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MTH512 Homework 4 **6 May 2023** **Advanced Linear Algebra**

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1. Assume A, B are similar $n \times n$ matrices, say $A = Q^{-1}BQ$

(i) We know $C_A(\alpha) = C_B(\alpha)$. Prove $m_A(\alpha) = m_B(\alpha)$.

Answer:
$m_A(\alpha) = a_k\alpha^k + a_{k-1}\alpha^{k-1} + ... + a_1\alpha + a_0$.
$m_b(\alpha) = b_k\alpha^k + b_{k-1}\alpha^{k-1} + ... + b_1\alpha + b_0$.
We have

$$0_{n\times n} = M_A(A) = a_kA^k + a_{k-1}A^{k-1} + ... + a_1A + a_0I$$
$$= a_k(Q^{-1}BQ)^k + a_{k-1}(Q^{-1}BQ)^{k-1} + ... + a_1(Q^{-1}BQ) + a_0I$$
$$= a_kQ^{-1}B^kQ + a_{k-1}Q^{-1}B^{k-1}Q + ... + a_1Q^{-1}BQ + a_0I$$
$$= Q^{-1}(a_kB^kQ + a_{k-1}B^{k-1}Q + ... + a_1BQ + a_0Q)$$
$$= Q^{-1}(a_kB^k + a_{k-1}B^{k-1} + ... + a_1B + a_0)Q = Q^{-1}m_A(B)Q$$

$m_A(B) = 0_{n\times n} \Rightarrow m_B(\alpha)|m_A(\alpha)$. It is clear that $B = QAQ^{-1}$. Repeating the process with $M_B(B)$, we get $0_{n\times n} = M_B(B) = QM_B(A)Q^{-1}$
$m_B(A) = 0_{n\times n} \Rightarrow m_A(\alpha)|m_B(\alpha)$.
Since $m_B(\alpha)|m_A(\alpha)$ and $m_A(\alpha)|m_B(\alpha)$, $m_B(\alpha) = m_A(\alpha)$.

(ii) Assume that a is an eigenvalue of A and $v_1, v_2, ..., v_k$ is a basis for $E_a(A)$. Prove that $\{Q_{v_1}, Q_{v_2}, ..., Q_{v_k}\}$ is a basis for $E_a(B)$. [Hint: Observe that $BQ = QA$ and since Q is invertible,$Qw = 0_n$iff $w = 0_n$]

Answer: Let a be an eigenvalue of A and $v_1, v_2, ..., v_k$ be a basis for $E_a(A)$. Then a be an eigenvalue of B and $dim(E_a(B)) = dim(E_a(A)) = k$.
$A = Q^{-1}BQ \Rightarrow QA = BQ \Rightarrow (QA)v_i = (BQ)v_i$.
$\Rightarrow Q(Av_i) = B(Qv_i) \Rightarrow a(Qv_i) = B(Qv_i)$.
Since Q is invertible and $v_i \neq 0$, Qv_i is a nonzero vector. Therefore, Qv_i is an eigenvector of B for $1 \leq i \leq k$. Therefore the set $\{Qv_1, Qv_2, ..., Qv_k\}$ forms a basis for $E_a(B)$. show ?

We show Qv_1,..., Qv_k are independent.

Assumme c_1Qv_1 + ... + c_kQv_k = 0. Thus Q(cv_1 + ...+ Qv_k) = 0. Since Q is invertible, c_1v_1 + ...+ c_kv_k = 0. Since v_1, ..., v_k are independent, c_1 = ... = c_k = 0

2. Let $A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$

(i) Find $m_A(\alpha)$. and all eigenvalues of A.

Answer: $A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} \Rightarrow C_A(\alpha) = |\alpha I_2 - A| = \alpha^2 - 4 = (\alpha - 2)(\alpha +$

$2) = m_A(\alpha)$. This is the product of two linear factors, therefore A is diagonalizable with eigenvalues $\alpha = 2, -2$.

(ii) If A is diagnolizable, then find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

Answer: We have $D = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$. To find Q, we find eigenvectors associated with each eigenvalue.

For $\alpha = 2$, we have $(2I - A) = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$. We find the null space of this matrix using an online calculator, which gives us $E_2(A) = span\{(2,1)\}$.

For $\alpha = -2$, we have $(2I - A) = \begin{bmatrix} -2 & -4 \\ -1 & -2 \end{bmatrix}$. We find the null space of this matrix using an online calculator, which gives us $E_2(A) = span\{(-2,1)\}$.

Therefore $Q = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$ and $Q^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{-1}{4} & \frac{1}{2} \end{bmatrix}$

(iii) Find A^{16} and A^{15}. [Hint: note that if $A = QDQ^{-1}$, then $Am = QD^mQ^{-1}$ and $A^{15} = A^{-1}A^{16}$]

Answer:

$$A^{16} = QD^{16}Q^{-1} = \begin{bmatrix} 2^{16} & 0 \\ 0 & 2^{16} \end{bmatrix} = \begin{bmatrix} 65536 & 0 \\ 0 & 65536 \end{bmatrix}$$

$$A^{15} = A^{-1}A^{16} = \begin{bmatrix} 0 & 2^{16} \\ 2^{14} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 65536 \\ 16384 & 0 \end{bmatrix}$$

(iv) Convince me that $A^9 + A^7 + 4A^5 + 2A^3 + 7I_2 = c_1A + C_2I_2$ for some real numbers c_1, c_2. Then find $A^9 + A^7 + 4A^5 + 2A^3 + 7I_2$.[Hint: Let $f(\alpha) = \alpha^9 + \alpha^7 + 4\alpha^5 + 2\alpha^3 + 7$. Then $f(\alpha) = q(\alpha)m_A(\alpha) + r(\alpha)$ such that $deg(r) < deg(m(\alpha))$. Since $m_A(A) = 0_{2\times 2}$, we have $f(A) = r(A)$.]

Answer: Using polynomial long division, we have

$$\frac{\alpha^9 + \alpha^7 + 4\alpha^5 + 2\alpha^3 + 7}{\alpha^2 - 4} = (\alpha^7 + 5\alpha^5 + 24\alpha^3 + 98\alpha) + \frac{392\alpha + 7}{\alpha^2 - 4}$$

$$\Rightarrow f(\alpha) = q(\alpha)m_A(\alpha) + (392\alpha + 7)$$

Therefore, $A^9 + A^7 + 4A^5 + 2A^3 + 7I_2 = 392A + 7I_2 = \begin{bmatrix} 7 & 1568 \\ 392 & 7 \end{bmatrix}$

3. (i) Let $f(\alpha)$ be a polynomial such that $f(A) = 0_{n\times n}$. Convince me that $m_A(\alpha)|f(\alpha)$.

Answer: Let $f(\alpha)$ be a polynomial such that $f(A) = 0_{n\times n}$. We know that $deg(f(\alpha)) \geq deg(m_A(\alpha))$. Otherwise, $f(\alpha)$ would be the minimum polynomial for A. Suppose $f(\alpha) = q(\alpha)m_A(\alpha) + r(\alpha)$. Then $f(A) = q(A)m_A(A) + r(A) = 0 + r(A) = r(A) = 0$. But since $deg(r(\alpha))$ $\boxed{<}$ $m_A(\alpha)$ that means $r(\alpha)$ would be the minimal polynomial, contradiction. Hence, $r(\alpha) = 0\ \forall\ \alpha$ (i.e. $r(\alpha)$ is the zero polynomial). Therefore, $m_A(\alpha)|f(\alpha)$.

(ii) Up to similarity, classify all 5×5 matrices such that $A^2 - 5A = -6I_5$. [Hint: Let $f(\alpha) = \alpha^2 - 5\alpha + 6$. Hence, by hypothesis, $f(A) = 0_{5\times 5}$. By (i), $m_A(\alpha)|f(\alpha)$. Hence $m_A(\alpha) = \alpha - 2$ OR $m_A(\alpha) = \alpha - 3$ OR $m_A(\alpha) = f(\alpha)$.]

Answer: There are three possibilities for $m_A(\alpha)$. Either $m_A(\alpha) = \alpha - 2$ OR $m_A(\alpha) = \alpha - 3$ OR $m_A(\alpha) = f(\alpha)$.

For $m_A(\alpha) = \alpha - 2$, $A \simeq C(\alpha-2)\oplus C(\alpha-2)\oplus C(\alpha-2)\oplus C(\alpha-2)\oplus C(\alpha-2)$. In fact, A = 2I_5

Similarly, **For** $m_A(\alpha) = \alpha - 3$, $A \simeq C(\alpha-3)\oplus C(\alpha-3)\oplus C(\alpha-3)\oplus C(\alpha-3)\oplus C(\alpha-3)$. Again, A = 3I_5

For $m_A(\alpha) = f(\alpha) = (\alpha-3)(\alpha-2)$, we have several possibilities for $C_A(\alpha)$. $C_A(\alpha) = (\alpha-3)^3(\alpha-2)^2$ OR $C_A(\alpha) = (\alpha-3)^2(\alpha-2)^3$ OR $C_A(\alpha) = (\alpha-3)^4(\alpha-2)^1$ OR $C_A(\alpha) = (\alpha-3)^1(\alpha-2)^4$.

For $C_A(\alpha) = (\alpha-3)^3(\alpha-2)^2$, We have $A \simeq C(\alpha-3)\oplus C((\alpha-3)(\alpha-2))\oplus C((\alpha-3)(\alpha-2))$
$= C(\alpha-3)\oplus C(\alpha^2-5\alpha+6)\oplus C(\alpha^2-5\alpha+6)$.

For $C_A(\alpha) = (\alpha-3)^2(\alpha-2)^3$, We have $A \simeq C(\alpha-2)\oplus C((\alpha-3)(\alpha-2))\oplus C((\alpha-3)(\alpha-2))$
$= C(\alpha-2)\oplus C(\alpha^2-5\alpha+6)\oplus C(\alpha^2-5\alpha+6)$.

For $C_A(\alpha) = (\alpha-3)^4(\alpha-2)$, We have $A \simeq C(\alpha-3)\oplus C(\alpha-3)\oplus C((\alpha-3))\oplus C((\alpha-3)(\alpha-2))$
$= C(\alpha-3)\oplus C(\alpha-3)C(\alpha-3)\oplus \boxed{\oplus} C(\alpha^2-5\alpha+6)$.

For $C_A(\alpha) = (\alpha-3)(\alpha-2)^4$, We have $A \simeq C(\alpha-2)\oplus C(\alpha-2)\oplus C((\alpha-2))\oplus C((\alpha-3)(\alpha-2))$
$= C(\alpha-2)\oplus C(\alpha-2)C(\alpha-2)\oplus C(\alpha^2-5\alpha+6)$.

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(iii) Let A be a 4×4 such that A is similar to $H = \begin{bmatrix} 0 & 0 & 0 & 9 \\ 1 & 0 & 0 & -18 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

(a) Find $C_A(\alpha)$ and $m_A(\alpha)$.

Answer: $A \simeq H = \begin{bmatrix} 0 & 0 & 0 & 9 \\ 1 & 0 & 0 & -18 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow A \simeq C(\alpha^4 - 2\alpha^3 - 8\alpha^2 + 18\alpha - 9)$

$\Rightarrow C_A(\alpha) = m_A(\alpha) = \alpha^4 - 2\alpha^3 - 8\alpha^2 + 18\alpha - 9$

(b) For each eigenvalue a of A find $dim(E_a(A))$[Hint: $C_A(\alpha) = (\alpha^2 - 2\alpha + 1)(\alpha^2 - 9)$]

Answer: $C_A(\alpha) = (\alpha^2 - 1)^2(\alpha - 3)(\alpha + 3)$. Therefore the eigenvalues a are $3, -3$, and 1 and $dim(E_a(A)) = 1$ for each eigenvalue of the A.

(c) Theoretically, how do you construct the columns of the invertible matrix Q where $Q^{-1}AQ = H$?

Answer: By class notes, there exists w in $\mathbb{R}^4$ such that the columns of Q would be $\{w, Aw, A^2w, A^3w\}$ and $\{w, Aw, A^2w, A^3w\}$ is a basis of $\mathbb{R}^4$.

4. (1) Give me an example of a matrix A such that $C_A(\alpha) = (\alpha - 1)^5(\alpha - 4)^3$ and $m_A(\alpha) = (\alpha - 1)^2(\alpha - 4)^2$ where $E_1(A) = 3$ [Hint: Write down your answer as $A = C() \oplus C() \oplus ... \oplus C()$]

Answer: $A = C(\alpha - 1) \oplus C((\alpha - 1)^2(\alpha - 4)) \oplus C((\alpha - 1)^2(\alpha - 4)^2)$

(2) Give me an example of a matrix A such that $C_A(\alpha) = (\alpha - 1)^5(\alpha - 4)^3$ and $m_A(\alpha) = (\alpha - 1)^2(\alpha - 4)^2$ where $E_1(A) = 4$

Answer: $A = C(\alpha-1) \oplus C(\alpha-1) \oplus C((\alpha-1)(\alpha-4)) \oplus C((\alpha-1)^2(\alpha-4)^2)$

(3) Is there a matrix A such that $C_A(\alpha) = (\alpha - 1)^5(\alpha - 4)^3$ and $m_A(\alpha) = (\alpha - 1)^2(\alpha - 4)^2$ where $E_4(A) = 3$? Explain Briefly

Answer: We know that $dim(E_4(C(f_i))) = 1$ for each f_i such that $\Pi_{i=1}^{k} f_i = C_A(\alpha) = (\alpha - 1)^5(\alpha - 4)^3$ and $f_k = m_A(\alpha) = (\alpha - 1)^2(\alpha - 4)^2$ and $f_1|...|f_k$. Since the multiplicity of 4 is 3, and 4 is repeated twice in the minimum polynomial, we can only have one more polynomial with $f_{k-1} = (\alpha - 1)^j(\alpha - 4)$ where $1 \leq j \leq 2$. so, we cannot , i.e., there is no such matrix

5. Given A is similar to $H = C(\alpha-4) \oplus C(\alpha-4) \oplus C(\alpha^2+\alpha-20) \oplus C(\alpha^2+\alpha-20)$

(i) Find $C_A(\alpha)$ and $m_A(\alpha)$.

Answer: $C_A(\alpha) = (\alpha-4)(\alpha-4)(\alpha^2+\alpha-20)(\alpha^2+\alpha-20) = (\alpha-4)^4(\alpha-5)^2$.

$m_A(\alpha) = \alpha^2+\alpha-20 = (\alpha-4)(\alpha+5)$

(ii) For each eigenvalue a of A find $dim(E_a(A))$

Answer: $dim(E_4(A)) = 4$

$dim(E_{-5}(A)) = 2$

(iii) Is A diagonalizable? explain briefly

Answer: Yes, because the minimum polynomial is a product of distinct linear factors with the roots being the distinct eigenvalues. (Another answer: Yes, because $dim(E_a(A))$ is equal to the multiplicity of a for each a eigenvalue of A).

(iv) Explicitly, write down the entries of H.

Answer: $\begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 20 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

6. Given A is similar to $J = J_1(2) \oplus J_1(2) \oplus J_3(2) \oplus J_2(5) \oplus J_4(5) \oplus J_5(7)$.

(i) Find $C_A(\alpha)$ and $m_A(\alpha)$

Answer: We have $C_A(\alpha) = (\alpha-2)(\alpha-2)(\alpha-2)^3(\alpha-5)^2(\alpha-5)^4(\alpha-7)^5 = (\alpha-2)^5(\alpha-5)^6(\alpha-7)^5$

We know that the multiplicity of a in the minimum polynomial is the number associated with the biggest Jordan block for a.

We have $m_A(\alpha) = (\alpha-2)^3(\alpha-5)^4(\alpha-7)^5$

(ii) For each eigenvalue a of A find $dim(E_a(A))$

Answer: We know that $dim(E_a(A))$ is the number of Jordan blocks for a.

$dim(E_2(A)) = 3$

$dim(E_5(A)) = 2$

$dim(E_7(A)) = 1$

(iii) A is similar to a matrix H in rational form. Find H. [Hint: write $H = C() \oplus ... \oplus C()$]

Answer: $H = C(\alpha-2) \oplus C((\alpha-2)(\alpha-5)^2) \oplus C((\alpha-2)^3(\alpha-5)^4(\alpha-7)^5)$

7. Let $T : \mathbb{R}^2 \to P^2$ such that $T(a,b) = bx + 2a + b$. Define $< f1, f2 >_{P^2} = \int_0^1 f_1 f_2 \, dx$ and $< q_1, q_2 >_{\mathbb{R}^2} = q_1 \cdot q_2$. Find T^a (the adjoint operator of T)
Answer: We know that $T^a(cx+d) = (m,n)$. We shall find m, n (in terms of c, d). We have

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$$< T(a,b), cx+d >_{P^2} = < (a,b), (m,n) >_{\mathbb{R}^2}$$

$$\int_0^1 (bx + 2a + b)(cx + d) \, dx = am + bn$$

$$\frac{bc}{3} + ac + \frac{bc}{2} + \frac{bd}{2} + 2ad + bd = am + bn$$

$$b(\frac{5c}{6} + \frac{3d}{2}) + a(2d + c) = am + bn$$

$$\Rightarrow m = (2d + c), n = (\frac{5c}{6} + \frac{3d}{2})$$

Therefore, $T^a(cx+d) = (2d + c, \frac{5c}{6} + \frac{3d}{2})$

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QUESTION 1. Assume A, B are similar $n \times n$ matrices, say $A = Q^{-1}BQ$.

$B = QAQ^{-1}$

(i) We know $C_A(\alpha) = C_B(\alpha)$. Prove $m_A(\alpha) = m_B(\alpha)$.

Take any polynomial $f(\alpha) = \alpha^K + a_{K-1}\alpha^{K-1} + \dots + a_1\alpha + a_0$

$$f(A) = f(Q^{-1}BQ)$$

$$(Q^{-1}BQ)^2 = (Q^{-1}BQ)(Q^{-1}BQ) = (Q^{-1}B)(QQ^{-1})(BQ)$$
$$= (Q^{-1}B)(I)(BQ) = (Q^{-1}B)(BQ)$$
$$= Q^{-1}B^2Q$$

by same method $(Q^{-1}BQ)^n = Q^{-1}B^nQ$

$$\Rightarrow f(A) = Q^{-1}B^KQ + a_{K-1}Q^{-1}B^{K-1}Q + \dots + a_1Q^{-1}BQ + a_0I$$
$$= Q^{-1}(B^K + a_{K-1}B^{K-1} + aB + a_0I)Q$$
$$= Q^{-1}f(B)Q$$

Let $m_A(\alpha)$ and $m_B(\alpha)$ be the minimal polynomials of A and B respectively. then $m_A(B) = m_A(QAQ^{-1}) = Q\,m_A(A)\,Q^{-1}$
$$= Q\,0_{n\times n}\,Q^{-1} = 0_{n\times n}$$

So $m_B(\alpha) \mid m_A(\alpha)$

Similarly $m_B(A) = 0_{n\times n} \Rightarrow m_A(\alpha) \mid m_B(\alpha)$

$$\Rightarrow m_B(\alpha) = m_A(\alpha).$$

(ii) Assume that a is an eigenvalue of A and $\{v_1, v_2, ..., v_k\}$ is a basis for $E_a(A)$. Prove that $\{Qv_1, Qv_2, ..., Qv_k\}$ is a basis for $E_a(B)$. [Hint: Observe that $BQ = QA$ and since Q is invertible, $Qw = 0_n$ iff $w = 0_n$]

To show the $BQv_i = aQv_i \quad \forall\, i \quad 1 \le i \le k$

$BQv_i = QAv_i = Q(av_i)$ (since a is an eigenvalue of A).

$= a(Qv_i)$

$\therefore$ Qv_i is an eigenvector of B.

$\begin{pmatrix} v_1 & v_2 & \cdots & v_k \\ & & & \end{pmatrix} w = 0_v$ iff $w = 0_v$

then

$Q\begin{pmatrix} v_1 & v_2 & \cdots & v_k \\ & & & \end{pmatrix} w = 0_v$ iff $w = 0_v$

$\Rightarrow$ Qv_i are idependent for $1 \le i \le k$

$\Rightarrow$ $\{Qv_1, \ldots, Qv_k\}$ are basis for $E_a(B)$

Since Q and L are invertibe, QL is invertible. Note that QL = MATRIX(Qv_1..... Qv_k). QLw = 0 iff w = 0. see another solution (similar idea as my comments as in Jamila's solution)

QUESTION 2. Let $A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$

(i) Find $m_A(\alpha)$ and all eigenvalues of A.

$C_A(\alpha) = |\alpha I - A| = \begin{vmatrix} \alpha & -4 \\ -1 & \alpha \end{vmatrix} = \alpha^2 - 4 \Rightarrow \alpha = \pm 2$ are the eigenvalues

$\Rightarrow m_A(\alpha) = (\alpha - 2)(\alpha + 2)$

(ii) If A is diagnolizable, then find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

$$E_2(A) = \text{span}\{(2,1)\}$$
$$E_{-2}(A) = \text{span}\{(-2,1)\}$$

$$Q = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}, \quad Q^{-1} = \begin{pmatrix} 1/4 & 1/2 \\ -1/4 & 1/2 \end{pmatrix}$$

$$D = Q^{-1}AQ = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

(iii) Find A^{16} and A^{15}. [Hint: note that if $A = QDQ^{-1}$, then $A^m = QD^mQ^{-1}$ and $A^{15} = A^{-1}A^{16}$]

$$A^{16} = QD^{16}Q^{-1} = Q\begin{pmatrix} 65536 & 0 \\ 0 & 65536 \end{pmatrix}Q^{-1}$$
$$= \begin{pmatrix} 65536 & 0 \\ 0 & 65536 \end{pmatrix}$$

$$A^{15} = \overbrace{\begin{pmatrix} 0 & 1 \\ 1/4 & 0 \end{pmatrix}}^{A^{-1}}\overbrace{\begin{pmatrix} 65536 & 0 \\ 0 & 65536 \end{pmatrix}}^{A^{16}} = \begin{pmatrix} 0 & 65536 \\ 16384 & 0 \end{pmatrix}$$

(iv) Convince me that $A^9 + A^7 + 4A^5 + 2A^3 + 7I_2 = c_1A + C_2I_2$ for some real numbers c_1, c_2. Then find $A^9 + A^7 + 4A^5 + 2A^3 + 7I_2$.[Hint: Let $f(\alpha) = \alpha^9 + \alpha^7 + 4\alpha^5 + 2\alpha^3 + 7$. Then $f(\alpha) = q(\alpha)m_A(\alpha) + r(\alpha)$ such that $deg(r) < deg(m(\alpha))$. Since $m_A(A) = 0_{2\times 2}$, we have $f(A) = r(A)$.]

$$\alpha^7 + 5\alpha^3 + 5\alpha^5 + 24\alpha^3 + 98\alpha \} \; q(\alpha)$$

$$\alpha^2 - 4 \;\big)\; \alpha^9 + \alpha^7 + 4\alpha^5 + 2\alpha^3 + 7$$
$$-\ \alpha^9 + 4\alpha^7$$
$$0 + 5\alpha^7 + 4\alpha^5$$
$$-\ 5\alpha^7 + 20\alpha^5$$
$$0 + 24\alpha^5 + 2\alpha^3$$
$$-\ 24\alpha^5 + 96\alpha^3$$
$$0 + 98\alpha^3 + 7$$
$$-\ 98\alpha^3 + 392\alpha$$
$$392\alpha + 7$$

$$\Rightarrow f(\alpha) = q(\alpha)m_A(\alpha) + r(\alpha)$$
$$f(A) = r(A)$$
$$\Rightarrow c_1 = 392$$
$$c_2 = 7$$

$$A^9 + A^7 + 4A^5 + 2A^3 + 7I_2 = 392A + 7I$$

$$= \begin{pmatrix} 0 & 1568 \\ 392 & 0 \end{pmatrix} + \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 7 & 1568 \\ 392 & 7 \end{pmatrix}$$

QUESTION 3. (i) Let $f(\alpha)$ be a polynomial such that $f(A) = 0_{n\times n}$. Convince me that $m_A(\alpha) \mid f(\alpha)$.

Divide $f(\alpha)$ by $m_A(\alpha)$. So $f(\alpha) = q(\alpha)\, m_A(\alpha) + r(\alpha)$

$$f(A) = q(A)\, m_A(A) + r(A)$$
$$0_{n\times n} = 0_{n\times n} + r(A) \Rightarrow r(A) = 0_{n\times n}$$

$0 \leq \deg(r(\alpha)) < \deg(m_A(\alpha))$ (but $m_A(\alpha)$ is the minimal such that $m_A(A) = 0$).

So $r(\alpha) = 0_V$

$\Rightarrow m_A(\alpha) \mid f(\alpha)$.

(ii) Up to similarity, classify all 5×5 matrices such that $A^2 - 5A = -6I_5$. [Hint : Let $f(\alpha) = \alpha^2 - 5\alpha + 6$. Hence, by hypothesis, $f(A) = 0_{5\times 5}$. By (i), $m_A(\alpha) \mid f(\alpha)$. Hence $m_A(\alpha) = \alpha - 2$ OR $m_A(\alpha) = \alpha - 3$ OR $m_A(\alpha) = f(\alpha)$.]

- if $m_A(\alpha) = \alpha - 2$.

  then, $C_A(\alpha) = (\alpha - 2)^5$

  $A \approx C(\alpha-2) \oplus C(\alpha-2) \oplus C(\alpha-2) \oplus C(\alpha-2)$

- if $m_A(\alpha) = \alpha - 3$

  then, $C_A(\alpha) = (\alpha-3)^5$

  $A \approx C(\alpha-3) \oplus C(\alpha-3) \oplus C(\alpha-3) \oplus C(\alpha-3)$

- if $m_A(\alpha) = \alpha^2 - 5\alpha + 6 = (\alpha-2)(\alpha-3)$

  <u>case 1</u>: $C_A(\alpha) = (\alpha-2)^3(\alpha-3)^2$

  $A \approx C(\alpha-2) \oplus C((\alpha-2)(\alpha-3)) \oplus C((\alpha-2)(\alpha-3))$

Case2: $C_A(\alpha) = (\alpha-2)^2(\alpha-3)^3$

$A \approx C(\alpha-3) \oplus C((\alpha-2)(\alpha-3)) \oplus C((\alpha-2)(\alpha-3))$

Case3: $C_A(\alpha) = (\alpha-2)^4(\alpha-3)^1$

$A \approx C(\alpha-2) \oplus C(\alpha-2) \oplus C(\alpha-2) \oplus C((\alpha-2)(\alpha-3))$

Case4: $C_A(\alpha) = (\alpha-2)^1(\alpha-3)^4$

$A \approx C(\alpha-3) \oplus C(\alpha-3) \oplus C(\alpha-3) \oplus C((\alpha-2)(\alpha-3))$

(iii) Let A be a 4×4 such that A is similar to $H = \begin{bmatrix} 0 & 0 & 0 & 9 \\ 1 & 0 & 0 & -18 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

a. Find $C_A(\alpha)$ and $m_A(\alpha)$.

b. For each eigenvalue a of A find $dim(E_a(A))$[Hint: $C_A(\alpha) = (\alpha^2 - 2\alpha + 1)(\alpha^2 - 9)$]

c. Theoretically, how do you construct the columns of the invertible matrix Q where $Q^{-1}AQ = H$?

a. $C_A(\alpha) = \alpha^4 - 2\alpha^3 - 8\alpha^2 + 18\alpha - 9 = (\alpha-1)^2(\alpha+3)(\alpha-3)$

H is not diagonalizable. so $m_A(\alpha) \neq (\alpha-1)(\alpha+3)(\alpha-3)$

5 $m_A(\alpha) = C_A(\alpha)$ ✓

b. $\dim(E_1(A)) = 1$, $\dim(E_3(A)) = 1$, $\dim(E_{-3}(A)) = 1$

c. Let v_1 be the eigenvector of $\alpha = 1$. then we expand it to a basis $\{v_1, v_2, v_3, v_4\}$ for $\mathbb{R}^4$. then $Q = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ & & & \end{pmatrix}$

X 0

See class notes

See Jamilah's

QUESTION 4. (1) Give me an example of a matrix A such that $C_A(\alpha) = (\alpha-1)^5(\alpha-4)^3$ and $m_A(\alpha) = (\alpha-1)^2(\alpha-4)^2$ where $E_1(A) = 3$ [Hint: Write down your answer as $A = C() \oplus C() \oplus \cdots \oplus C()$]

$f_K = (\alpha-1)^2(\alpha-4)^2$
$f_2 = (\alpha-1)^2(\alpha-4)$
$f_1 = (\alpha-1)$

5

$A = C(\alpha-1) \oplus C((\alpha-1)(\alpha-4)) \oplus C((\alpha-1)^2(\alpha-4)^2)$

(2) Give me an example of a matrix A such that $C_A(\alpha) = (\alpha-1)^5(\alpha-4)^3$ and $m_A(\alpha) = (\alpha-1)^2(\alpha-4)^2$ where $E_1(A) = 4$.

$f_4 = (\alpha-1)^2(\alpha-4)^2$
$f_3 = (\alpha-1)(\alpha-4)$
$f_2 = (\alpha-1)$
$f_1 = (\alpha-1)$

5

$A = C(\alpha-1) \oplus C(\alpha-1) \oplus C((\alpha-1)(\alpha-4)) \oplus C((\alpha-1)^2(\alpha-4)^2)$

(3) Is there a matrix A such that $C_A(\alpha) = (\alpha-1)^5(\alpha-4)^3$ and $m_A(\alpha) = (\alpha-1)^2(\alpha-4)^2$ where $E_4(A)) = 3$? Explain briefly

No, since $\dim(E_4(A)) = 3$, $(\alpha-4)$ must be a factor for 3 f_i's for $0 \le i \le K$

but $f_K = m_A(\alpha) = (\alpha-1)^2(\alpha-4)^2$.

So we only need one more $(\alpha-1)$ to satisfy $f_1 \cdot f_2 \cdot \ldots \cdot f_K = C_A(\alpha)$.

4

QUESTION 5. Given A is similar to $H = C(\alpha-4) \oplus C(\alpha-4) \oplus C(\alpha^2+\alpha-20) \oplus C(\alpha^2+\alpha-20)$

(i) Find $C_A(\alpha)$ and $m_A(\alpha)$.

(ii) For each eigenvalue a of A find $dim(E_a(A))$

(iii) Is A diagnolizable? explain briefly

(iv) Explicitly, write down the entries of H.

(i) $C_A(\alpha) = (\alpha-4)^4(\alpha+5)^2$

$m_A(\alpha) = (\alpha-4)(\alpha+5)$

(ii) $\dim(E_4(A)) = 4$, $\dim(E_{-5}(A)) = 2$

(iii) Yes, because $m_A(\alpha)$ is the product of distinct linear factors.

(iv) $C(\alpha-4) = (4)$

$$C(\alpha^2+\alpha-20) = \begin{pmatrix} 0 & 20 \\ 1 & -1 \end{pmatrix}$$

$$H = (4) \oplus (4) \oplus \begin{pmatrix} 0 & 20 \\ 1 & -1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 20 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \oplus \begin{pmatrix} 0 & 20 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 20 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 20 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Question 6:

(2) Given A is similar to $J = J_1(2) \oplus J_1(2) \oplus J_3(2) \oplus J_2(5) \oplus J_4(5) \oplus J_5(7)$.

(i) Find $C_A(\alpha)$ and $m_A(\alpha)$.

(ii) For each eigenvalue a of A find $dim(E_a(A))$.

(iii) A is similar to a matrix H in rational form. Find H. [Hint: write $H = C() \oplus \cdots \oplus C()$]

(i) $C_A(\alpha) = (\alpha-2)(\alpha-2)(\alpha-2)^3\ (\alpha-5)^2\ \ (\alpha-5)^4\ (\alpha-7)^5$

$m_A(\alpha) = (\alpha-2)^3 (\alpha-5)^4\ (\alpha-7)^5$

(ii) $\dim(E_2(A)) = 3$

$\dim(E_5(A)) = 2$

$\dim(E_7(A)) = 1$

(iii) $f_k = (\alpha-2)^3 (\alpha-5)^4 (\alpha-7)^5$

$C_A(\alpha) = (\alpha-2)^5 (\alpha-5)^6 (\alpha-7)^5$

$f_1 \cdot f_2 \cdot \ldots \cdot f_{k-1} = (\alpha-2)^2 (\alpha-5)^2$

$f_3 = m_A(\alpha)$

$f_2 = (\alpha-2)\ (\alpha-5)^2$

$f_1 = (\alpha-2)$

$H = C(\alpha-2) \oplus C((\alpha-2)(\alpha-5)^2) \oplus C(m_A(\alpha))$

QUESTION 7. Let $T : R^2 \to P_2$ such that $T(a,b) = bx + 2a + b$. Define $< f_1, f_2 >_{P_2} = \int_0^1 f_1 f_2 \, dx$ and $< q_1, q_2 >_{R^2} = q_1 \cdot q_2$. Find T^a (the adjoint operator of T)

$T : \mathbb{R}^2 \to P^2 \qquad V = \mathbb{R}^2 \ , \ (a,b) \in \mathbb{R}^2$

$T^a : P_2 \to \mathbb{R}^2 \qquad W = P_2 \ , \ a_1 x + a_2 \in P_2$

$$\langle T(v), w \rangle_W = \langle v, T^a(w) \rangle_V$$

$$\langle T(a,b) , a_1 x + a_2 \rangle_{P_2} = \langle (a,b) , T^a(a_1 x + a_2) \rangle_{\mathbb{R}^2}$$

$$\langle bx + 2a + b, \ a_1 x + a_2 \rangle_{P_2} = \int_0^1 (bx + 2a + b)(a_1 x + a_2) \, dx$$

$$= \frac{ba_1}{3} + \frac{(ba_2 + 2aa_1 + ba_1)}{2} + (2aa_2 + ba_2) \quad \ldots \quad (1)$$

$\Rightarrow$ Let $T^a(a_1 x + a_2) = (a_3, a_4)$

$$\Rightarrow \langle (a,b) , T^a(a_1 x + a_2) \rangle_{\mathbb{R}^2} = \langle (a,b), (a_3, a_4) \rangle_{\mathbb{R}^2}$$

$$= aa_3 + ba_4 \quad \ldots \quad (2)$$

$(1) = (2)$

$$\frac{ba_1}{3} + \frac{(ba_2 + 2aa_1 + ba_1)}{2} + (2aa_2 + ba_2) = aa_3 + ba_4$$

$$\left(\frac{a_1}{3} + \frac{3a_2}{2} + \frac{a_1}{2} \right) b + (a_1 + 2a_2) a = aa_3 + ba_4$$

$$\Rightarrow \quad a_3 = a_1 + 2a_2$$

$$a_4 = \frac{a_1}{3} + \frac{3a_2}{2} + \frac{a_1}{2}$$

$$T^a(a_1 x + a_2) = \left(a_1 + 2a_2 , \ \frac{5a_1}{6} + \frac{3a_2}{2} \right)$$