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MTH512 Homework 5 18 May 2023 Advanced Linear Algebra

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1. Given A is similar to $H = C(\alpha-4) \oplus C(\alpha-4) \oplus C(\alpha^2+\alpha-20) \oplus C(\alpha^2+\alpha-20)$. Then A is similar to a matrix J where J is in Jordan form.

   (i) Explicitly, write down the entries of J.
   Answer: $C_A(\alpha) = (\alpha-4)^4(\alpha-5)^2$, $m_A(\alpha) = (\alpha-4)(\alpha-5)$. Therefore, $J = J_1(4) \oplus J_1(4) \oplus J_1(4) \oplus J_1(4) \oplus J_1(-5) \oplus J_1(-5)$. Explicitly,

$$J = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{bmatrix}$$

   (ii) What is the dimension of each generalized eigenspace?
   Answer: $dim(G-E_4(A)) = dim(E_4(A)) = 4$, $dim(G-E_{-5}(A)) = dim(E_{-5}(A)) = 2$

   (iii) Theoretically, how do you construct the columns of the invertible matrix Q where $Q^{-1}AQ = J$?
   Answer: Since every generalized eigenvector is of order 1 for each eigenvalue, (i.e. $dim(G-E_a(A)) = dim(E_a(A))$ for each a), we can choose v_1, v_2, v_3, v_4 which are independent eigenvectors corresponding to the eigenvalue 4, and w_1, w_2 which are independent eigenvectors corresponding to the eigenvalue 5. These vectors form a basis for $\mathbb{R}^6$.

2. Given A is 4×4 and A is similar to $J_4(3)$.

   (i) Find $C_A(\alpha)$ and $m_A(\alpha)$.
   Answer: $C_A(\alpha) = m_A(\alpha) = (\alpha-3)^4$.

   (ii) find $dim(E_3(A))$ and $dim(G-E_3(A))$.
   Answer: $dim(E_3(A)) = 1$ and $dim(G-E_3(A)) = 4$.

   (iii) Theoretically, how do you construct the columns of the invertible matrix Q where $Q^{-1}AQ = J_4(3)$?
   Answer: Given v is a generalized eigenvector of order 4 corresponding to the eigenvalue 3, the columns of the matrix Q are $\{(A-3I_4)^3v, (A-3I_4)^2v, (A-3I_4)v, v\}$.

(iv) Find the rational form of A.
Answer: $R = C((\alpha - 3)^4)$

3. (1) Give me an example of a matrix A in Jordan form such that $C_A(\alpha) = (\alpha-1)^5(\alpha-4)^3$ and $m_A(\alpha) = (\alpha-1)^2(\alpha-4)2$ where $dim(E_1(A)) = 3$
[Hint: Write down your answer as $A = J() \oplus J() \oplus ... \oplus J()$]
Answer: $J = J_1(1) \oplus J_2(1) \oplus J_2(1) \oplus J_1(4) \oplus J_2(4)$

(2) Give me an example of a matrix A in Jordan form such that $C_A(\alpha) = (\alpha-1)^5(\alpha-4)^3$ and $m_A(\alpha) = (\alpha-1)^2(\alpha-4)2$ where $dim(E_1(A)) = 4$
Answer: $J = J_1(1) \oplus J_1(1) \oplus J_1(1) \oplus J_2(1) \oplus J_1(4) \oplus J_2(4)$

4. Assume A is 4×4 s.t. $C_A(\alpha) = (\alpha-3)^4$.

(i) Find all possible Jordan forms of A. For each form, find $dim(E_3(A))$
Answer: Since $C_A(\alpha) = (\alpha-3)^4$, we have $m_A(\alpha) = (\alpha-3)$ OR $m_A(\alpha) = (\alpha-3)^2$ OR $m_A(\alpha) = (\alpha-3)^3$ OR $m_A(\alpha) = (\alpha-3)^4$
For $m_A(\alpha) = (\alpha-3)$, we have $J = J_1(3) \oplus J_1(3) \oplus J_1(3) \oplus J_1(3) = 3I_4$ $(dim(E_3(A)) = 4)$ $\longrightarrow A = 3I_4$
For $m_A(\alpha) = (\alpha-3)^2$, we have $J = J_2(3) \oplus J_1(3) \oplus J_1(3)$ $(dim(E_3(A)) = 3)$ OR $J = J_2(3) \oplus J_2(3)$ $(dim(E_3(A)) = 2)$
For $m_A(\alpha) = (\alpha-3)^3$, we have $J = J_1(3) \oplus J_3(3)$ $(dim(E_3(A)) = 2)$
For $m_A(\alpha) = (\alpha-3)^4$, we have $J = J_4(3)$ $(dim(E_3(A)) = 1)$

(ii) Find all possible Rational forms of A.
Answer:
For $m_A(\alpha) = (\alpha-3)$, we have $R = C(\alpha-3) \oplus C(\alpha-3) \oplus C(\alpha-3) \oplus C(\alpha-3) = 3I_4$
For $m_A(\alpha) = (\alpha-3)^2$, we have $R = C(\alpha-3) \oplus C(\alpha-3) \oplus C((\alpha-3)^2)$ OR $R = C((\alpha-3)^2) \oplus C((\alpha-3)^2)$
For $m_A(\alpha) = (\alpha-3)^3$, we have $R = C((\alpha-3)) \oplus C((\alpha-3)^3)$
For $m_A(\alpha) = (\alpha-3)^4$, we have $R = C((\alpha-3)^4)$

5. (Least square problem) The following is inconsistent system, i.e., it has no solution. $\begin{bmatrix} 3 & -1 \\ 1 & 9 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = w = \begin{bmatrix} 2 \\ -7 \\ 6 \end{bmatrix}$

. Find the best solution for x, y, i.e., find x, y and $d \in span\{v_1, v_2\}$ ($v1$ = first column, v_2 = second column) such that $\begin{bmatrix} 3 & -1 \\ 1 & 9 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = d$ is consistent and $|w-d|$ is minimum. Note that $<>$ is the normal dot product on $\mathbb{R}^n$. [Hint: observe that the columns of the coefficient matrix are orthogonal, hence by HW, you know how to find d, thus $x = \frac{<w,v_1>}{|v_1|^2}$ and $y = \frac{<w,v_2>}{|v_2|^2}$, v_1 is the first column, v_2 is the second column].

Answer: We have

$$d = \frac{< w, v_1 >}{|v_1|^2}v_1 + \frac{< w, v_2 >}{|v_2|^2}v_2$$

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Therefore, $x = \frac{<w,v_1>}{|v_1|^2} = \frac{11}{14}$, $y = \frac{<w,v_2>}{|v_2|^2}v_2 = \frac{-83}{91}$

6. Let A $= \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$, $w = \begin{bmatrix} 5 \\ -5 \\ 5 \end{bmatrix}$ such that $A \begin{bmatrix} x \\ y \end{bmatrix} = w$ is inconsistent. Find the best solution for x, y. [hint: $v_1 ==$ first column, $v_2 =$ second column. Note that the columns of A are not orthogonal. Hence find orthogonal points Q_1, Q_2 such that $M = span\{Q_1, Q_2\} = span\{v_1, v_2\}$. Now find d in M, the closets to w and $A \begin{bmatrix} x \\ y \end{bmatrix} = d$. Another method, no need for Q_1, Q_2 and d, just solve $A^T A \begin{bmatrix} x \\ y \end{bmatrix} = A^T w$]

Answer: We have $A^T A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, $A^T w = \begin{bmatrix} 5 \\ -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ \frac{-5}{2} \end{bmatrix}$ 5

7. . Let $T : P_5 \to P_5$ be a L. T such that $T(ax^4 + bx^3 + cx^2 + dx + e) = (a+2b+c+3d+4e)x^4 + (2a+3b+c+5d)x^3 + (a+b+6c+d+e)x^2 + (3a+5b+c+7d+2e)x + (4a+c+2d+6e)$. Convince me that T is diagnolizable (hint: Writing the question is harder than the answer, find the co-linear and stare).

Answer: Let $L : \mathbb{R}^5 \to \mathbb{R}^5$ be the colinear of T,

$$M_L = \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 2 & 3 & 1 & 5 & 0 \\ 1 & 1 & 6 & 1 & 1 \\ 3 & 5 & 1 & 7 & 2 \\ 4 & 0 & 1 & 2 & 6 \end{bmatrix}$$

By staring, M_L is symmetric. $\Rightarrow L$ is diagonalizable $\Rightarrow T$ is diagonalizable.

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8. Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be a L. T. Assume the normal dot product on $\mathbb{R}^3$. Convince me that $T+T^a+2I_3 : R3 \to R3$ is diagnolizable, i.e., the standard matrix N of $T+T^a+2I_3$ is diagnolizable

   Answer: We have $M_{T+T^a+2I_3} = M_T + M_T^T + 2I_3$. $M_{T+T^a+2I_3}^T = (M_T + M_T^T + 2I_3)^T = M_T + M_T^T + 2I_3 = M_{T+T^a+2I_3}$. Hence, $T+T^a+2I_3$ is symmetric. Therefore, $T+T^a+2I_3$ is diagonalizable.

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QUESTION 1. Given A is similar to $H = C(\alpha - 4) \oplus C(\alpha - 4) \oplus C(\alpha^2 + \alpha - 20) \oplus C(\alpha^2 + \alpha - 20)$. Then A is similar to a matrix J where J is in Jordan form.

(i) Explicitly, write down the entries of J.

$A \approx J_1(4) \oplus J_1(4) \oplus J_1(4) \oplus J_1(-5) \oplus J_1(4) \oplus J_1(-5)$

$$J = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{pmatrix}$$

✓ 5

(ii) What is the dimension of each generalized eigenspace?

dim_4(A)

5 $\dim(G\text{-}E_4(A)) = 4$, $\dim(G\text{-}E_{-5}(A)) = 2$ = dim_-5(A)

(iii) Theoretically, how do you construct the columns of the invertible matrix Q where $Q^{-1}AQ = J$?

$$Q = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ & & & & & \end{pmatrix}$$

- where v_1, v_2, v_3, v_4 are the eigenvectors associated to the eigenvalue 4.
- v_5, v_6 are the eigenvectors associated to the eigenvalue -5

✓ 5

QUESTION 2. Given A is 4×4 and A is similar to $J_4(3)$.

(i) Find $C_A(\alpha)$ and $m_A(\alpha)$.

$C_A(\alpha) = (\alpha - 3)^4$, $m_A(\alpha) = (\alpha - 3)^4$ ✓ 5

(ii) find $dim(E_3(A))$ and $dim(G - E_3(A))$

$\dim(E_3(A)) = 1$ ✓ $\dim(G\text{-}E_3(A)) = 4$ ✓ 5

(iii) Theoretically, how do you construct the columns of the invertible matrix Q where $Q^{-1}AQ = J_4(3)$?

$Q = \{ (A-3I_4)^3 v,\ (A-3I_4)^2 v,\ (A-3I_4) v,\ v \}$

where v is the generalized eigenvector associated to 3

(iv) Find the rational form of A.

$A \approx C((\alpha-3)^4)$

QUESTION 3. (1) Give me an example of a matrix A in Jordan form such that $C_A(\alpha) = (\alpha-1)^5(\alpha-4)^3$ and $m_A(\alpha) = (\alpha-1)^2(\alpha-4)^2$ where $dim(E_1(A)) = 3$ [Hint: Write down your answer as $A = J() \oplus J() \oplus \cdots \oplus J()$]

$A \approx J_2(1) \oplus J_2(1) \oplus J_1(1) \oplus J_2(4) \oplus J_1(4)$

(2) Give me an example of a matrix A in Jordan form such that $C_A(\alpha) = (\alpha-1)^5(\alpha-4)^3$ and $m_A(\alpha) = (\alpha-1)^2(\alpha-4)^2$ where $dim(E_1(A)) = 4$.

$A \approx J_2(1) \oplus J_2(4) \oplus J_1(1) \oplus J_1(1) \oplus J_1(1) \oplus J_1(4)$

QUESTION 4. Assume A is 4×4 s.t. $C_A(\alpha) = (\alpha-3)^4$.

(i) Find all possible Jordan forms of A. For each form, find $dim(E_3(A))$

Case 1: $J_4(3)$ $\quad \dim(E_3(A)) = 1$

Case 2: $J_3(3) \oplus J_1(3)$ $\quad \dim(E_3(A)) = 2$

Case 3: $J_2(3) \oplus J_2(3)$ $\quad \dim(E_3(A)) = 2$

Case 4: $J_2(3) \oplus J_1(3) \oplus J_1(3)$ $\quad \dim(E_3(A)) = 3$

Case 5: $J_1(3) \oplus J_1(3) \oplus J_1(3) + J_1(3)$ $\quad \dim(E_3(A)) = 4$

(ii) Find all possible Rational forms of A.

Case 1: $C((\alpha-3)^4)$

Case 2: $C((\alpha-3)) \oplus C((\alpha-3)^3)$

Case 3: $C((\alpha-3)^2) \oplus C((\alpha-3)^2)$

Case 4: $C(\alpha-3) \oplus C(\alpha-3) \oplus C((\alpha-3)^2)$

Case 5: $C(\alpha-3) \oplus C(\alpha-3) \oplus C(\alpha-3) \oplus C(\alpha-3)$

QUESTION 5. (Least square problem) The following is inconsistent system, i.e., it has no solution. $\begin{bmatrix} 3 & -1 \\ 1 & 9 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = w = \begin{bmatrix} 2 \\ -7 \\ 6 \end{bmatrix}$. Find the best solution for x, y, i.e., find x, y and $d \in span\{v_1, v_2\}$ (v_1 == first column, v_2 = second column) such that $\begin{bmatrix} 3 & -1 \\ 1 & 9 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = d$ is consistent and $|w - d|$ is minimum. Note that $< >$ is the normal dot product on R^n. [Hint: observe that the columns of the coefficient matrix are orthogonal, hence by HW, you know how to find d, thus $x =< w, v_1 > /|v_1|^2$ and $y =< w, v_2 > /|v_2|^2$, v_1 is the first column, v_2 is the second column].

$$\langle w, v_1 \rangle = (2, -7, 6) \cdot (3, 1, 2) = 11$$

$$x = \frac{\langle w, v_1 \rangle}{|v_1|^2} = \frac{11}{14}$$

$$y = \frac{\langle w, v_2 \rangle}{|v_2|^2} = \frac{-83}{91}$$

QUESTION 6. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$, $w = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$ such that $A \begin{bmatrix} x \\ y \end{bmatrix} = w$ is inconsistent. Find the best solution for x, y. [hint: v_1 == first column, v_2 = second column. Note that the columns of A are not orthogonal. Hence find orthogonal points Q_1, Q_2 such that $M = span\{Q_1, Q_2\} = span\{v_1, v_2\}$. Now find d in M, the closets to w and solve $A \begin{bmatrix} x \\ y \end{bmatrix} = d$. Another method, no need for Q_1, Q_2 and d, just solve $A^T A \begin{bmatrix} x \\ y \end{bmatrix} = A^T w$]

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \end{pmatrix}$$

read my last message on ilearn/ but the method correct / note that x = 5, y = 0 are the exact solution

$$\begin{aligned} 3x + y &= 15 \\ x + 3y &= 5 \end{aligned} \Rightarrow \begin{aligned} x &= 5 \\ y &= 0 \end{aligned} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

QUESTION 7. Let $T : P_5 \to P_5$ be a L. T such that $T(ax^4 + bx^3 + cx^2 + dx + e) = (a + 2b + c + 3d + 4e)x^4 + (2a + 3b + c + 5d)x^3 + (a + b + 6c + d + e)x^2 + (3a + 5b + c + 7d + 2e)x + (4a + c + 2d + 6e)$. Convince me that T is diagnolizable (hint: Writing the question is harder than the answer, find the co-linear and stare).

$$M_T = \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 1 & 2 & 1 & 3 & 4 \\ 2 & 3 & 1 & 5 & 0 \\ 1 & 1 & 6 & 1 & 1 \\ 3 & 5 & 1 & 7 & 2 \\ 4 & 0 & 1 & 2 & 6 \end{pmatrix} \end{array}$$

M_T is symmetric

$\Rightarrow$ M_T is diagonalizable

$\Rightarrow$ T is diagonalizable

QUESTION 8. Let $T : R^3 \to R^3$ be a L. T. Assume the normal dot product on R^3. Convince me that $T + T^a + 2I_3 : R^3 \to R^3$ is diagnolizable, i.e., the standard matrix N of $T + T^a + 2I_3$ is diagnolizable .

$L: T + T^a + 2I_3 \Rightarrow N = A + A^T + 2I_3$

$$N^T = A^T + (A^T)^T + 2(I_3)^T,$$
$$= A^T + A + 2I_3 = N$$

where A is the standard matrix representation of T.

$\Rightarrow$ N is symmetric

$\Rightarrow$ L is diagonalizable