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WEAKLY (m,n)-CLOSED IDEALS AND (m,n)-VON NEUMANN REGULAR RINGS

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Abstract. Let R be a commutative ring with $1 \neq 0$, I a proper ideal of R, and m and n positive integers. In this paper, we define I to be a *weakly (m,n)-closed ideal* if $0 \neq x^m \in I$ for $x \in R$ implies $x^n \in I$, and R to be an *(m,n)-von Neumann regular ring* if for every $x \in R$, there is an $r \in R$ such that $x^m r = x^n$. A number of results concerning weakly (m,n)-closed ideals and (m,n)-von Neumann regular rings are given.

1. Introduction

Let R be a commutative ring with $1 \neq 0$, I a proper ideal of R, and n a positive integer. As in [2], I is an *n-absorbing* (resp., *strongly n-absorbing*) *ideal* of R if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$ (resp., $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of R), then there are n of the x_i's (resp., n of the I_i's) whose product is in I. As in [4], I is a *semi-n-absorbing ideal* of R if $x^{n+1} \in I$ for $x \in R$ implies $x^n \in I$; and for positive integers m and n, I is an *(m,n)-closed ideal* of R if $x^m \in I$ for $x \in R$ implies $x^n \in I$. And, as in [15], I is a *weakly n-absorbing* (resp., *strongly weakly n-absorbing*) *ideal* of R if whenever $0 \neq x_1 \cdots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$ (resp., $0 \neq I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of R), then there are n of the x_i's (resp., n of the I_i's) whose product is in I.

In this paper, we define I to be a *weakly semi-n-absorbing ideal* of R if $0 \neq x^{n+1} \in I$ for $x \in R$ implies $x^n \in I$. More generally, for positive integers m and n, we define I to be a *weakly (m,n)-closed ideal* of R if $0 \neq x^m \in I$ for $x \in R$ implies $x^n \in I$. Thus I is a weakly semi-n-absorbing ideal if and only if I is a weakly $(n+1,n)$-closed ideal. Moreover, an (m,n)-closed ideal is a weakly (m,n)-closed ideal, and the two concepts agree when R is reduced. Every proper ideal is weakly (m,n)-closed for $m \leq n$; so we usually assume that $m > n$.

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The above definitions all concern generalizations of prime ideals. A 1-absorbing ideal is just a prime ideal, and a weakly 1-absorbing ideal is just a weakly prime ideal (a proper ideal I of R is a *weakly prime ideal* if $0 \neq xy \in I$ for $x, y \in R$ implies $x \in I$ or $y \in I$). A proper ideal is a radical ideal if and only if it is $(2,1)$-closed. However, a weakly $(2,1)$-closed ideal need not be a weakly radical ideal (a proper ideal I of R is a *weakly radical ideal* if $0 \neq x^n \in I$ for $x \in R$ and n a positive integer implies $x \in I$) (see Example 2.3(b)).

Weakly prime ideals and weakly radical ideals were studied in [1], and weakly radical (semiprime) ideals have been studied in more detail in [6]. The concept of 2-absorbing ideals was introduced in [5] and then extended to n-absorbing ideals in [2]. Related concepts include 2-absorbing primary ideals (see [9]), weakly 2-absorbing ideals (see [11]), weakly 2-absorbing primary ideals (see [10]), and (m,n)-closed ideals (see [4]). Other generalizations and related concepts are investigated in [1], [6], [8], [11], [12], [13], and [15]. For a survey on n-absorbing ideals, see [7].

Let R be a commutative ring and m and n positive integers. We define R to be an *(m,n)-von Neumann regular ring* if for every $x \in R$, there is an $r \in R$ such that $x^m r = x^n$. Thus a $(2,1)$-von Neumann regular ring is just a von Neumann regular ring. In this paper, we study weakly (m,n)-closed ideals, (m,n)-von Neumann regular rings, and the connections between the two concepts.

Let m and n be positive integers with $m > n$. Among the many results in this paper, we show in Theorem 2.6 that if I is a weakly (m,n)-closed, but not (m,n)-closed, ideal of R, then $I \subseteq Nil(R)$. In Theorem 2.11, we determine when a proper ideal of $R_1 \times R_2$ is weakly (m,n)-closed, but not (m,n)-closed; and in Theorem 2.12, we investigate when a proper ideal of $R(+)M$ is weakly (m,n)-closed, but not (m,n)-closed. In Section 3, we introduce and investigate (m,n)-von Neumann regular elements and (m,n)-von Neumann regular rings. It is shown in Theorem 3.5 that every proper ideal of R is weakly (m,n)-closed if and only if every non-nilpotent element of R is (m,n)-von Neumann regular and $w^m = 0$ for every $w \in Nil(R)$. In Theorem 3.7, we show that every proper ideal of R is (m,n)-closed if and only if R is (m,n)-von Neumann regular. Finally, we define the concepts of n-regular and ω-regular commutative rings as a way to measure how far a zero-dimensional commutative ring is from being von Neumann regular.

We assume throughout this paper that all rings are commutative with $1 \neq 0$, all R-modules are unitary, and $f(1) = 1$ for all ring homomorphisms $f : R \longrightarrow T$. For such a ring R, let $Nil(R)$ be its ideal of nilpotent elements, $Z(R)$ its set of zero-divisors, $U(R)$ its group of units, $\mathrm{char}(R)$ its characteristic, and $\dim(R)$ its (Krull) dimension. Then R is *reduced* if $Nil(R) = \{0\}$ and R is *quasilocal* if it has exactly one maximal ideal. As usual, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Z}_n$ will denote the positive integers, integers, and integers modulo n, respectively. Several of our results use the $R(+)M$ construction as in [14]. Let R be a commutative ring and M an R-module. Then $R(+)M = R \times M$ is a commutative ring with identity $(1,0)$

under addition defined by $(r,m)+(s,n)=(r+s,m+n)$ and multiplication defined by $(r,m)(s,n)=(rs,rn+sm)$. Note that $(\{0\}(+)M)^2=\{0\}$; so $\{0\}(+)M \subseteq Nil(R(+)M)$.

2. Properties of weakly (m,n)-closed ideals

In this section, we give some basic properties of weakly (m,n)-closed ideals and investigate weakly (m,n)-closed ideals in several classes of commutative rings. We start by recalling the definitions of weakly semi-n-absorbing and weakly (m,n)-closed ideals.

Definition 2.1. Let R be a commutative ring, I a proper ideal of R, and m and n positive integers.

(1) I is a *weakly semi-n-absorbing ideal* of R if $0 \neq x^{n+1} \in I$ for $x \in R$ implies $x^n \in I$.

(2) I is a *weakly (m,n)-closed ideal* of R if $0 \neq x^m \in I$ for $x \in R$ implies $x^n \in I$.

The proof of the next result follows easily from the definitions, and thus will be omitted.

Theorem 2.2. *Let R be a commutative ring and m and n positive integers.*

(1) *If I is a weakly n-absorbing ideal of R, then I is weakly semi-n-absorbing (i.e., weakly $(n+1,n)$-closed).*

(2) *If I is a weakly (m,n)-closed ideal of R, then I is weakly (m,n')-closed for every positive integer $n' \geq n$.*

(3) *If I is a weakly n-absorbing ideal of R, then I is weakly (m,n)-closed for every positive integer m.*

(4) *An intersection of weakly (m,n)-closed ideals of R is weakly (m,n)-closed.*

While an (m,n)-closed ideal is always weakly (m,n)-closed, the converse need not hold. If an ideal is (m,n)-closed, then it is also (m',n')-closed for all positive integers $m' \leq m$ and $n' \geq n$ [4, Theorem 2.1(3)]. However, a weakly (m,n)-closed ideal need not be weakly (m',n)-closed for $m' < m$. We next give two examples to illustrate these differences.

Example 2.3. (a) Let $R=\mathbb{Z}_8$ and $I=\{0,4\}$. Then I is weakly $(3,1)$-closed since $x^3=0$ for every nonunit x in R. However, I is not $(3,1)$-closed since $2^3=0 \in I$ and $2 \notin I$, and I is not weakly $(2,1)$-closed since $0 \neq 2^2=4 \in I$ and $2 \notin I$.

(b) Let $R=\mathbb{Z}_{16}$ and $I=\{0,8\}$. Then I is weakly $(2,1)$-closed since 8 is not a square in $\mathbb{Z}_{16}$. However, I is not $(2,1)$-closed since $4^2=0 \in I$ and $4 \notin I$, and I is not a weakly radical ideal (and thus not weakly prime) since $0 \neq 2^3=8 \in I$ and $2 \notin I$.

The following definition will be useful for studying weakly (m,n)-closed ideals that are not (m,n)-closed (cf. [6, Definition 2.2]).

Definition 2.4. Let R be a commutative ring, m and n positive integers, and I a weakly (m,n)-closed ideal of R. Then $a \in R$ is an (m,n)-*unbreakable-zero element* of I if $a^m = 0$ and $a^n \notin I$. (Thus I has an (m,n)-*unbreakable-zero element if and only if I is not (m,n)-closed.)*

Theorem 2.5 (cf. [6, Theorem 2.3])**.** *Let R be a commutative ring, m and n positive integers, and I a weakly (m,n)-closed ideal of R. If a is an (m,n)-unbreakable-zero element of I, then $(a+i)^m = 0$ for every $i \in I$.*

Proof. Let $i \in I$. Then

$$(a+i)^m = a^m + \sum_{k=1}^{m} \binom{m}{k} a^{m-k} i^k = 0 + \sum_{k=1}^{m} \binom{m}{k} a^{m-k} i^k \in I,$$

and similarly, $(a+i)^n \notin I$ since $a^n \notin I$. Thus $(a+i)^m = 0$ since I is weakly (m,n)-closed. □

Theorem 2.6 (cf. [1, p. 839] and [6, Theorems 2.4 and 2.5])**.** *Let R be a commutative ring, m and n positive integers, and I a weakly (m,n)-closed ideal of R. If I is not (m,n)-closed, then $I \subseteq Nil(R)$. Moreover, if I is not (m,n)-closed and $char(R) = m$ is prime, then $i^m = 0$ for every $i \in I$.*

Proof. Since I is a weakly (m,n)-closed ideal of R that is not (m,n)-closed, I has an (m,n)-unbreakable-zero element a. Let $i \in I$. Then $a^m = 0$, and $(a+i)^m = 0$ by Theorem 2.5; so $a, a+i \in Nil(R)$. Thus $i = (a+i) - a \in Nil(R)$; so $I \subseteq Nil(R)$.

The "moreover" statement is clear since $0 = (a+i)^m = a^m + i^m = i^m$ when $\mathrm{char}(R) = m$ is prime. □

The next two theorems are the analogs of the results for (m,n)-closed ideals in [4, Theorem 2.8] and [4, Theorem 2.10], respectively. Their proofs are similar, and thus will be omitted.

Theorem 2.7. *Let R be a commutative ring, I a proper ideal of R, $S \subseteq R \setminus \{0\}$ a multiplicative set, and m and n positive integers. If I is a weakly (m,n)-closed ideal of R, then I_S is a weakly (m,n)-closed ideal of R_S.*

Theorem 2.8. *Let $f : R \longrightarrow T$ be a homomorphism of commutative rings and m and n positive integers.*

1. *If f is injective and J is a weakly (m,n)-closed ideal of T, then $f^{-1}(J)$ is a weakly (m,n)-closed ideal of R. In particular, if R is a subring of T and J is a weakly (m,n)-closed ideal of T, then $J \cap R$ is a weakly (m,n)-closed ideal of R.*
2. *If f is surjective and I is a weakly (m,n)-closed ideal of R containing $kerf$, then $f(I)$ is a weakly (m,n)-closed ideal of T. In particular, if I is a weakly (m,n)-closed ideal of R and $J \subseteq I$ is an ideal of R, then I/J is a weakly (m,n)-closed ideal of R/J.*

In the following theorems, we determine when an ideal of $R_1 \times R_2$ is weakly (m,n)-closed, but not (m,n)-closed. (Recall that an ideal of $R_1 \times R_2$ has the form $I_1 \times I_2$ for ideals I_1 of R_1 and I_2 of R_2.) It is easy to determine when an ideal of $R_1 \times R_2$ is (m,n)-closed.

Theorem 2.9 (cf. [4, Theorem 2.12]). *Let $R = R_1 \times R_2$, where R_1 and R_2 are commutative rings, J a proper ideal of R, and m and n positive integers. Then the following statements are equivalent.*

(1) *J is an (m,n)-closed ideal of R.*

(2) *$J = I_1 \times R_2$, $R_1 \times I_2$, or $I_1 \times I_2$ for (m,n)-closed ideals I_1 of R_1 and I_2 of R_2.*

Proof. This follows directly from the definitions. □

The analog of (1) ⇒ (2) of Theorem 2.9 clearly holds for weakly (m,n)-closed ideals by Theorem 2.8(2), but our next theorem shows that the analog of (2) ⇒ (1) does not hold for weakly (m,n)-closed ideals.

Theorem 2.10. *Let $R = R_1 \times R_2$, where R_1 and R_2 are commutative rings, I_1 a proper ideal of R_1, and m and n positive integers. Then the following statements are equivalent.*

(1) *$I_1 \times R_2$ is a weakly (m,n)-closed ideal of R.*

(2) *I_1 is an (m,n)-closed ideal of R_1.*

(3) *$I_1 \times R_2$ is an (m,n)-closed ideal of R.*

A similar result holds for $R_1 \times I_2$ when I_2 is a proper ideal of R_2.

Proof. (1) ⇒ (2) I_1 is a weakly (m,n)-closed ideal of R_1 by Theorem 2.8(2). If I_1 is not an (m,n)-closed ideal of R_1, then I_1 has an (m,n)-unbreakable-zero element a. Thus $(0,0) \neq (a,1)^m \in I_1 \times R_2$, but $(a,1)^n \notin I_1 \times R_2$, a contradiction. Hence I_1 is an (m,n)-closed ideal of R_1.

(2) ⇒ (3) This is clear (cf. [4, Theorem 2.12]).

(3) ⇒ (1) This is clear by definition. □

Theorem 2.11. *Let $R = R_1 \times R_2$, where R_1 and R_2 are commutative rings, J a proper ideal of R, and m and n positive integers. Then the following statements are equivalent.*

(1) *J is a weakly (m,n)-closed ideal of R that is not (m,n)-closed.*

(2) *$J = I_1 \times I_2$ for proper ideals I_1 of R_1 and I_2 of R_2 such that either*

  (a) *I_1 is a weakly (m,n)-closed ideal of R_1 that is not (m,n)-closed, $y^m = 0$ whenever $y^m \in I_2$ for $y \in R_2$ (in particular, $i^m = 0$ for every $i \in I_2$), and if $0 \neq x^m \in I_1$ for some $x \in R_1$, then I_2 is an (m,n)-closed ideal of R_2, or*

  (b) *I_2 is a weakly (m,n)-closed ideal of R_2 that is not (m,n)-closed, $y^m = 0$ whenever $y^m \in I_1$ for $y \in R_1$ (in particular, $i^m = 0$ for every $i \in I_1$), and if $0 \neq x^m \in I_2$ for some $x \in R_2$, then I_1 is an (m,n)-closed ideal of R_1.*

Proof. (1) $\Rightarrow$ (2) Since J is not an (m,n)-closed ideal of R, by Theorem 2.10 we have $J = I_1 \times I_2$, where I_1 is a proper ideal of R_1 and I_2 is a proper ideal of R_2. Since J is not an (m,n)-closed ideal of R, either I_1 is a weakly (m,n)-closed ideal of R_1 that is not (m,n)-closed or I_2 is a weakly (m,n)-closed ideal of R_2 that is not (m,n)-closed. Assume that I_1 is a weakly (m,n)-closed ideal of R_1 that is not (m,n)-closed. Thus I_1 has an (m,n)-unbreakable-zero element a. Assume that $y^m \in I_2$ for $y \in R_2$. Since a is an (m,n)-unbreakable-zero element of I_1 and $(a,y)^m \in J$, we have $(a,y)^m = (0,0)$. Hence $y^m = 0$ (in particular, $i^m = 0$ for every $i \in I_2$). Now assume that $0 \neq x^m \in I_1$ for some $x \in R_1$. Let $y \in R_2$ such that $y^m \in I_2$. Then $(0,0) \neq (x,y)^m \in J$. Thus $y^n \in I_2$, and hence I_2 is an (m,n)-closed ideal of R_2. Similarly, if I_2 is a weakly (m,n)-closed ideal of R_2 that is not (m,n)-closed, then $y^m = 0$ whenever $y^m \in I_1$ for $y \in R_1$ (in particular, $i^m = 0$ for every $i \in I_1$), and if $0 \neq x^m \in I_2$ for some $x \in R_2$, then I_1 is an (m,n)-closed ideal of R_1.

(2) $\Rightarrow$ (1) Suppose that I_1 is a weakly (m,n)-closed proper ideal of R_1 that is not (m,n)-closed, $y^m = 0$ whenever $y^m \in I_2$ for $y \in R_2$ (in particular, $i^m = 0$ for every $i \in I_2$), and if $0 \neq x^m \in I_1$ for some $x \in R_1$, then I_2 is an (m,n)-closed ideal of R_2. Let a be an (m,n)-unbreakable-zero element of I_1. Then $(a,0)$ is an (m,n)-unbreakable-zero element of J. Thus J is not an (m,n)-closed ideal of R. Now assume that $(0,0) \neq (x,y)^m = (x^m, y^m) \in J$ for $x \in R_1$ and $y \in R_2$. Then $(0,0) \neq (x,y)^m = (x^m, 0) \in J$ and $0 \neq x^m \in I_1$. Since I_1 is a weakly (m,n)-closed ideal of R_1 and I_2 is an (m,n)-closed ideal of R_2, we have $(x,y)^n \in J$. Similarly, assume that I_2 is a weakly (m,n)-closed ideal of R_2 that is not (m,n)-closed, $y^m = 0$ whenever $y^m \in I_1$ for $y \in R_1$ (in particular, $i^m = 0$ for every $i \in I_1$), and if $0 \neq x^m \in I_2$ for some $x \in R_2$, then I_1 is an (m,n)-closed ideal of R_1. Then again, J is a weakly (m,n)-closed ideal of R that is not (m,n)-closed. $\square$

We next consider when certain ideals of $R(+)M$ are weakly (m,n)-closed.

Theorem 2.12. *Let R be a commutative ring, I a proper ideal of R, M an R-module, and m and n positive integers. Then the following statements are equivalent.*

(1) *$I(+)M$ is a weakly (m,n)-closed ideal of $R(+)M$ that is not (m,n)-closed.*
(2) *I is a weakly (m,n)-closed ideal of R that is not (m,n)-closed and $m(a^{m-1}M) = 0$ for every (m,n)-unbreakable-zero element a of I.*

Proof. (1) $\Rightarrow$ (2) Let $J = I(+)M$. Assume that $0 \neq r^m \in I$ for $r \in R$. Thus $(0,0) \neq (r,0)^m = (r^m, 0) \in J$. Hence $(r,0)^n = (r^n, 0) \in J$; so $r^n \in I$. Thus I is a weakly (m,n)-closed ideal of R. Since J is not (m,n)-closed, J, and hence I, has an (m,n)-unbreakable-zero element; so I is not (m,n)-closed. Let a be an (m,n)-unbreakable-zero element of I and $x \in M$. Then $(a,x)^m = (a^m, m(a^{m-1}x)) \in J$. Since $a^n \notin I$, we have $(a,x)^m = (a^m, m(a^{m-1}x)) = (0,0)$. Thus $m(a^{m-1}M) = 0$.

(2) $\Rightarrow$ (1) Since I is a weakly (m,n)-closed ideal of R that is not (m,n)-closed, I has an (m,n)-unbreakable-zero element a. Hence $(a,0)$ is an (m,n)-unbreakable-zero element of $J = I(+)M$. Thus J is not an (m,n)-closed ideal of A. Suppose that $(0,0) \neq (r,y)^m = (r^m, m(r^{m-1}y)) \in J$. Then r is not an (m,n)-unbreakable-zero element of I by hypothesis. Hence $(r^n, n(r^{n-1}y)) = (r,y)^n \in J$; so J is a weakly (m,n)-closed ideal of A that is not (m,n)-closed. $\square$

We end this section with another way to construct weakly (m,n)-closed ideals that are not (m,n)-closed. See [4, Theorems 3.1 and 3.8] for similar results for (m,n)-closed ideals.

Theorem 2.13. *Let R be an integral domain and $I = p^kR$ a principal ideal of R, where p is a prime element of R and k a positive integer. Let m be a positive integer such that $m < k$, and write $k = mq + r$ for integers q, r, where $q \geq 1$ and $0 \leq r < m$. Then $J = I/p^cR$ is a weakly (m,n)-closed ideal of R/p^cR that is not (m,n)-closed for positive integers $n < m$ and $c \geq k+1$ if and only if $r \neq 0$, $k+1 \leq c \leq m(q+1)$, and $n(q+1) < k$.*

Proof. Suppose that J is a weakly (m,n)-closed ideal of R/p^cR that is not (m,n)-closed for positive integers $n < m$ and $c \geq k+1$. It is clear that $r \neq 0$, for if $r = 0$, then $0 \neq (p^q)^m + p^cR \in J$, but $(p^q)^n + p^cR \notin J$. Since $q+1$ is the smallest positive integer such that $(p^{(q+1)})^m + p^cR \in J$ and J is not (m,n) closed, we have $0 = (p^{(q+1)})^m + p^cR \in J$ and $(p^{(q+1)})^n + p^cR \notin J$. Thus $n(q+1) < k$ and $k+1 \leq c \leq (q+1)m$.

Conversely, assume that $r \neq 0$, $k+1 \leq c \leq m(q+1)$, and $n(q+1) < k$. Let $x \in R/p^cR$ such that $x^m \in J$. Then $x = p^iy + p^cR$ for some $y \in R$ such that $p^{(i+1)} \nmid y$ in R. Since $x^m = (p^i)^m + p^cR \in J$, we have $i \geq q+1$. Thus by hypothesis, $x^m = 0$ in R/p^cR. Since $0 = (p^{(q+1)})^m + p^cR \in J$ and $n(q+1) < k$, we have $(p^{(q+1)})^n + p^cR \notin J$. Hence J is not (m,n)-closed. $\square$

Example 2.14. (a) Let $R = \mathbb{Z}$, $I = 2^{12}\mathbb{Z}$, and $J = I/2^{13}\mathbb{Z}$. Then by Theorem 2.13, J is a weakly $(5,3)$-closed ideal of $\mathbb{Z}/2^{13}\mathbb{Z}$ that is not $(5,3)$-closed.

(b) Let R, I, and J be as in part (a) above. Then $J(+)J$ is a weakly $(5,3)$-closed ideal of $\mathbb{Z}/2^{13}\mathbb{Z}(+)J$ that is not $(5,3)$-closed by Theorem 2.12.

3. (m,n)-von Neumann regular rings

In this section, we introduce the concepts of (m,n)-von Neumann regular elements and (m,n)-von Neumann regular rings and use them to determine when every proper ideal of R is (m,n)-closed or weakly (m,n)-closed. We also define the related concepts of n-regular and ω-regular commutative rings. First, we handle the case for ideals contained in $Nil(R)$.

Theorem 3.1. *Let R be a commutative ring and m and n positive integers with $m > n$. Then every ideal of R contained in $Nil(R)$ is weakly (m,n)-closed if and only if $w^m = 0$ for every $w \in Nil(R)$.*

Proof. Suppose that every ideal of R contained in $Nil(R)$ is weakly (m,n)-closed, but $w^m \neq 0$ for some $w \in Nil(R)$. Let $J = w^m R \subseteq Nil(R)$. Then J is weakly (m,n)-closed and $0 \neq w^m \in J$; so $w^n \in J$ and $w^n \neq 0$ since $n < m$. Thus $w^n = w^m a$ for some $a \in R$, and hence $w^n(1 - w^{m-n}a) = 0$. Then $1 - w^{m-n}a \in U(R)$ since $w^{m-n}a \in Nil(R)$); so $w^n = 0$, a contradiction. Thus $w^m = 0$ for every $w \in Nil(R)$.

Conversely, suppose that $w^m = 0$ for every $w \in Nil(R)$. Then every ideal of R contained in $Nil(R)$ is weakly (m,n)-closed by definition. □

Recall that $x \in R$ is a *von Neumann regular element* of R if $x^2 r = x$ for some $r \in R$. Similarly, $x \in R$ is a *π-regular element* of R if $x^{2n} r = x^n$ for some $r \in R$ and positive integer n. Thus R is a von Neumann regular ring (resp., π-regular ring) if and only if every element of R is von Neumann regular (resp., π-regular). It is well known that R is π-regular (resp., von Neumann regular) if and only if $\dim(R) = 0$ (resp., R is reduced and $\dim(R) = 0$) [14, Theorem 3.1, p. 10]. A ring R is a *strongly π-regular ring* if there is a positive integer n such that for every $x \in R$, we have $x^{2n} r = x^n$ for some $r \in R$. For a recent article on von Neumann regular and related elements of a commutative ring, see [3]. These concepts are generalized in the next definition.

Definition 3.2. Let R be a commutative ring and m and n positive integers. Then $x \in R$ is an *(m,n)-von Neumann regular* element of R (or *(m,n)-vnr* for short) if $x^m r = x^n$ for some $r \in R$. If every element of R is (m,n)-vnr, then R is an *(m,n)-von Neumann regular* ring.

Thus a commutative ring R is von Neumann regular if and only if it is $(2,1)$-von Neumann regular, and R is strongly π-regular if and only if it is $(2n,n)$-von Neumann regular for some positive integer n. The next theorem gives some basic facts about (m,n)-vnr elements.

Theorem 3.3. *Let R be a commutative ring, $x \in R$, and m and n positive integers.*

(1) *x is (m,n)-vnr for $m \leq n$ (so we usually assume that $m > n$).*
(2) *If x is (m,n)-vnr, then x is (m',n')-vnr for all positive integers $m' \leq m$ and $n' \geq n$.*
(3) *If $x \in U(R)$ or $x = 0$, then x is (m,n)-vnr for all positive integers m and n.*
(4) *If $x \in R \setminus (Z(R) \cup U(R))$, then x is (m,n)-vnr if and only if $m \leq n$.*
(5) *If $x^n = 0$, then x is (m,n)-vnr for every positive integer m.*
(6) *If $x^k = 0$ and $x^{k-1} \neq 0$ for an integer $k \geq 2$, then x is (m,n)-vnr if and only if $m \leq n$ or $n \geq k$.*
(7) *If x is (m,n)-vnr with $m > n$, then x is $(m+1,n)$-vnr. Moreover, in this case, x is (m',n')-vnr for all positive integers m' and $n' \geq n$. Thus R is von Neumann regular if and only if R is (m,n)-von Neumann regular for all positive integers m and n.*

Proof. The proofs of (1)-(3) and (5) are clear.

(4) By (1), x is (m,n)-vnr for $m \leq n$. If $m > n$, then $x^m r = x^n$ for $r \in R$ implies $x^{m-n}r = 1$. Thus $x \in U(R)$, a contradiction.

(6) Suppose that $x^m r = x^n$ for $r \in R$, but $m > n$ and $n < k$. Then $x^{k-1} = x^n(x^{k-n-1}) = (x^m r)(x^{k-n-1}) = x^k(x^{m-n-1}r) = 0$, a contradiction. Thus $m \leq n$ or $n \geq k$. The converse is clear.

(7) Let x be (m,n)-vnr with $m > n$. Then $x^m r = x^n$ for $r \in R$ implies $x^n = x^m r = x^n(x^{m-n}r) = (x^m r)(x^{m-n}r) = x^{m+1}(x^{m-n-1}r^2)$ with $x^{m-n-1}r^2 \in R$. Thus x is $(m+1,n)$-vnr. The "moreover" statement follows by induction and (2). $\square$

Corollary 3.4. *Let R be a commutative ring and m and n positive integers with $m > n$. Then R is (m,n)-von Neumann regular if and only if R is (m',n')-von Neumann regular for all positive integers m' and $n' \geq n$. In particular, if R is (m,n)-von Neumann regular, then R is strongly π-regular, and thus $\dim(R) = 0$.*

We next determine when every proper ideal of R is weakly (m,n)-closed.

Theorem 3.5. *Let R be a commutative ring and m and n positive integers with $m > n$. Then the following statements are equivalent.*

(1) *Every proper ideal of R is weakly (m,n)-closed.*
(2) *Every non-nilpotent element of R is (m,n)-vnr and $w^m = 0$ for every $w \in Nil(R)$.*

Proof. (1) $\Rightarrow$ (2) Since every ideal of R contained in $Nil(R)$ is weakly (m,n)-closed, $w^m = 0$ for every $w \in Nil(R)$ by Theorem 3.1. Let $x \in R \setminus Nil(R)$. If $x \in U(R)$, then x is (m,n)-vnr by Theorem 3.3(3). If $x \notin U(R)$, then $I = x^m R$ is weakly (m,n)-closed and $0 \neq x^m \in I$; so $x^n \in I$. Thus $x^n = x^m r$ for some $r \in R$, and hence x is (m,n)-vnr.

(2) $\Rightarrow$ (1) Let I be a proper ideal of R and $0 \neq x^m \in I$ for $x \in R$. Then $x \notin Nil(R)$; so x is (m,n)-vnr. Thus $x^m r = x^n$ for some $r \in R$; so $x^n = x^m r \in I$. Hence I is weakly (m,n)-closed. $\square$

In view of Theorem 3.5, we have the following result.

Corollary 3.6. *Let R be a reduced commutative ring and m and n positive integers. Then the following statements are equivalent.*

(1) *Every proper ideal of R is weakly (m,n)-closed.*
(2) *Every proper ideal of R is (m,n)-closed.*
(3) *R is (m,n)-von Neumann regular.*

The following result is the analog of Theorem 3.5 for (m,n)-closed ideals.

Theorem 3.7. *Let R be a commutative ring and m and n positive integers. Then the following statements are equivalent.*

(1) *Every proper ideal of R is (m,n)-closed.*

(2) *R is (m,n)-von Neumann regular.*

Proof. (1) $\Rightarrow$ (2) Let $x \in R$. If $x \in U(R)$, then x is (m,n)-vnr by Theorem 3.3(3). If $x \notin U(R)$, then $I = x^m R$ is (m,n)-closed and $x^m \in I$. Thus $x^n \in I$; so $x^n = x^m r$ for some $r \in R$. Hence x is (m,n)-vnr, and thus R is (m,n)-von Neumann regular.

(2) $\Rightarrow$ (1) Let I be a proper ideal of R and $x^m \in I$ for $x \in R$. Since x is (m,n)-vnr, $x^m r = x^n$ for some $r \in R$. Thus $x^n = x^m r \in I$; so I is (m,n)-closed. □

Of course, we are mainly interested in the case when $m > n$. The next theorem incorporates Theorem 3.7 with another characterization ([4, Theorem 2.14]) of when every proper ideal is (m,n)-closed. Note that in Theorem 3.8(3) below, there are no conditions on m other than $m > n$.

Theorem 3.8. *Let R be a commutative ring and m and n positive integers with $m > n$. Then the following statements are equivalent.*

(1) *Every proper ideal of R is (m,n)-closed.*
(2) *R is (m,n)-von Neumann regular.*
(3) $\dim(R) = 0$ *and $w^n = 0$ for every $w \in Nil(R)$.*

Proof. (1) $\Leftrightarrow$ (2) is Theorem 3.7 and (1) $\Leftrightarrow$ (3) is [4, Theorem 2.14]. □

Theorem 3.8 gives a nice ring-theoretic characterization of (m,n)-von Neumann regular rings (for $m > n$). This can now be used to give a characterization of strongly π-regular commutative rings which strengthens Corollary 3.4.

Theorem 3.9. *Let R be a commutative ring. Then the following statements are equivalent.*

(1) *R is strongly π-regular.*
(2) *There are positive integers m and n with $m > n$ such that R is (m,n)-von Neumann regular.*
(3) *There is a positive integer n such that R is (m,n)-von Neumann regular for every positive integer m.*
(4) $\dim(R) = 0$ *and there is a positive integer n such that $w^n = 0$ for every $w \in Nil(R)$.*

Proof. (1) $\Rightarrow$ (2) A strongly π-regular ring is $(2n,n)$-von Neuman regular for some positive integer n.

(2) $\Rightarrow$ (3) This follows from Corollary 3.4.

(3) $\Rightarrow$ (1) In particular, R is $(2n,n)$-von Neumann regular, and thus strongly π-regular.

(2) $\Leftrightarrow$ (4) This is just (2) $\Leftrightarrow$ (3) of Theorem 3.8. □

We next investigate in more detail the pairs (m,n) for which a commutative ring R or an $x \in R$ is (m,n)-von Neumann regular.

Definition 3.10. Let R be a commutative ring, $x \in R$, and k a positive integer.

(1) $\mathcal{V}(R,x) = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid x \text{ is } (m,n)\text{-vnr}\}$.
(2) $\mathcal{V}(R) = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid R \text{ is } (m,n)\text{-von Neumann regular}\}$.
(3) $\mathcal{B}_k = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid m \leq n \text{ or } n \geq k\}$.
(4) $\mathcal{B}_\omega = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid m \leq n\}$.

Then $\mathcal{V}(R) = \bigcap_{x \in R} \mathcal{V}(R,x)$ and

$$\mathbb{N} \times \mathbb{N} = \mathcal{B}_1 \supsetneq \mathcal{B}_2 \supsetneq \cdots \supsetneq \mathcal{B}_\omega.$$

Theorem 3.11. *Let R be a commutative ring and $x \in R$.*

(1) *$\mathcal{V}(R,x) = \mathcal{B}_k$, where k is the smallest positive integer such that $(i,k) \in \mathcal{V}(R,x)$ for some $i > k$. (Thus k is the smallest positive integer such that x is (m,k)-vnr for every positive integer m.) If no such k exists, then $\mathcal{V}(R,x) = \mathcal{B}_\omega$.*
(2) *$\mathcal{V}(R) = \mathcal{B}_k$, where k is the smallest positive integer such that $(i,k) \in \mathcal{V}(R,x)$ for some $i > k$ and every $x \in R$. (Thus k is the smallest positive integer such that x is (m,k)-vnr for every $x \in R$ and positive integer m.) If no such k exists, then $\mathcal{V}(R) = \mathcal{B}_\omega$.*

Proof. (1) follows directly from Theorem 3.3(7). Thus (2) holds by definition. □

These ideas can also be used to classify zero-dimensional commutative rings.

Definition 3.12. Let R be a commutative ring and n a positive integer.

(1) R is n-regular if $\mathcal{V}(R) = \mathcal{B}_n$, i.e., n is the smallest positive integer such that for every $x \in R$ and positive integer m, $x^n = x^m r_m$ for some $r_m \in R$.
(2) R is ω-regular if for every $x \in R$, $\mathcal{V}(R,x) = \mathcal{B}_{n_x}$ for some positive integer n_x, but $\mathcal{V}(R) = \mathcal{B}_\omega$.

A commutative ring R is von Neumann regular if and only if it is 1-regular, and R is strongly π-regular if and only if it is n-regular for some positive integer n. Note that R is π-regular if and only if every $x \in R$ is (m,n)-vnr for some positive integers m and n with $m > n$, but a π-regular ring may be ω-regular (see Example 3.13(d)). Thus R is α-regular for α a positive integer or ω if and only if R is π-regular, if and only if $\dim(R) = 0$. So, in some sense, this concept measures how far a zero-dimensional commutative ring is from being von Neumann regular.

We next give several examples. In particular, we show that if α is any positive integer or ω, there is a quasilocal commutative ring R_α that is α-regular.

Example 3.13. Let R be a commutative ring.

(a) Suppose that there is an $x \in R \setminus (Z(R) \cup U(R))$ (so $\dim(R) > 0$). Then $\mathcal{V}(R) = \mathcal{V}(R, x) = \mathcal{B}_\omega$ by Theorem 3.3(4). Thus R is not ω-regular or n-regular for any positive integer n.

(b) Suppose that R is quasilocal with maximal ideal $M = (x)$ with $x^k = 0$ and $x^{k-1} \neq 0$ for an integer $k \geq 2$. Then $\mathcal{V}(R) = \mathcal{B}_k$ by Theorem 3.3(3),(6); so R is k-regular. This also holds for $k = 1$ since a field is von Neumann regular. In particular, for a prime p and any positive integer k, $\mathcal{V}(\mathbb{Z}_{p^k}) = \mathcal{B}_k$, and thus $\mathbb{Z}_{p^k}$ is k-regular.

(c) Let R_1 and R_2 be commutative rings. Then $x = (x_1, x_2) \in R_1 \times R_2$ is (m, n)-vnr if and only if x_1 and x_2 are (m, n)-vnr in R_1 and R_2, respectively. Thus $\mathcal{V}(R_1 \times R_2) = \mathcal{B}_k$, where $\mathcal{V}(R_1) = \mathcal{B}_{k_1}$, $\mathcal{V}(R_2) = \mathcal{B}_{k_2}$, and $k = \max\{k_1, k_2\}$; so $R_1 \times R_2$ is $\max\{k_1, k_2\}$-regular when R_1 and R_2 are k_1-regular and k_2-regular, respectively. In particular, for distinct primes $p_1, \dots, p_r$, positive integers $k_1, \dots, k_r$, and $k = \max\{k_1, \dots, k_r\}$, $\mathcal{V}(\mathbb{Z}_{p_1^{k_1}} \times \cdots \times \mathbb{Z}_{p_r^{k_r}}) = \mathcal{B}_k$, and hence $\mathbb{Z}_{p_1^{k_1}} \times \cdots \times \mathbb{Z}_{p_r^{k_r}}$ is k-regular.

(d) Let $R = \mathbb{Z}_2[\{X_n\}_{n\in\mathbb{N}}]/(\{X_n^{n+1}\}_{n\in\mathbb{N}}) = \mathbb{Z}_2[\{x_n\}_{n\in\mathbb{N}}]$. Then R is a zero-dimensional quasilocal commutative ring with maximal ideal $Nil(R) = (\{x_n\}_{n\in\mathbb{N}})$; so R is π-regular. Thus every $x \in R$ has $\mathcal{V}(R, x) = \mathcal{B}_k$ for some positive integer k and $\mathcal{V}(R, x_n) = \mathcal{B}_{n+1}$ by Theorem 3.3(3),(6); so $\mathcal{V}(R) = \mathcal{B}_\omega$. Hence R is ω-regular.

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