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MTH512 Homework 2 12 March 2023 Advanced Linear Algebra

1. Let W be a finite dimensional vector space over a field F such that $dim(W) = n \geq 2024$. Given $T, L : W \to W$ are F-homomorphisms such that

$$dim(Range(T)) = n - 2023$$

and $(T \circ L)(w) = 0_w$ for every $w \in W$. Let $d = dim(Range(L))$. Find the maximum value of d, and explain briefly.

Answer: Since $(T \circ L)(w) = 0_w$ for every $w \in W$, $L(w) \in Ker(T) \forall w \in W$. Hence,

$$Range(L) \subseteq Ker(T)$$

We know that

$$dim(Range(T)) + dim(Ker(T)) = dim(W) = n$$

Therefore, the maximum value for $d = dim(Range(L))$ is going to be:

$$dim(Ker(T)) = dim(W) - dim(Range(T)) = n - (n - 2023) = 2023$$

Therefore, $d \leq 2023$.

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2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be an $\mathbb{R}$-homomorphism. Given $1, -1, 2$ are the eigenvalues of T, such that $E_1(T) = span\{(5, 2, 1)\}$, $E_{-1}(T) = span\{(-5, -1, -1)\}$, and $E_2(T) = span\{(0, 0, 7)\}$. Let $L = T^2 + I$ (where I is the identity map). Then $L : \mathbb{R}^3 \to \mathbb{R}^3$ is an $\mathbb{R}$-Homomorphism.

   (i) Find all eigenvalues of L.
   We know that the eigenvalues a for $T^2 + I$ are given by $a = \alpha^2 + 1$ where α is an eigenvalue of T. Therefore, the eigenvalues of L are $a_1 = (1)^2 + 1 = 2$, $a_2 = (-1)^2 + 1 = 2$, and $a_3 = (2)^2 + 1 = 5$.

   show it

   (ii) For each eigenvalue a of L, find $E_a(L)$.
   Answer: The eigenvectors of T are also eigenvectors of L.
   for $a = 5$, $L((0, 0, 2)) = 5(0, 0, 2) = (0, 0, 1)$. $\Rightarrow E_5(L) = span\{(0, 0, 1)\}$
   for $a = 2$, $L((5, 2, 1)) = (10, 4, 2)$ and $L((-5, -1, -1)) = (-10, -1, -1) \Rightarrow E_2(L) = span\{(5, 2, 1), (-5, -1, -1)\}$.

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   (iii) Let $F = T^2 - I : \mathbb{R}^3 \to \mathbb{R}^3$, then F is an $\mathbb{R}$-homomorphism. Find $Ker(F)$ and $Range(F)$ (nice question)[hint: Find $E_a(F)$ for each eigenvalue a of

F, then stare and scratch your head]
Answer: We have:

$$a_1 = (1)^2 - 1 = 0,\ a_2 = (-1)^2 - 1 = 0, a_3 = (2)^2 - 1 = 3$$

for $a = 3$, $E_3(F) = span\{(0,0,1)\}$.
for $\alpha = 0$, $E_0(F) = span\{(5,2,1), (-5,-1,-1)\}$.
Notice that $E_0(F)$ does not intersect $E_3(F)$. Therefore, we have:

$$Ker(F) = Nul(M_T) = E_0(F) = span\{(5,2,1), (-5,-1,-1)\}$$

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Since $dim(Range(F)) + dim(ker(F)) = 3$, $dim(Range(F)) = 1$. Due to $\mathbb{R}$-isomorphism, We have:

$$Range(F) = span\{(0,0,1)\}$$

Since the eigenspace is invariant under L.

3. Let $T : P2 \to P2$ such that $T(ax+b) = (2a+3b)x+a+b$, and $L : P2 \to P2$ such that $L(ax+b) = (a+4b)x+a+3b$. I claim that $(T \circ L)^{-1} : P2 \to P2$ exists and $(T \circ L)^{-1}(ax+b) =$ (something)$x+$ (something else). Find the something and the something else. [Hint: use the concept of co-linear and translation]
Answer: We know that $P_2 \simeq \mathbb{R}^2$. The co-linear transformation of $T$is $T' : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T((a,b)) = (2a+3b, a+b)$ and the co-linear transformation of L is $L' : \mathbb{R}^2 \to \mathbb{R}^2$ such that $L((a,b)) = (a+4b, a+3b)$. Then, $M_{T'} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$ and $M_{L'} = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}$. We have

$$M_{T' \circ L'} = M_{T'} M_{L'} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 17 \\ 2 & 7 \end{bmatrix}$$

$$M_{(T' \circ L')^{-1}} = M^{-}1_{T' \circ L'} = \begin{bmatrix} 7 & -17 \\ -2 & 5 \end{bmatrix}$$

Therefore, $(T' \circ L')^{-1} = (7a - 17b, -2a + 5b)$. Translating back, we get $(T \circ L)^{-1} = (7a - 17b)x - 2a + 5b$.

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4. Let $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ such that

$$T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} 2a+b+3d & 2b+c+d \\ 0 & 0 \end{bmatrix}$$

(i) For each eigenvalue a of T find $E_a(T)$. [Hint: use the concept of co-linear and translate]

Answer: We know that $\mathbb{R}^{2\times 2} \simeq \mathbb{R}^4$. The co-linear of T is $T' : \mathbb{R}^4 \to \mathbb{R}^4$ such that $T'((a,b,c,d)) = (2a+b+3d, 2b+c+d, 0, 0$. The standard matrix representation of T' is $M_{T'} = \begin{bmatrix} 2 & 1 & 0 & 3 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. For the eigenvalues: Find the roots of $C_{M_{T'}}(a) = det(aI_4 - M_{T'}) = a^2(a-2)^2$. The eigenvalues are $a_1 = 0$ and $a_2 = 2$. To find the eigenspace: For $a = 2$, we find the nullspace of $2I_4 - M_{T'} = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \Rightarrow E_2(T') = span\{(1,0,0,0)\} \Rightarrow$

$E_2(T) = span\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\}$.

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For $a = 0$, we find the nullspace of $0I_4 - M_{T'} = \begin{bmatrix} -2 & 1 & 0 & 3 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$

$E_0(T') = span\{(1,2,4,0),(7,2,0,4)\} \Rightarrow E_0(T) = span\{\begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} 7 & 2 \\ 0 & 4 \end{bmatrix}\}$.

calculation ? see solution

By staring, $\begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix}$ and $\begin{bmatrix} 7 & 2 \\ 0 & 4 \end{bmatrix}$

(ii) Find $Ker(T)$ [Hint: it should be copy-paste from (i)]

Answer:

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$$Ker(T) = E_0(T) = span\{\begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} 7 & 2 \\ 0 & 4 \end{bmatrix}\}$$

Ok on wrong

(iii) Consider the dual-operator T^*. Find a basis for $Range(T^*)$ and a basis for $Ker(T^*)$. [Hint: use the concept of co-linear and translate]

Answer: To find the dual operator $T^* : (\mathbb{R}^{2\times 2})^* \to (\mathbb{R}^{2\times 2})^*$ we have

$$T^*(c_1e_1^* + c_2e_2^* + c_3e_3^* + c_4e_4^*) \sim \begin{bmatrix} 2 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

$$\sim (2c_1, c_1 + 2c_2, c_2, 3c_1 + c_2)$$

$$\sim 2c_1(\overline{e}_1)^* + (c_1 + 2c_2)(\overline{e}_2)^* + c_2(\overline{e}_3)^* + (3c_1 + c_2)(\overline{e}_4)^*$$

Where $\{(\overline{e}_1)^*, (\overline{e}_2)^*, (\overline{e}_3)^*, (\overline{e}_4)^*\}$ is the standard ordered basis of $(R^{2\times 2})^*$.

$$Range(T^*) = span\{2(\overline{e}_1)^* + (\overline{e}_2)^* + 3(\overline{e}_4)^*, 2(\overline{e}_2)^* + (\overline{e}_3)^* + (\overline{e}_4)^*\}$$

$$Ker(T^*) = span\{e_3^*, e_4^*\}$$

By staring, the set $\{2(\overline{e}_1)^* + (\overline{e}_2)^* + 3(\overline{e}_4)^*, 2(\overline{e}_2)^* + (\overline{e}_3)^* + (\overline{e}_4)^*\}$ and the set $\{e_3^*, e_4^*\}$ are linearly independent sets. Therefore, a basis for $Range(T^*)$ is $\{2(\overline{e}_1)^* + (\overline{e}_2)^* + 3(\overline{e}_4)^*, 2(\overline{e}_2)^* + (\overline{e}_3)^* + (\overline{e}_4)^*\}$ and a basis for $Ker(T^*)$ is $\{e_3^*, e_4^*\}$.

5. Give me an example of a matrix $A \in \mathbb{R}^{2\times 2}$ such that A has no real eigenvalues, but all eigenvalues of A^2 are real numbers.

Answer: Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A has no real eigenvalues as $C_A(\alpha) = \alpha^2 + 1$ has no real roots. However, A^2 has real eigenvalues as $C_{A^2}(\alpha) = (\alpha - 1)^2$ has one real root $\alpha = 1$.