S$ such that $x * x^{-1} = e$. Since $x^{-1} \in S$, then by (3), there is an element $(x^{-1})^{-1} \in S$ such that $x^{-1} * (x^{-1})^{-1} = e$. We then have,

$$x^{-1} * x = x^{-1} * x * e = x^{-1} * x * (x^{-1} * (x^{-1})^{-1}) =$$
$$x^{-1} * (x * x^{-1}) * (x^{-1})^{-1} = x^{-1} * e * (x^{-1})^{-1} = x^{-1} * (x^{-1})^{-1} = e.$$

Also, $e * x = (x * x^{-1}) * x = x * (x^{-1} * x) = x * e = x$ by the above. Therefore, $(S, *)$ is a group. □

Exercise 2. *Let $X = \{1, 2, 3\}$. Find all invertible elements of X^X.*

We have $3! = 6$ such functions. Let $f_n : X \longrightarrow X$, for $n \in \{1, 2, \ldots, 6\}$. Then,

$$f_1(1) = 1, f_1(2) = 2, f_1(3) = 3$$
$$f_2(1) = 1, f_2(2) = 3, f_2(3) = 2$$
$$f_3(1) = 2, f_3(2) = 1, f_3(3) = 3$$
$$f_4(1) = 2, f_4(2) = 3, f_4(3) = 1$$
$$f_5(1) = 3, f_5(2) = 1, f_5(3) = 2$$
$$f_6(1) = 3, f_6(2) = 2, f_6(3) = 1$$