ON n-SEMIPRIMARY IDEALS AND n-PSEUDO VALUATION DOMAINS

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Abstract. Let R be a commutative ring with $1 \neq 0$ and n a positive integer. A proper ideal I of R is an *n-semiprimary ideal* of R if whenever $x^n y^n \in I$ for $x, y \in R$, then $x^n \in I$ or $y^n \in I$. Let R be an integral domain with quotient field K. A proper ideal I of R is an *n-powerful ideal* of R if whenever $x^n y^n \in I$ for $x, y \in K$, then $x^n \in R$ or $y^n \in R$; and I is an *n-powerful semiprimary ideal* of R if whenever $x^n y^n \in I$ for $x, y \in K$, then $x^n \in I$ or $y^n \in I$. If every prime ideal of R is an n-powerful semiprimary ideal of R, then R is an *n-pseudo-valuation domain* (*n-PVD*). In this paper, we study the above concepts and relate them to several generalizations of pseudo-valuation domains.

1. introduction

Let R be a commutative ring with $1 \neq 0$ and n a positive integer. Recall that an ideal I of R is a *semiprimary ideal* of R if $\sqrt{I}$ is a prime ideal of R. In this paper, we introduce and study n-semiprimary ideals (resp., n-powerful semiprimary ideals in integral domains), where a proper ideal I of R is *n-semiprimary* (resp., *n-powerful semiprimary*) if whenever $x^n y^n \in I$ for $x, y \in R$ (resp., $x, y \in K$, the quotient field of R), then $x^n \in I$ or $y^n \in I$. These concepts generalize prime ideals and are generalized by semiprimary ideals. We also investigate several other "n" generalizations obtained by replacing x with x^n in the definition.

In Section 2, we give some basic properties of n-semiprimary ideals. For example, we show that an n-semiprimary ideal is semiprimary, and the converse holds when R is Noetherian. We also show that an n-semiprimary ideal is m-semiprimary for every integer $m \geq n$. In Section 3, we characterize n-semiprimary ideals in several classes of commutative rings. In particular, we investigate n-semiprimary ideals in zero-dimensional commutative rings, Dedekind domains, valuation domains, and idealizations. In Section 4, we study n-powerful semiprimary ideals in integral domains and introduce n-pseudo-valuation domains (n-PVDs), a generalization of pseudo-valuation domains (PVDs). We also study n-valuation domains (n-VDs). In the final section, Section 5, we introduce pseudo n-valuation domains (PnVDs), another generalization of PVDs. Many examples are given throughout the paper to illustrate the theory.

Throughout, R will be a commutative ring with $1 \neq 0$, $\sqrt{I} = \{x \in R \mid x^n \in I$ for some $n \in \mathbb{N}\}$ for I an ideal of R, ideal of nilpotent elements $nil(R) = \sqrt{\{0\}}$, group of units $U(R)$, (Krull) dimension $dim(R)$, and characteristic $char(R)$. An overring

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of an integral domain R with quotient field K is a subring of K containing R, and we denote the integral closure of R (in K) by $\overline{R}$. In particular, if I is an ideal of R, then $(I : I) = \{x \in K \mid xI \subseteq I\}$ is an overring of R. Other definitions will be given throughout the paper as needed. As usual, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{F}_{p^n}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ will denote the set of positive integers, the rings of integers and integers mod n, the finite field with p^n elements, and the fields of rational numbers, real numbers, and complex numbers, repectively. For any undefined terminology, see [23], [26], [27], or [28].

2. Basic properties of n-semiprimary ideals

In this section, we give some basic properties of n-semiprimary ideals. We begin with the definition.

Definition 2.1. Let I be a proper ideal of a commutative ring R and n a positive integer. Then I is an *n-semiprimary ideal* of R if whenever $x^n y^n \in I$ for $x, y \in R$, then $x^n \in I$ or $y^n \in I$.

Note that a 1-semiprimary ideal is just a prime ideal. For convenience, call a commutative ring R an *n-ring* if $x^n y^n = 0$ for $x, y \in R$ implies $x^n = 0$ or $y^n = 0$. Then a 1-ring is just an integral domain, R is an n-ring if and only if $\{0\}$ is an n-semiprimary ideal of R, and R/I is an n-ring if and only I is an n-semiprimary ideal of R. We start with some elementary results that follow directly from the definitions.

Theorem 2.2. *Let I be a proper ideal of a commutative ring R.*

(a) *Let I be an n-semiprimary ideal of R. Then I is an mn-semiprimary ideal of R for every positive integer m.* (*See Theorem 2.14 for a stronger result.*)

(b) *Let $J \subseteq I$ be proper ideals of R. Then I is an n-semiprimary ideal of R if and only if I/J is an n-semiprimary ideal of R/J.*

(c) *Let I be an n-semiprimary ideal of R and S a multiplicatively closed subset of R with $I \cap S = \emptyset$. Then I_S is an n-semiprimary ideal of R_S.*

We next show that an n-semiprimary ideal is indeed semiprimary.

Theorem 2.3. *Let I be an n-semiprimary ideal of a commutative ring R. Then $\sqrt{I}$ is a prime ideal of R and $x^n \in I$ for every $x \in \sqrt{I}$. In particular, I is a semiprimary ideal of R, and $x \in \sqrt{I}$ if and only if $x^n \in I$.*

Proof. Let $xy \in \sqrt{I}$ for $x, y \in R$. Then there is a positive integer k such that $(x^k)^n (y^k)^n = (xy)^{kn} \in I$. Thus $x^{kn} = (x^k)^n \in I$ or $y^{kn} = (y^k)^n \in I$ since I is an n-semiprimary ideal of R. Hence $x \in \sqrt{I}$ or $y \in \sqrt{I}$; so $\sqrt{I}$ is a prime ideal of R. Let $x \in \sqrt{I}$ and m be the least positive integer such that $x^{mn} \in I$. Then $x^n (x^{m-1})^n = x^n x^{(m-1)n} = x^{mn} \in I$, and thus $x^n \in I$ or $x^{(m-1)n} \in I$ since I is an n-semiprimary ideal of R. Hence $m = 1$; so $x^n \in I$. The "in particular" statement is clear. □

The following is an example of a semiprimary ideal of a commutative ring R that is not an n-semiprimary ideal for any positive integer n. Note that R is *not* Noetherian. In fact, Corollary 2.6 shows that semiprimary ideals in a commutative Noetherian ring are n-semiprimary for all large n.

Example 2.4. Let $R = \mathbb{Z}_2[\{X_n\}_{n=1}^{\infty}]$ and $I = (\{X_n^n\}_{n=1}^{\infty})$. Then $\sqrt{I} = (\{X_n\}_{n=1}^{\infty})$ is a prime ideal of R; so I is a semiprimary ideal of R. However, I is not an n-semiprimary ideal of R for any positive integer n since $X_{2n}^n X_{2n}^n = X_{2n}^{2n} \in I$, but $X_{2n}^n \notin I$.

The next theorem gives a sufficient condition for a semiprimary ideal to be an n-semiprimary ideal. As a consequence, n-absorbing semiprimary ideals are n-semiprimary and semiprimary ideals in commutative Noetherian rings are n-semiprimary for all large n.

Theorem 2.5. *Let I be a proper ideal of a commutative ring R such that $P = \sqrt{I}$ is a prime ideal of R and $P^n \subseteq I$ for a positive integer n. Then I is an m-semiprimary ideal of R for every integer $m \geq n$. In particular, Q^n is an m-semiprimary ideal of R for every prime ideal Q of R and integer $m \geq n$.*

Proof. Let $x^n y^n \in I \subseteq P$ for $x, y \in R$. Then $x \in P$ or $y \in P$. Thus $x^n \in P^n \subseteq I$ or $y^n \in P^n \subseteq I$, and hence I is an n-semiprimary ideal of R. Moreover, $P^m \subseteq P^n \subseteq I$ for every integer $m \geq n$; so I is also an m-semiprimary ideal of R for every integer $m \geq n$. The "in particular" statement is clear. □

Corollary 2.6. *Let I be a semiprimary ideal of a commutative Noetherian ring R. Then there is a positive integer n such that I is an m-semiprimary ideal of R for every integer $m \geq n$.*

Proof. Since I is a semiprimary ideal of R, $P = \sqrt{I}$ is a prime ideal of R, and $P^n \subseteq I$ for some positive integer n since P is finitely generated. Thus I is an m-semiprimary ideal of R for every integer $m \geq n$ by Theorem 2.5. □

Recall ([15], [9]) that a proper ideal I of a commutative ring R is an *n-absorbing ideal* of R if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$, then the product of n of the x_i*'s* is in I (for a related concept, also see [10]). Both n-semiprimary and n-absorbing ideals generalize prime ideals, but in rather different ways. An n-semiprimary ideal need not be an n-absorbing ideal (see Example 2.9); and an n-absorbing ideal need not be n-semiprimary since, for example, (6) is a 2-absorbing ideal of $\mathbb{Z}$, but not a 2-semiprimary ideal since $\sqrt{(6)} = (6)$ is not a prime ideal of $\mathbb{Z}$. However, we next show that if $\sqrt{I}$ is a prime ideal, then an n-absorbing ideal I is n-semiprimary.

Corollary 2.7. *Let I be an n-absorbing ideal of a commutative ring R. If $\sqrt{I}$ is a prime ideal of R, then I is an m-semiprimary ideal of R for every integer $m \geq n$. In particular, an n-absorbing ideal is n-semiprimary if and only if it is semiprimary.*

Proof. Let $P = \sqrt{I}$ be a prime ideal of R. Then $P^n = (\sqrt{I})^n \subseteq I$ since I is an n-absorbing ideal of R ([18], [22]). Thus I is an m-semiprimary ideal of R for every integer $m \geq n$ by Theorem 2.5. The "in particular" statement now follows from Theorem 2.3. □

Corollary 2.8. *Let $P_1 \subseteq \cdots \subseteq P_k$ be prime ideals of a commutative ring R and $n_1, \ldots, n_k$ positive integers. Then $I = P_1^{n_1} \cdots P_k^{n_k}$ is an m-semiprimary ideal of R for every integer $m \geq n_1 + \cdots + n_k$.*

Proof. Note that $\sqrt{I} = P_1$ is a prime ideal of R and $P_1^n \subseteq P_1^{n_1} \cdots P_k^{n_k} = I$, where $n = n_1 + \cdots + n_k$. Thus I is an m-semiprimary ideal of R for every integer $m \geq n$ by Theorem 2.5. □

The converse of Theorem 2.5 need not be true, i.e., if I is an n-semiprimary ideal of R for some integer $n \geq 2$, then $(\sqrt{I})^n$ need not be a subset of I. Let $p \geq 2$ be a prime integer. In the following example, we show that there is a proper ideal I of a commutative ring R such that I is a p-semiprimary ideal of R, but $(\sqrt{I})^p \nsubseteq I$, and thus I is not a p-absorbing ideal of R ([18], [22]).

Example 2.9. Let $p \geq 2$ be a prime integer, $R = \mathbb{Z}_p[X, Y]$, and $I = (X^p, Y^p)$. Then I is a proper ideal of R with prime ideal $P = \sqrt{I} = (X, Y)$ and $P^p \nsubseteq I$ since $YX^{p-1} \notin I$. Thus I is not a p-absorbing ideal of R ([18], [22]). Let $f^p g^p \in I \subseteq (X, Y)$ for $f, g \in R$. Then $f \in (X, Y)$ or $g \in (X, Y)$; so $f^p \in I$ or $g^p \in I$, and hence I is a p-semiprimary ideal of R.

Recall [19] that a proper ideal I of a commutative ring R is a *uniformly primary ideal* of R if there is a positive integer n such that whenever $xy \in I$ for $x, y \in R$, then $x \in I$ or $y^n \in I$. If I is a uniformly primary ideal of R for a positive integer n, then we say that I is an *n-primary ideal* of R. By the following theorem, an n-primary ideal is also n-semiprimary.

Theorem 2.10. *Let I be an n-primary ideal of a commutative ring R. Then I is an n-semiprimary ideal of R.*

Proof. Let $x^n y^n \in I$ for $x, y \in R$ with $x^n \notin I$, and let m be the least positive integer such that $x^n y^m \in I$. Then $(x^n y^{m-1})y = x^n y^m \in I$. Since $x^n y^{m-1} \notin I$ and I is an n-primary ideal of R, we have $y^n \in I$. Thus I is an n-semiprimary ideal of R. □

In the following example, we show that there is a commutative ring R with ideals $\{I_n\}_{n=2}^{\infty}$ such that every I_n is an n-semiprimary ideal of R with $(\sqrt{I_n})^n \subseteq I_n$, but I_n is not a primary ideal of R. In particular, I_n is not an m-primary ideal of R for any positive integer m.

Example 2.11. Let $R = \mathbb{Z}_2[X, Y]$. For every integer $n \geq 2$, $I_n = (XY, Y^n)$ is an ideal of R with prime ideal $P = \sqrt{I_n} = (Y)$. Thus I_n is an n-semiprimary ideal of R by Theorem 2.5 since $P^n \subseteq I_n$. However, $YX \in I_n$, $Y \notin I_n$, and $X^m \notin I_n$ for every positive integer m; so I_n is not a primary ideal of R, and hence I_n is not an m-primary ideal of R for any positive integer m.

The next definition generalizes the "n-semiprimary" concept from elements to ideals.

Definition 2.12. Let I be a proper ideal of a commutative ring R and n a positive integer. Then I is a *strongly n-semiprimary ideal* of R if whenever $J^n K^n \subseteq I$ for proper ideals J and K of R, then $J^n \subseteq I$ or $K^n \subseteq I$.

A strongly 1-semiprimary ideal is just a prime ideal, a strongly n-semiprimary ideal is an n-semiprimary ideal, and a strongly n-semiprimary ideal is also strongly mn-semiprimary for every positive integer m. However, the following example shows that an n-semiprimary ideal need not be strongly n-semiprimary.

Example 2.13. Let $R = \mathbb{Z}_2[X, Y]$ and $I = (X^2, Y^2)$. By Example 2.9, I is a 2-semiprimary ideal of R with prime ideal $P = \sqrt{I} = (X, Y)$. Clearly, $P^2 P^2 = P^4 \subseteq I$, but $P^2 \nsubseteq I$. Thus I is not a strongly 2-semiprimary ideal of R. Note that I is an n-semiprimary ideal of R for every integer $n \geq 3$ by Theorem 2.5 since $P^3 \subseteq I$, and hence I is an n-semiprimary ideal of R for every integer $n \geq 2$.

We have already observed in Theorem 2.2 that an n-semiprimary ideal is also mn-semiprimary for every positive integer m. We next give a much stronger result.

Theorem 2.14. *Let I be an n-semiprimary ideal of a commutative ring R.*

(a) *If $x^m y^k \in I$ for $x, y \in R$ and positive integers m and k, then $x^n \in I$ or $y^n \in I$. In particular, if $x^m \in I$ for $x \in R$ and m a postive integer, then $x^n \in I$.*

(b) *I is an m-semiprimary ideal of R for every positive integer $m \geq n$.*

Proof. (a) Let $x^m y^k \in I$ for $x, y \in R$; we may assume that $m \geq k$. Then $(xy)^m = x^m y^m = (x^m y^k) y^{m-k} \in I$. Thus $xy \in \sqrt{I}$; so $x^n y^n = (xy)^n \in I$ by Theorem 2.3. Hence $x^n \in I$ or $y^n \in I$ since I is an n-semiprimary ideal of R. The "in particular" statement is clear.

(b) Let $x^m y^m \in I$ for $x, y \in R$ with $m \geq n$. Then $x^n \in I$ or $y^n \in I$ by part (a). Thus $x^m = x^{m-n} x^n \in I$ or $y^m = y^{m-n} y^n \in I$ since $m \geq n$; so I is an m-semiprimary ideal of R. □

An ideal may be n-semiprimary for many different values of n. We now make that statement more precise. For a proper ideal I of a commutative ring R, let $W_R(I) = \{n \in \mathbb{N} \mid I \text{ is an } n\text{-semiprimary ideal of } R\}$ and $\delta_R(I) = minW_R(I)$ (let $\delta_R(I) = \infty$ if $W_R(I) = \emptyset$). Then $W_R(I) = [\delta_R(I), \infty) \cap \mathbb{N}$ by Theorem 2.14(b).

3. n-SEMIPRIMARY IDEALS IN SOME CLASSES OF RINGS

In this section, we study n-semiprimary ideals in several important classes of commutative rings. We have already observed in Corollary 2.6 that for commutative Noetherian rings, a semiprimary ideal is n-semiprimary for all large n. The first two results concern the case when $dim(R) = 0$.

Theorem 3.1. *Let $I \supseteq nil(R)$ be an ideal of a commutative ring R with $dim(R) = 0$. Then I is an n-semiprimary ideal of R if and only if I is a prime ideal of R (i.e., I is a 1-semiprimary ideal of R).*

Proof. A prime ideal is certainly n-semiprimary for every positive integer n. Conversely, we show that an n-semiprimary ideal I of R is a prime ideal of R. Let $xy \in I$ for $x, y \in R$; so $x^n y^n \in I$. Then $x^n \in I$ or $y^n \in I$; say $x^n \in I$. Since $dim(R) = 0$, we have $x = eu + w$ for an idempotent $e \in R$, $u \in U(R)$, and $w \in nil(R)$ [13, Corollary 1]. Thus $x^n = (eu+w)^n = eu^n + a_1 eu^{n-1} w + a_2 eu^{n-2} w^2 + \cdots + a_{n-1} euw^{n-1} + w^n = e(u^n + a_1 u^{n-1} w + a_2 u^{n-2} w^2 + \cdots + a_{n-1} uw^{n-1}) + w^n \in I$, where the a_i's are positive integers, and $v = u^n + a_1 u^{n-1} w + a_2 u^{n-2} w^2 + \cdots + a_{n-1} uw^{n-1} \in U(R)$. Hence $x^n = (eu + w)^n = ev + w^n$ with $w^n \in nil(R) \subseteq I$. Thus $ev = x^n - w^n \in I$, and hence $eu = (ev)(v^{-1} u) \in I$. Thus $x = eu + w \in I$; so I is a prime ideal of R. □

Corollary 3.2. *Let R be a commutative von-Neumann regular ring. Then a proper ideal I of R is an n-semiprimary ideal of R if and only if I is a prime ideal of R.*

Proof. A commutative ring R is von Neumann regular if and only if $nil(R) = \{0\}$ and $dim(R) = 0$ [26, page 5]. □

However, if I is an n-semiprimary ideal of a zero-dimensional commutative ring R for some integer $n \geq 2$ and $nil(R) \nsubseteq I$, then I need not be a prime ideal of R. We have the following example.

Example 3.3. Let $R = \mathbb{Z}_4 \times \mathbb{Z}_2$. Then $dim(R) = 0$ and $I = \{0\} \times \mathbb{Z}_2$ is a 2-semiprimary ideal of R with $nil(R) = \{0, 2\} \times \{0\} \nsubseteq I$. However, I is not a prime ideal of R.

It is easy to determine the n-semiprimary ideals in a Dedekind domain R since every nonzero proper ideal of R is (uniquely) a product of prime (maximal) ideals [28, Theorem 6.16].

Theorem 3.4. *Let I be a nonzero proper ideal of a Dedekind domain R. Then I is an n-semiprimary ideal of R if and only if $I = P^k$, where $P = \sqrt{I}$ is a prime (maximal) ideal of R and $n \geq k$. Moreover, $\delta_R(I) = n$ if and only if $I = P^n$.*

Proof. Let I be a nonzero proper ideal of a Dedekind domain R. Then $\sqrt{I} = P$ is a prime (maximal) ideal if and only if $I = P^k$ for some positive integer k. Thus by Theorem 2.3 and Theorem 2.5, I is n-semiprimary if and only if $I = P^k$ for some positive integer k, where $n \geq k$. The "in particular" statement is clear. □

Next, we give a characterization of Dedekind domains in terms of 2-semiprimary ideals.

Theorem 3.5. *Let R be a Noetherian integral domain. Then the following statements are equivalent.*

(1) *R is a Dedekind domain.*
(2) *If I is an ideal of R with $\delta_R(I) = 2$, then $I = M^2$ for some maximal ideal M of R.*

Proof. (1) ⇒ (2) This follows directly from Theorem 3.4.

(2) ⇒ (1) Let I be an ideal of R with $M^2 \subseteq I \subsetneq M$ for a maximal ideal M of R. Then I is 2-semiprimary by Theorem 2.5 and not prime (maximal); so $\delta_R(I) = 2$. Thus $I = M^2$ by hypothesis. Hence there are no ideals of R strictly between M and M^2 for every maximal ideal M of R; so R is a Dedekind domain by [28, Theorem 6.20]. □

It is also easy to describe the n-semiprimary ideals in a valuation domain. Recall that every proper ideal in a valuation domain is semiprimary [23, Theorem 17.1(2)].

Theorem 3.6. *Let I be a proper ideal of a valuation domain R with $P = \sqrt{I}$.*

(a) *I is an n-semiprimary ideal of R if and only if $P^n \subseteq I$.*

(b) *If P is idempotent, then I is an n-semiprimary ideal of R if and only if $I = P$.*

(c) *If P is not idempotent, then I is an n-semiprimary ideal of R for some positive integer n. Moreover, every ideal of R between P and the prime ideal directly below P is an n-semiprimary ideal for some positive integer n.*

Proof. (a) If $P^n \subseteq I$, then I is n-semiprimary by Theorem 2.5. Conversely, suppose that I is n-semiprimary. Then $x^n \in I$ for every $x \in P$ by Theorem 2.3; so $P^n = \{rx^n \mid r \in R, x \in P\} \subseteq I$ (cf. [12, Proposition 2.1 and Corollary 2.2]).

(b) This follows directly from part (a).

(c) If $P = \sqrt{I}$ is not idempotent, then $P^n \subseteq I$ for some positive integer n [23, Theorem 17.1(5)], and thus I is n-semiprimary by Theorem 2.5. For the "moreover" statement, $P^n \subseteq I$ for some positive integer n since the prime ideal directly below P is $Q = \cap_{n=1}^{\infty} P^n$ [23, Theorem 17.1(3)(4)]. □

The following example illustrates the possible behavior of n-semiprimary ideals in valuation domains R with $dim(R) \leq 2$. The details follow directly from Theorem 3.6 and well-known facts about the value group of a valuation domain (cf. [23, Chapter 3]). It is interesting to compare Theorem 3.6 (resp., Example 3.7) with [9, Theorem 5.5] (resp., [9, Example 5.6]) which concerns n-absorbing ideals in a valuation domain. There are n-semiprimary ideals that are not n-absorbing ideals in some valuation domains R since I is an n-semiprimary (resp., n-absorbing) ideal of a valuation domain R if and only if $P^n \subseteq I$ (resp., $P^n = I$).

Example 3.7. (a) Let R be a one-dimensional valuation domain with maximal ideal M. If M is principal, then R is a DVR, and thus every proper ideal of R is an n-semiprimary ideal for some positive integer n. If M is not principal, then $M^2 = M$, and hence $\{0\}$ and M are the only proper ideals of R that are n-semiprimary for some positive integer n.

(b) Let R be a two-dimensional valuation domain with prime ideals $\{0\} \subsetneq P \subsetneq M$ and value group G. If $G = \mathbb{Z} \oplus \mathbb{Z}$ (all direct sums have the lexicographic order), then $P^2 \neq P$ and $M^2 \neq M$; so every proper ideal of R is n-semiprimary for some positive integer n. If $G = \mathbb{Q} \oplus \mathbb{Q}$, then $P^2 = P$ and $M^2 = M$; so $\{0\}$, P, and M are the only ideals of R that are n-semiprimary for some positive integer n. If $G = \mathbb{Z} \oplus \mathbb{Q}$, then $P^2 \neq P$ and $M^2 = M$; so M and every ideal of R contained in P is n-semiprimary for some positive integer n, but no ideal properly between P and M is n-semiprimary for any positive integer n. If $G = \mathbb{Q} \oplus \mathbb{Z}$, then $P^2 = P$ and $M^2 \neq M$; so every ideal of R between P and M is n-semiprimary for some positive integer n, but $\{0\}$ and P are the only ideals of R contained in P that are n-semiprimary for some positive integer n.

We end this section with two results on idealization. Let M be an R-module over a commutative ring R. The *idealization* of M is the commutative ring $R(+)M = R \times M$ with addition and multiplication defined by $(a, m) + (b, n) = (a + b, m + n)$ and $(a, m)(b, n) = (ab, bm+an)$, respectively, and identity $(1, 0)$ (cf. [2], [26, Section 25]). Note that $(\{0\}(+)M)^2 = \{0\}$; so $\{0\}(+)M \subseteq nil(R(+)M)$.

Theorem 3.8. *Let I be a proper ideal of a commutative ring R, M an R-module, and $S = IM$ a submodule of M. If I is an n-semiprimary ideal of R, then $I(+)S$ is an $(n + 1)$-semiprimary ideal of $R(+)M$. Moreover, if $I(+)S$ is an n-semiprimary ideal of $R(+)M$, then I is an n-semiprimary ideal of R.*

Proof. Let I be an n-semiprimary ideal of R and $(a, m)^{n+1}(b, h)^{n+1} = (a^{n+1}b^{n+1}, z) \in I(+)S$ for $(a, m), (b, h) \in R(+)M$. Then $a^n \in I$ or $b^n \in I$ by Theorem 2.14(a) since I is an n-semiprimary ideal of R. We may assume that $a^n \in I$; so $(n + 1)a^n m \in IM = S$. Thus $(a, m)^{n+1} = (a^{n+1}, (n + 1)a^n m) \in I(+)S$; so $I(+)S$ is an $(n + 1)$-semiprimary ideal of $R(+)M$. The "moreover" statement is clear. □

Theorem 3.9. *Let I a proper ideal of a commutative ring R with $char(R) = n \geq 2$, M an R-module, and S a submodule of M. Then $I(+)S$ is an n-semiprimary ideal of $R(+)M$ if and only if I is an n-semiprimary ideal of R.*

Proof. If $J = I(+)S$ is an n-semiprimary ideal of $A = R(+)M$, then clearly I is an n-semiprimary ideal of R. Conversely, assume that I is an n-semiprimary ideal of R. Let $(a, m)^n(b, h)^n = (a^n b^n, z) \in J$ for $(a, m), (b, h) \in A$. Then $a^n \in I$ or $b^n \in I$ since I is an n-semiprimary ideal of R; assume that $a^n \in I$. Since $char(R) = n \geq 2$, we have $na^{n-1}m = 0 \in S$. Thus $(a, m)^n = (a^n, na^{n-1}m) = (a^n, 0) \in J$; so J is an n-semiprimary ideal of A. □

4. n-powerful semiprimary ideals and n-PVDs

In this section, we study n-powerful semiprimary ideals in integral domains and two generalizations of valuation domains, namely, n-pseudo-valuation domains (n-PVDs) and n-valuation domains (n-VDs).

Recall [17] (resp., [24]) that a proper ideal I of an integral domain R with quotient field K is *powerful* (resp., *strongly prime*) if whenever $xy \in I$ for $x, y \in K$, then $x \in R$ or $y \in R$ (resp., $x \in I$ or $y \in I$). We begin with an "n" generalization.

Definition 4.1. Let R be an integral domain with quotient field K and n a positive integer. A proper ideal I of R is an *n-powerful ideal* of R if whenever $x^n y^n \in I$ for $x, y \in K$, then $x^n \in R$ or $y^n \in R$; and I is an *n-powerful semiprimary ideal* of R if whenever $x^n y^n \in I$ for $x, y \in K$, then $x^n \in I$ or $y^n \in I$.

Thus a 1-powerful (resp., 1-powerful semiprimary) ideal is just a powerful (resp., strongly prime) ideal, and an n-powerful (resp., n-powerful semiprimary) ideal is also an mn-powerful (resp., mn-powerful semiprimary) ideal for every positive integer m. It is well known that prime ideals in a valuation domain are strongly prime ideals. From this observation, it easily follows that n-semiprimary ideals in a valuation domain are also n-powerful semiprimary ideals; so Theorem 3.6 and Example 3.7 also hold for n-powerful semiprimary ideals. However, an n-semiprimary ideal need not be an n-powerful semiprimary ideal. For example, let $R = \mathbb{Z}_2[[X^2, X^3]]$. Then its maximal ideal $M = (X^2, X^3)$ is a prime (1-semiprimary) ideal, but not a strongly prime (1-powerful semiprimary) ideal. Also, see Example 4.5 for a 2-semiprimary ideal that is not 2-powerful semiprimary.

We next give a stronger result.

Theorem 4.2. *Let R be an integral domain with quotient field K.*

(a) *Let I be an n-semiprimary ideal of R. If $\sqrt{I}$ is a strongly prime ideal of R, then I is an n-powerful semiprimary ideal of R.*

(b) *Let $I \subseteq J$ be proper ideals of R. If J is an n-powerful ideal of R, then I is an n-powerful ideal of R.*

(c) *Let I be an n-powerful (resp.,n-powerful semiprimary) ideal of R and S a multiplicatively closed subset of R with $I \cap S = \emptyset$. Then I_S is an n-powerful (resp., n-powerful semiprimary) ideal of R_S.*

Proof. (a) Let $P = \sqrt{I}$ and $x^n y^n \in I \subseteq P$ for $x, y \in K$. Then $x \in P$ or $y \in P$ since P is a strongly prime ideal of R. Thus $x^n \in I$ or $y^n \in I$ by Theorem 2.3; so I is an n-powerful semiprimary ideal of R.

(b) Let $x^n y^n \in I \subseteq J$ for $x, y \in K$. Then $x^n \in R$ or $y^n \in R$ since J is an n-powerful deal of R. Thus I is an n-powerful ideal of R.

(c) This follows easily from the definitions. □

Note that every ideal in a valuation domain is powerful; so an n-powerful ideal need not be n-powerful semiprimary. However, for prime ideals, these two concepts coincide.

Theorem 4.3. *Let I be a prime ideal of an integral domain R with quotient field K. Then I is an n-powerful semiprimary ideal of R if and only if I is an n-powerful ideal of R.*

Proof. If I is an n-powerful semiprimary ideal of R, then I is certainly an n-powerful ideal. Conversely, assume that I is an n-powerful prime ideal of R. Let $x^n y^n \in I$ for $x, y \in K$. First, suppose that $x^n, y^n \in R$. Since I is a prime ideal of R, then $x^n \in I$ or $y^n \in I$. Thus we may assume that $x^n \notin R$, and hence $y^n \in R$ since I is an n-powerful ideal of R. Since $x^n \notin R$ and I is an n-powerful ideal of R, we have $x^{2n} = x^n x^n \notin I$. Assume that $x^{2n} \in R$; so $y^{2n}, x^{2n} \in R$. Since $x^{2n} y^{2n} \in I$ and $x^{2n} \notin I$, we have $y^{2n} \in I$. Since $y^n \in R$, I is a prime ideal of R, and $y^n y^n = y^{2n} \in I$, we have $y^n \in I$. Now, assume that $x^{2n} \notin R$. Since $(y^2/xy)^n (x^2)^n = (y^{2n}/x^n y^n) x^{2n} = x^n y^n \in I$, $x^{2n} \notin R$, and I is an n-powerful ideal of R, we have $y^{2n}/x^n y^n \in R$. Thus $y^{2n} = x^n y^n (y^{2n}/x^n y^n) \in I$. Since $y^n \in R$, I is a prime ideal of R, and $y^{2n} = y^n y^n \in I$, we have $y^n \in I$. Hence I is an n-powerful semiprimary ideal of R. □

Theorem 4.4. *Let $P \subseteq Q$ be prime ideals of an integral domain R. If Q is an n-powerful semiprimary ideal of R, then P is an n-powerful semiprimary ideal of R.*

Proof. Let Q be an n-powerful ideal of R; so P is an n-powerful ideal of R by Theorem 4.2(b). Thus P is an n-powerful semiprimary ideal of R by Theorem 4.3. □

Let I be a proper ideal of an integral domain R. As the "powerful" analogs of $W_R(I)$ and $\delta_R(I)$, we define $\overline{W}_R(I) = \{n \in \mathbb{N} \mid I$ is an n-powerful semiprimary ideal of $R\}$ and $\overline{\delta}_R(I) = min \overline{W}_R(I)$ (let $\overline{\delta}_R(I) = \infty$ if $\overline{W}_R(I) = \emptyset$). Note that $\overline{W}_R(I) \subseteq W_R(I)$ and $\delta_R(I) \leq \overline{\delta}_R(I)$. The next example shows that the analogs of Theorem 2.14(b) and Theorem 4.2(b) do not hold for n-powerful semiprimary ideals. In particular, if I is an n-powerful semiprimary ideal, then I is an n-semiprimary ideal. Thus I is also an m-semiprimary ideal for every integer $m \geq n$, but I need not be an m-poweful semiprimary ideal.

Example 4.5. Let $R = F[[X^2, X^5]] = F + FX^2 + X^4 F[[X]]$, where F is a field. Then R is quasilocal with maximal ideal $M = (X^2, X^5) = FX^2 + X^4 F[[X]]$ and quotient field $K = F[[X]][1/X]$. Clearly M is a 2-semiprimary ideal of R, but not a 3-powerful semiprimary ideal of R since $X^3 X^3 = X^6 \in M$, but $X^3 \notin M$. Moreover, M is a 2-powerful semiprimary ideal of R if and only if $char(F) = 2$, and M is an n-powerful semiprimary ideal of R for every integer $n \geq 4$. So, for $R = \mathbb{Z}_2[[X^2, X^5]]$, M is a 2-powerful semiprimary ideal, but not a 3-powerful semiprimary ideal, and $\overline{W}_R(M) = \mathbb{N} \setminus \{1, 3\}$. Thus the "powerful" analog of Theorem 2.14(b) fails for M. Let $I = X^4 F[[X]]$. Then I is a 2-semiprimary ideal of R, but not a 2-powerful semiprimary ideal of R since $X^2 X^2 \in I$, but $X^2 \notin I$. So the "semiprimary" analog of Theorem 4.2(b) fails for $I \subseteq J = M$ when $char(F) = 2$.

Recall [24] that an integral domain R is a *pseudo-valuation domain* (PVD) if every prime ideal of R is strongly prime. A PVD is neccessarily quasilocal [24, Corollary 1.3]. A quasilocal integral domain R with maximal ideal M is a PVD $\Leftrightarrow$ M is strongly prime [24, Theorem 1.4], and R is a PVD $\Leftrightarrow$ $(M : M)$ is a valuation domain with maximal ideal M [11, Proposition 2.5]. Let $T = K + M$ be a valuation domain, where K is a field and M is the maximal ideal of T. Then for a proper subfield k of K, the subring $R = k + M$ is a PVD which is not a valuation domain [24, Example 2.1]. By Theorem 4.2(a), every n-semiprimary ideal in a PVD is an n-powerful semiprimary ideal.

We next give an "n" generalization of PVDs.

Definition 4.6. Let R be an integral domain and n a positive integer. Then R is an *n-pseudo-valuation domain* (*n-PVD*) if every prime ideal of R is an n-powerful semiprimary ideal of R.

Note that a 1-PVD is just a PVD and an n-PVD is also an mn-PVD for every positive integer m. The next several results show that n-PVDs behave very much like PVDs (cf. [1], [6], [8], [11], [17], [24], and [25]).

Theorem 4.7. *Let R be an n-PVD. Then R is quasilocal.*

Proof. By way of contradiction, assume that M and N are distinct maximal ideals of R. Let $x \in M \setminus N$ and $y \in N \setminus M$. Then $(x/y)^n(y^2)^n = (x^n/y^n)y^{2n} = x^n y^n \in M$, and thus $(x/y)^n \in M$ since M is an n-powerful semiprimary ideal of R and $(y^2)^n \notin M$. Hence $x^n = (x/y)^n y^n \in N$; so $x \in N$, a contradiction. Thus R is quasilocal. □

In view of Theorem 4.3, Theorem 4.4, and the proof of Theorem 4.7, we have the following result.

Corollary 4.8. *An integral domain R is an n-PVD if and only if some maximal ideal of R is an n-powerful semiprimary ideal of R, if and only if some maximal ideal of R is an n-powerful ideal of R.*

Recall ([20], [14]) that a prime ideal P of a commutative ring R is a *divided prime ideal* of R if $x \mid p$ (in R) for every $x \in R \setminus P$ and $p \in P$ (i.e., (x) is comparable to P for every $x \in R$), and R is a *divided ring* if every prime ideal of R is divided. We next give the "n" generalization.

Definition 4.9. Let R be a commutative ring and n a positive integer. Then a prime ideal P of R is an *n-divided prime ideal* of R if $x^n \mid p^n$ (in R) for every $x \in R \setminus P$ and $p \in P$. Moreover, R is an *n-divided ring* if every prime ideal of R is an n-divided prime ideal of R.

A 1-divided prime ideal (resp., ring) is just a divided prime ideal (resp., ring), and an n-divided prime ideal is mn-divided for every positive integer m. Thus an n-divided ring is mn-divided for every positive integer m.

The next several results show that n-divided rings behave very much like divided rings (cf. [14], [20]).

Theorem 4.10. *Let R be an n-divided commutative ring. Then the set of prime ideals of R is linearly orderd by inclusion. In particular, R is quasilocal.*

Proof. Let P and Q be prime ideals of an n-divided commutative ring R with $P \not\subseteq Q$. We show that $Q \subseteq P$. Let $x \in P \setminus Q$; then $x^n \mid q^n$ for every $q \in Q$ since Q is an n-divided prime ideal of R. Thus $q^n \in (x^n) \subseteq P$; so $q \in P$ for every $q \in Q$. Hence $Q \subseteq P$. □

Theorem 4.11. *Let P a prime ideal of an integral domain R. If P is an n-powerful semiprimary ideal of R, then P is an n-divided prime ideal of R. Moreover, the set of prime ideals of R that are contained in P is linearly ordered by inclusion.*

Proof. Let $x \in R \setminus P$ and $p \in P$. Then $(p/x)^n x^n = (p^n/x^n)x^n = p^n \in P$. Thus $p^n/x^n \in P$ since $x^n \notin P$ and P is an n-powerful semiprimary ideal of R. Hence

$p^n = (p^n/x^n)x^n$; so $x^n \mid p^n$ (in R). Thus P is an n-divided prime ideal of R. Now suppose that F and H are distinct prime ideals of R contained in P. Then F and H are n-powerful semiprimary ideals of R by Theorem 4.4, and hence are n-divided prime ideals. The proof of Theorem 4.10 shows that F and H are comparable under inclusion. □

Corollary 4.12. *Let R be an n-PVD. Then R is an n-divided domain and the set of prime ideals of R is linearly ordered by inclusion. Moreover, if R is Noetherian, then $dim(R) \leq 1$.*

Proof. We need only prove the "moreover" statement; it follows directly from [27, Theorem 144]. □

Let R be an integral domain with quotient field K, $S \subseteq R$, and n a positive integer. Define $E_n(S) = \{x \mid x^n \notin S, x \in K\}$ and $A_n(S) = \{x^n \mid x^n \in S, x \in K\}$. We next use these two sets to give another characterization of n-powerful semiprimary ideals. Note that actually $x^{-n}d \in A_n(P)$ in Theorem 4.13 and Corollary 4.15(4), and $x^{-n}d \in A_n(M)$ in Corollary 4.16(3).

Theorem 4.13. *Let P a prime ideal of an integral domain R with quotient field K. Then P is an n-powerful semiprimary ideal of R if and only if $x^{-n}d \in P$ for every $x \in E_n(P)$ and $d \in A_n(P)$.*

Proof. Suppose that $x^{-n}d \in P$ for every $x \in E_n(P)$ and $d \in A_n(P)$. Let $x^n y^n \in P$ for $x, y \in K$ with $x^n \notin P$; so $x \in E_n(P)$. Since $x^n y^n = (xy)^n \in A_n(P)$, we have $y^n = x^{-n}(x^n y^n) \in P$. Thus P is an n-powerful semiprimary ideal of R.

Conversely, suppose that P is an n-powerful semiprimary ideal of R. Let $d \in A_n(P)$; so $d = a^n \in P$ for some $a \in K$ and $x^n(x^{-1}a)^n = x^n x^{-n} a^n = a^n \in P$ for every $0 \neq x \in K$. Suppose that $x \in E_n(P)$. Then $x^n \notin P$; so $(x^{-1}a)^n \in P$ since P is an n-powerful semiprimary ideal of R. Thus $x^{-n}d = x^{-n}a^n = (x^{-1}a)^n \in P$. □

The proof of the following result is similar to that of Theorem 4.13, and thus will be omitted.

Theorem 4.14. *Let I a proper ideal of an integral domain R. Then I is an n-powerful ideal of R if and only if $x^{-n}d \in R$ for every $x \in E_n(R)$ and $d \in A_n(I)$.*

In view of Theorem 4.3, Theorem 4.13, and Theorem 4.14, we have the following result.

Corollary 4.15. *Let P be a prime ideal of an integral domain R. Then the following statements are equivalent.*

(1) *P is an n-powerful semiprimary ideal of R.*
(2) *P is an n-powerful ideal of R.*
(3) *$x^{-n}d \in R$ for every $x \in E_n(R)$ and $d \in A_n(P)$.*
(4) *$x^{-n}d \in P$ for every $x \in E_n(P)$ and $d \in A_n(P)$.*

In view of Corollary 4.8, Theorem 4.13, and Theorem 4.14, we have the following result.

Corollary 4.16. *Let R be a quasilocal integral domain with maximal ideal M. Then the following statements are equivalent.*

(1) *R is an n-PVD.*
(2) *$x^{-n}d \in R$ for every $x \in E_n(R)$ and $d \in A_n(M)$.*

(3) *$x^{-n}d \in M$ for every $x \in E_n(M)$ and $d \in A_n(M)$.*

If R is a PVD, then R/P is also a PVD for P a prime ideal of R [21, Lemma 4.5(i)]. The analogous result holds for n-PVDs.

Theorem 4.17. *Let P be a prime ideal of an n-PVD R. Then R/P is an n-PVD.*

Proof. Let M be the maximal ideal of R, K the quotient field of R, $F = R_P/PR_P$ the quotient field of $A = R/P$, and $H_n(M/P) = \{x^n \in M/P \mid x \in F\}$. Suppose that $x = a+P, y = b+P \in A$, and $x^n \nmid y^n$ in A. Then $a^n \nmid b^n$ in R; so $b^n \mid a^n d$ in R for every $d \in A_n(M)$ by Corollary 4.16. Thus $y^n \mid x^n h$ in A for every $h \in H_n(M/P)$; so A is an n-PVD by Corollary 4.16 again. □

Let n be a positive integer. Recall that an integral domain R with quotient field K is *n-root closed* if whenever $x^n \in R$ for $x \in K$, then $x \in R$; and R is *root closed* if R is n-root closed for every positive integer n. For example, an integrally closed integral domain is root closed. Note that R is mn-root closed if and only if R is m-root closed and n-root closed. Thus $\mathcal{C}(R) = \{n \in \mathbb{N} \mid R \text{ is } n\text{-root closed}\}$ is a multiplicative submonoid on $\mathbb{N}$ generated by some set of prime numbers. Moreover, for S any multiplicative submonoid of $\mathbb{N}$ generated by a set of prime numbers, $S = \mathcal{C}(R)$ for some integral domain R [7, Theorem 2.7].

For n-root closed integral domains, the n-PVD and PVD concepts coincide.

Theorem 4.18. *Let R be an n-root closed integral domain with quotient field K. Then R is an n-PVD if and only if R is a PVD. In particular, an integrally closed n-PVD is a PVD.*

Proof. If R is a PVD, then clearly R is an n-PVD. Conversely, let R be an n-root closed n-PVD with maximal ideal M. We show that M is a powerful ideal of R. Let $xy \in M$ for $x, y \in K$ and $x \notin R$. Then $x^n y^n \in M$ and $x^n \notin R$ since R is n-root closed. Thus $y^n \in M \subseteq R$ since M is an n-powerful semiprimary ideal of R, and hence $y \in R$ since R is n-root closed. Thus M is a powerful ideal of R; so M is a strongly prime ideal of R (i.e., M is a 1-powerful semiprimary ideal of R) by Theorem 4.3. Hence R is a PVD. The “in particular” statement is clear. □

Recall ([4], [3], [5], [29]) that an integral domain R with quotient field K is an *almost valuation domain* if for every $0 \neq x \in K$, there is a positive integer n (depending on x) such that $x^n \in R$ or $x^{-n} \in R$. We have the following “n” generalization.

Definition 4.19. Let n be a positive integer. An integral domain R with quotient field K is an *n-valuation domain* (*n-VD*) if $x^n \in R$ or $x^{-n} \in R$ for every $0 \neq x \in K$.

It is clear that a valuation domain is an n-VD for every positive integer n, an n-root closed n-VD is a valuation domain, an n-VD is an almost valuation domain, an n-VD is also an mn-VD for every positive integer m, and an n-VD is an n-PVD. Moreover, an n-VD is quasilocal, an overring of an n-VD is also an n-VD, and a Noetherian n-VD has (Krull) dimension at most one.

We have the following elementary results about n-VDs which show that n-VDs behave very much like valuation domains (cf. [23, Chapter 3]). In [1, page 3], it was observed that R is a valuation domain if and only if R is a strongly prime ideal of R (here, and in Theorem 4.20(a)(5), we drop the usual assumption that a prime ideal is a proper ideal).

Theorem 4.20. *Let R be an integral domain with quotient field K and n a positive integer.*

(a) *The following statements are equivalent.*

(1) *R is an n-VD.*
(2) *$x^n \mid y^n$ or $y^n \mid x^n$ for every $0 \neq x, y \in K$.*
(3) *$x^n \mid y^n$ or $y^n \mid x^n$ for every $0 \neq x, y \in R$.*
(4) *Let G be the group of divisibility of R. Then for every $g \in G$, either $ng \geq 0$ or $ng < 0$.*
(5) *R is an n-powerful semiprimary ideal of R.*

(b) *Let R be an n-VD. Then R is an n-divided domain, and thus the prime ideals of R are linearly ordered by inclusion.*

(c) *Let R be an n-VD and $x \in K$. If x^n is integral over R, then $x^n \in R$.*

Proof. The proofs are essentially the same as for valuation domains. See [23, Theorem 16.3] for part (a) and [23, Theorem 17.5] for part (c). Part (b) follows from Corollary 4.12 since an n-VD is also an n-PVD. $\square$

An n-VD is always an n-PVD, but an n-PVD need not be an n-VD. Also, an almost valuation domain need not be an n-VD for any positive integer n.

Example 4.21. (a) Let $R = \mathbb{Q} + X\mathbb{R}[[X]]$. Then R is a PVD with maximal ideal $X\mathbb{R}[[X]]$ and quotient field $\mathbb{R}[[X]][1/X]$, and thus R is an n-PVD for every positive integer n. However, R is not an n-VD for any positive integer n since $\pi^n, \pi^{-n} \notin R$ for every positive integer n.

(b) Let $R = \mathbb{Z}_p + XF[[X]]$, where p is a positive prime integer and $F = \overline{\mathbb{Z}_p}$ is the algebraic closure of $\mathbb{Z}_p$. Then R is an almost valuation domain with maximal ideal $XF[[X]]$ and quotient field $F[[X]][1/X]$, but not an n-VD for any positive integer n. This follows from the fact that for every $0 \neq a \in F$, there is a positive integer n such that $a^n = 1$; but for every positive integer n, there is a $b \in F$ such that $b^n \notin \mathbb{Z}_p$ and $b^{-n} \notin \mathbb{Z}_p$. Note that R is also a PVD, and thus an n-PVD for every positive integer n.

In some cases, an overring of an n-PVD is also an n-VD.

Theorem 4.22. *Let R be an n-PVD with maximal ideal M, quotient field K, and V an overring of R such that $1/s \in V$ for some $0 \neq s \in M$. Then V is an n-VD, and thus V is an almost valuation domain.*

Proof. Let $x \in K$ with $x^n \notin V$; so $x \in E_n(R)$. Then $x^{-n}d \in M$ for every $d \in A_n(M)$ by Corollary 4.16. In particular, $a = x^{-n}s^n \in M$ since $d = s^n \in A_n(M)$, and thus $x^{-n} = a/s^n \in V$ since $1/s \in V$. Hence V is an n-VD, and thus V is an almost valuation domain. $\square$

By Theorem 4.2(c), if R is an n-PVD, then R_P is also an n-PVD for every nonmaximal prime ideal P of R. We next give a stronger result; R_P is an n-VD.

Theorem 4.23. *Let R be an n-PVD with maximal ideal M and $P \subsetneq M$ a prime ideal of R. Then R_P is an n-VD, and thus R_P is an almost valuation domain. Moreover, $x^n \in R$ for every $x \in P_P$, and hence $P_P \subsetneq \overline{R}$.*

Proof. Since $P \subsetneq M$, there is an $s \in M \setminus P$. Thus $1/s \in R_P$; so R_P is an n-VD (and hence also an almost valuation domain) by Theorem 4.22. Let $x \in P_P$; so $x = a/s$ for some $a \in P$ and $s \in R \setminus P$. Thus $s^n \mid a^n$ (in R) since P is an n-divided prime ideal of R by Theorem 4.11. Hence $x^n = a^n/s^n \in R$; so $P_P \subsetneq \overline{R}$. $\square$

We next show that n-divided principal prime ideals are actually maximal ideals.

Theorem 4.24. *Let R be an integral domain R with quotient field K and (nonzero) principal prime ideal P. If P is an n-divided ideal of R, then P is a maximal ideal of R. Moreover, if P is also an n-powerful semiprimary ideal of R, then P is a maximal ideal of R and R is an n-VD.*

Proof. Let $P = (p)$ for a prime element p of R. By way of contradiction, assume that P is not a maximal ideal; so there is a nonunit $x \in R \setminus P$. If P is an n-divided prime ideal of R, then there is a $y \in R$ with $p^n = x^n y p^n$ or $p^n = x^n w p^m$ for some positive integer $m < n$ and $w \in R \setminus P$. If $p^n = x^n y p^n$, then $1 = x^n y$, and thus $x \in U(R)$, a contradiction. If $p^n = x^n w p^m$, then $x^n w = p^{n-m} \in P$, which is a contradiction since $x \notin P$ and $w \notin P$. Hence P is a maximal ideal of R.

Now, suppose that $P = (p)$ is an n-powerful semiprimary ideal of R. Then P is an n-divided prime ideal of R by Theorem 4.11. Thus P is a maximal ideal of R, and hence R is an n-PVD by Corollary 4.8. Finally, we show that R is an n-VD. Let $x \in K$, and suppose that $x^n \notin R$. Then $x^n \notin P$, and thus $x^{-n}p^n \in P$ by Theorem 4.13. Since $x^{-n}p^n \in P = (p)$, we have $x^{-n}p^n = hp^n$ for some $h \in R$ or $x^{-n}p^n = dp^m$ for some positive integer $m < n$ and $d \in U(R)$. If $x^{-n}p^n = dp^m$, then $x^n = d^{-1}p^{n-m} \in R$, a contradiction. Thus $x^{-n}p^n = hp^n$ for some $h \in R$, and hence $x^{-n} = h \in R$. Thus R is an n-VD. □

We have already observed several parts of the next theorem. One interesting consequence is that if P is an n-powerful semiprimary prime ideal of an integral domain R with quotient field K, then $\{x \in K \mid x^m \in P$ for some positive integer $m\} = \{x \in K \mid x^n \in P\}$ (cf. Theorem 2.3).

Theorem 4.25. *Let P be a prime ideal of an integral domain R with quotient field K. If P is an n-powerful semiprimary ideal of R, then P is an mn-powerful semiprimary ideal of R for every positive integer m. Furthermore, if $x^m \in P$ for a positive integer m and $x \in K$, then $x^n \in P$. In particular, if R is an n-PVD, then R is an mn-PVD for every positive integer m.*

Proof. Let m be a positive integer. Assume that $x^{mn}y^{mn} \in P$ for $x, y \in K$. Then $(x^m)^n(y^m)^n \in P$. Since P is an n-powerful semiprimary ideal of R, $(x^m)^n = x^{mn} \in P$ or $(y^m)^n = y^{mn} \in P$. Thus P is an mn-powerful semiprimary ideal of R. Next, assume that $x^m \in P$ for $x \in K$ and some positive integer m; so $x^{mn} = (x^m)^n \in P$. Let d be the least positive integer such that $x^{dn} \in P$. Since $(x^{d-1})^n x^n = x^{dn} \in P$ and P is an n-powerful semiprimary ideal of R, we have $(x^{d-1})^n \in P$ or $x^n \in P$. Hence $d = 1$, and thus $x^n \in P$. The "in particular" statement is clear. □

The next several results concern integral overrings of an n-PVD. In particular, an integral overring of an n-PVD is an n-PVD, and the integral closure of an n-PVD is a PVD. Note that $\{x \in K \mid x^n \in M\} = \{x \in \overline{R} \mid x^n \in M\}$ in the next several results and $\sqrt{MB} = \sqrt{M\overline{R}} \cap B$ for B an integral overring of R.

Theorem 4.26. *Let R be an n-PVD with maximal ideal M and quotient field K. If B is an integral overring of R, then B is an n-PVD with maximal ideal $M_B = \sqrt{MB} = \{x \in B \mid x^n \in M\}$.*

Proof. Let $m \in M$. Then $\sqrt{mR}$ is a prime ideal of R since the prime ideals of R are linearly ordered (under inclusion) by Corollary 4.12, and thus $\sqrt{mR}$ is an

n-powerful semiprimary ideal of R since R is an n-PVD. We show that $\sqrt{mB}$ is an n-powerful semiprimary ideal of B and $\sqrt{mB} = \{x \in B \mid x^n \in \sqrt{mR}\}$. Let $x^n y^n \in \sqrt{mB}$ for $0 \neq x, y \in K$. Then $x^{nk} y^{nk} = (xy)^{nk} = fm$ for some positive integer k and $0 \neq f \in B$. Note that $f^{-n} \notin M$; for if $f^{-n} \in M$, then $1/a = f^n \in B$ for some $a \in M$, a contradiction since B is integral over R. Then $f^n m^n \in \sqrt{mR}$ since $f^{-n}(fm)^n = m^n \in \sqrt{mR}$, $f^{-n} \notin \sqrt{mR} \subseteq M$, and $\sqrt{mR}$ is an n-powerful semiprimary ideal of R. Thus $(x^{nk})^n (y^{nk})^n = (xy)^{nkn} = f^n m^n \in \sqrt{mR}$; so $x^{nkn} \in \sqrt{mR} \subseteq \sqrt{mB}$ or $y^{nkn} \in \sqrt{mR} \subseteq \sqrt{mB}$. Hence $x^n \in \sqrt{mR} \subseteq \sqrt{mB}$ or $y^n \in \sqrt{mR} \subseteq \sqrt{mB}$ by Theorem 4.25. Thus $\sqrt{mB}$ is an n-powerful semiprimary ideal of B, and hence a prime ideal of B by Theorem 2.3. A slight modification of the above proof also shows that $\sqrt{mB} = \{x \in B \mid x^n \in \sqrt{mR}\}$.

We next show that $M_B = \{x \in B \mid x^n \in M\}$ is an n-powerful semiprimary ideal of B. First, we show that M_B is an ideal of B. Let $x_1, x_2 \in M_B$; so $x_1^n = m_1 \in M$ and $x_2^n = m_2 \in M$. Thus $x_1 \in \sqrt{m_1 B}$ and $x_2 \in \sqrt{m_2 B}$. Since the prime ideals of R are linearly ordered, we may assume that $\sqrt{m_1 R} \subseteq \sqrt{m_2 R}$, and hence $\sqrt{m_1 B} \subseteq \sqrt{m_2 B}$. Thus $x_1 + x_2 \in \sqrt{m_2 B} = \{x \in B \mid x^n \in \sqrt{m_2 R}\} \subseteq M_B$. Next, let $x \in M_B$ and $y \in B$. Then $x^n = m_3 \in M$; so $x \in \sqrt{m_3 B}$. Thus $xy \in \sqrt{m_3 B} \subseteq M_B$; so M_B is an ideal of B. A similar argument to that for $\sqrt{mB}$ above shows that if $x^n y^n \in M_B$ for $0 \neq x, y \in K$, then $x^n \in \sqrt{mR} \subseteq M \subseteq M_B$ or $y^n \in \sqrt{mR} \subseteq M \subseteq M_B$. Hence M_B is an n-powerful semiprimary ideal of B, and thus M_B is a prime ideal of B since it is a radical ideal of B by Theorem 4.25. Hence M_B is a maximal ideal of B since B is integral over R and $M_B \cap R = M$; so B is an n-PVD by Corollary 4.8. Clearly $M_B = \{x \in B \mid x^n \in M\} \subseteq \sqrt{MB}$, and $\sqrt{MB} \subseteq M_B$ since $MB \subsetneq B$ as B is integral over R. Thus $M_B = \sqrt{MB}$. □

Corollary 4.27. *Let R be an n-PVD with maximal ideal M and quotient field K. Then $\overline{R}$ is a PVD (1-PVD) with maximal ideal $\sqrt{M\overline{R}} = \{x \in K \mid x^n \in M\}$.*

Proof. By Theorem 4.26, $\overline{R}$ is an n-PVD with maximal ideal $\sqrt{M\overline{R}} = M_{\overline{R}} = \{x \in \overline{R} \mid x^n \in M\} = \{x \in K \mid x^n \in M\}$. Thus $\overline{R}$ is a PVD by Theorem 4.18. □

Corollary 4.28. *Let P be a nonzero finitely generated prime ideal of an n-PVD R. Then $W = (P : P)$ is an n-PVD with maximal ideal $\sqrt{MW} = \{x \in W \mid x^n \in M\}$. In particular, if R is a Noetherian n-PVD with maximal ideal M, then $(M : M)$ is an n-PVD.*

Proof. Note that $W = (P : P)$ is integral over R since P is finitely generated. Thus W is an n-PVD with maximal ideal $\sqrt{MW} = \{x \in W \mid x^n \in M\}$ by Theorem 4.26. The "in particular" statement is clear. (However, recall that a Noetherian n-PVD R has $dim(R) \leq 1$ by Corollary 4.12). □

The converse of Corollary 4.27 also holds.

Theorem 4.29. *Let R be a quasilocal integral domain with maximal ideal M and quotient field K. Then R is an n-PVD if and only if $\overline{R}$ is a PVD with maximal ideal $\sqrt{M\overline{R}} = \{x \in K \mid x^n \in M\}$.*

Proof. Let R be an n-PVD. Then $\overline{R}$ is a PVD with maximal ideal $\sqrt{M\overline{R}} = \{x \in K \mid x^n \in M\}$ by Corollary 4.27. Conversely, suppose that $\overline{R}$ is a PVD with maximal ideal $N = \sqrt{M\overline{R}} = \{x \in K \mid x^n \in M\}$. Then $M = R \cap N$ since $M \subseteq N$. Let $x^n y^n = (xy)^n \in M$ for $x, y \in K$; so $xy \in N$. Thus $x \in N$ or $y \in N$ since N is a strongly prime ideal of $\overline{R}$. Hence $x^n \in M$ or $y^n \in M$; so M is an n-powerful semiprimary ideal of R. Thus R is an n-PVD by Corollary 4.8. ☐

Corollary 4.30. *Let R be a quasilocal integral domain with maximal ideal M and quotient field K. Then the following statements are equivalent.*

(1) *R is an n-PVD.*
(2) *$\overline{R}$ is a PVD with maximal ideal $\sqrt{M\overline{R}} = \{x \in K \mid x^n \in M\}$.*
(3) *$N = \sqrt{M\overline{R}} = \{x \in K \mid x^n \in M\}$ is a maximal ideal of $\overline{R}$ such that $(N : N)$ is a valuation domain with maximal ideal N.*

Proof. (1) $\Leftrightarrow$ (2) is Theorem 4.29, and (2) $\Leftrightarrow$ (3) is clear by [11, Proposition 2.5]. ☐

We have seen that integral overrings of an n-PVD are also n-PVDs. We next determine when every overring of an n-PVD is an n-PVD. Note that an integrally closed PVD need not be a valuation domain. For example, $R = \mathbb{Q} + X\mathbb{C}[[X]]$ is a PVD, and $\overline{R} = \overline{\mathbb{Q}} + X\mathbb{C}[[X]]$ is a PVD, but not a valuation domain, where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$. In this case, $\mathbb{Q}[\pi] + X\mathbb{C}[[X]]$ is a (non-integral) overring of R which is not an n-VD or n-PVD for any positive integer n.

Theorem 4.31. *Let R be an n-PVD with maximal ideal M. Then every overring of R is an n-PVD if and only if $\overline{R}$ is a valuation domain. Moreover, if $\overline{R}$ is a valuation domain, then every non-integral overring of R is an n-VD.*

Proof. Suppose that every overring of R is an n-PVD. Since $\overline{R}$ is a PVD by Theorem 4.18, the proof of [25, Proposition 2.7] shows that if $\overline{R}$ is not a valuation domain, then there is a non-quasilocal overring B of $\overline{R}$ (and hence B is an overring of R). Thus B cannot be an n-PVD by Theorem 4.7; so $\overline{R}$ is a valuation domain.

Conversely, suppose that $\overline{R}$ is a valuation domain with maximal ideal N. Let B be an overring of R. If B is integral over R, then B is an n-PVD by Theorem 4.26; so assume that B is not integral over R. Let $b \in B \setminus \overline{R}$. Then $b^{-1} \in N$ since $\overline{R}$ is a valuation domain; so $m = b^{-n} = (b^{-1})^n \in M$ by Corollary 4.27 since $\overline{R}$ is a valuation domain (and thus a PVD). Hence $1/m = b^n \in B$; so B is an n-VD, and thus an n-PVD, by Theorem 4.22. The "moreover" statement is clear. ☐

Let R be a 1-PVD (i.e., PVD) and P a prime ideal of R. Then $A_1(P) = P$; so $V = (A_1(P) : A_1(P)) = (P : P)$ is a 1-VD (i.e., valuation domain) by [8, Proposition 4.3], and it is easily checked that P is the maximal ideal of V. We have the following analogous result for n-PVDs.

Theorem 4.32. *Let R is an n-PVD, P a prime ideal of R, and $I = (A_n(P))$. Then $V = (I : I)$ is an n-VD with maximal ideal $\sqrt{IV} = \{x \in V \mid x^n \in I\}$. Moreover, $\sqrt{IV} = \{x \in V \mid x^n \in P\} = \sqrt{PV}$.*

Proof. Let $x \in K$ with $x^n \notin V$. Then $x^n \notin P$; so $x^{-n}I \subseteq I$ by Corollary 4.15. Thus $x^{-n} \in V$; so V is an n-VD with maximal ideal N_V. Let $y \in N_V$. Assume that $y^n \notin I$; so $y^n \notin P$. Thus $y^{-n}I \subseteq I$ by Corollary 4.15 again; so $y^{-n} \in V$. Hence $y \in U(V)$, a contradiction. Thus $N_V \subseteq \{x \in V \mid x^n \in I\} \subseteq \sqrt{IV}$. Also, $IV = I \subsetneq V$; so $\sqrt{IV} \subseteq N_V$. Hence $N_V = \sqrt{IV} = \{x \in V \mid x^n \in I\}$. Clearly $\{x \in V \mid x^n \in I\} \subseteq \{x \in V \mid x^n \in P\}$ since $I \subseteq P$. Also, $x^n \in P$ for $x \in V \Rightarrow x^n \in A_n(P)$; so $\{x \in V \mid x^n \in P\} \subseteq \{x \in V \mid x^n \in I\}$. Thus $\{x \in V \mid x^n \in I\} = \{x \in$

$V \mid x^n \in P\}$. Clearly $x \in P \Rightarrow x^n \in A_n(P) \subseteq I \Rightarrow x^n \in \sqrt{IV}$; so $P \subseteq \sqrt{IV}$, and hence $\sqrt{PV} \subseteq \sqrt{IV}$. Also, $\sqrt{IV} \subseteq \sqrt{PV}$ since $I \subseteq P$; so $\sqrt{IV} = \sqrt{PV}$. □

Recall that a quasilocal integral domain R with maximal ideal M is a PVD if and only if $(M : M)$ is a valuation domain with maximal ideal M [11, Proposition 2.5]. Example 4.34(c) below shows that if R is an n-PVD with maximal ideal M, then $(M : M)$ need not be an n-VD. And Example 4.34(d)(e) shows that $V = (M : M)$ may be an n-VD with maximal ideal $\sqrt{MV} = \{x \in V \mid x^n \in M\}$ when R is not an n-PVD. However, since $M = A_1(M)$, the next theorem may be viewed as the n-PVD analog. By adding the extra condition "$(*)_n$: if $x \in K$ is a nonunit of $\overline{R}$, then $x^n \in M$," we get a converse to Theorem 4.32. Note that $I = (A_n(M)) \subsetneq M$ in general (see Example 4.34(a)(b)).

Theorem 4.33. *Let R be a quasilocal integral domain with maximal ideal M, quotient field K, and $I = (A_n(M))$. Then the following statements are equivalent.*

(1) *R is an n-PVD.*
(2) *$V = (I : I)$ is an n-VD with maximal ideal $\sqrt{MV} = \{x \in V \mid x^n \in M\}$, and if $x \in K$ is a nonunit of $\overline{R}$, then $x^n \in M$.*

Proof. (1) $\Rightarrow$ (2) By Theorem 4.32, V is an n-VD with maximal ideal $\sqrt{MV} = \{x \in V \mid x^n \in M\}$. Let $x \in K$ be a nonunit of $\overline{R}$. Then $x^n \in M$ by Corollary 4.27.

(2) $\Rightarrow$ (1) Let $x \in K$. Suppose that $x \in E_n(M)$, i.e., $x^n \notin M$. First, assume that $x^n \in V$. Suppose that $x^n \in N = \{x \in V \mid x^n \in M\}$; so $x^{n^2} = (x^n)^n \in M$. Thus $x \in \overline{R}$ and x is a nonunit of $\overline{R}$; so $x^n \in M$ by hypothesis, a contradiction. Hence $x^n \in U(V)$, and thus $x^{-n}I \subseteq I$. Hence $x^{-n}d \in I \subseteq M$ for every $d \in A_n(M)$. Now, suppose that $x^n \notin V$. Then $x^{-n} \in V$ since V is an n-VD. Thus $x^{-n}I \subseteq I$, and hence $x^{-n}d \in I \subseteq M$ for every $d \in A_n(M)$. Thus $x^{-n}d \in M$ for every $x \in E_n(M)$ and $d \in A_n(M)$; so R is an n-PVD by Corollary 4.16. □

We end this section with several examples.

Example 4.34. (a) Let $R = \mathbb{Z}_2[[X^2, X^3]] = \mathbb{Z}_2 + X^2\mathbb{Z}_2[[X]]$. Then R is quasilocal with maximal ideal $M = (X^2, X^3) = X^2\mathbb{Z}_2[[X]]$ and quotient field $K = \mathbb{Z}_2[[X]][1/X]$. It is easily checked that R is an n-PVD if and only if $n \geq 2$ and an n-VD if and only if n is even. First, suppose that n is even. Then $I = (A_n(M)) = \mathbb{Z}_2X^n + X^{n+2}\mathbb{Z}_2[[X]] \subsetneq M$ and $V = (I : I) = R$ has maximal ideal $M_V = M$. Also, $M_V = \{x \in V \mid x^n \in M\} \subsetneq \{x \in K \mid x^n \in M\} = X\mathbb{Z}_2[[X]]$. Next, suppose that $n \geq 3$ is odd. Then $I = (A_n(M)) = X^n\mathbb{Z}_2[[X]] \subsetneq M$ and $V = (I : I) = \mathbb{Z}_2[[X]]$ has maximal ideal $M_V = X\mathbb{Z}_2[[X]] = \{x \in K \mid x^n \in M\}$.

(b) Let $R = F[[X^2, X^3]] = F + X^2F[[X]]$, where F is a field. Then R is quasilocal with maximal ideal $M = (X^2, X^3) = X^2F[[X]]$ and quotient field $F[[X]][1/X]$, and R is an n-PVD if and only if $n \geq 2$. If $char(F) = 2$, then $(A_n(M)) \subsetneq M$ for every integer $n \geq 2$. However, $M = (A_2(M))$ if $char(F) \neq 2$.

(c) Let $R = \mathbb{Z}_p + \mathbb{Z}_pX + X^2F[[X]]$, where $F = \overline{\mathbb{Z}_p}$ is the algebraic closure of $\mathbb{Z}_p$. Then R is quasilocal with maximal ideal $M = \mathbb{Z}_pX + X^2F[[X]]$ and quotient field $K = F[[X]][1/X]$. Moreover, R is an n-PVD if and only if $n \geq 2$ by Theorem 4.29 since $\overline{R} = F[[X]]$ is a PVD (in fact, a valuation domain). However, $V = (M : M) = \mathbb{Z}_p + XF[[X]]$ is an almost valuation domain with maximal ideal $XF[[X]] = \{x \in K \mid x^n \in M\}$, but V is not an n-VD for any positive integer n by Example 4.21(b). Note that V is a PVD, and thus an n-PVD for every positive integer n.

(d) Let F be a field and N a positive integer. Then $R_N = F + X^N F[[X]]$ is a quasilocal integral domain with maximal ideal $M_N = X^N F[[X]]$, quotient field $F[[X]][1/X]$, and integral closure $\overline{R_N} = F[[X]]$. Note that $V_N = (M_N : M_N) = F[[X]]$ is a valuation domain with maximal ideal $XF[[X]] = \{x \in V_N \mid x^N \in M_N\} = \sqrt{M_N V_N}$, and thus V_N is an n-VD for every positive integer n. However, R_N is an n-PVD if and only if $n \geq N$, and R_N satisfies condition $(*)_n$ if and only if $n \geq N$.

(e) Let $R = \mathbb{Z}_3 + \mathbb{Z}_3 X^9 + X^{12}\mathbb{Z}_3[[X]]$. Then R is a quasilocal integral domain with maximal ideal $M = \mathbb{Z}_3 X^9 + X^{12}\mathbb{Z}_3[[X]]$, quotient field $\mathbb{Z}_3[[X]][1/X]$, and integral closure $\overline{R} = \mathbb{Z}_3[[X]]$. Note that $V = (M : M) = \mathbb{Z}_3 + X^3\mathbb{Z}_3[[X]]$ is a 3-VD with maximal ideal $X^3\mathbb{Z}_3[[X]] = \sqrt{MV} = \{x \in V \mid x^3 \in M\}$. However, R is not a 3-PVD since $(X^2)^3(X^2)^3 \in M$, but $X^6 \notin M$, and R does not satisfy condition $(*)_3$ since $X^3 \notin M$.

5. Pseudo n-strongly prime ideals, PnVDs, and n-VDs

In this final section, we introduce and investigate pseudo n-valuation domains (PnVDs), yet another generalization of PVDs. We also give some more results on n-VDs.

Let R be an integral domain with quotient field K. Recall [16] that R is a *pseudo-almost valuation domain* (PAVD) if every prime ideal P of R is *pseudo-strongly prime*, i.e., if whenever $xyP \subseteq P$ for $x, y \in K$, then there is a positive integer n such that $x^n \in R$ or $y^n P \subseteq P$. Also, recall [17] that R is an *almost pseudo-valuation domain* (APVD) if every prime ideal P of R is *strongly primary*, i.e, if whenever $xy \in P$ for $x, y \in K$, then $x^n \in P$ for some positive integer n or $y \in P$. Note that valuation domain ⇒ PVD ⇒APVD ⇒ PAVD, and no implication is reversible [16, page 1168].

The following is an example of an n-PVD for some integer $n \geq 2$ which is neither an APVD, a PAVD, a PVD, nor an almost valuation domain.

Example 5.1. (cf. [16, Example 3.4]) Let $R = \mathbb{Q} + \mathbb{C}X^2 + X^4\mathbb{C}[[X]]$. Then R is quasilocal with maximal ideal $M = \mathbb{C}X^2 + X^4\mathbb{C}[[X]]$ and quotient field $K = \mathbb{C}[[X]][1/X]$. One can see that R is neither an APVD, a PAVD, a PVD, an almost valuation domain, nor an n-VD for any positive integer n. However, it is easily checked that R is a n-PVD for $n \geq 4$ and $\overline{R} = \overline{\mathbb{Q}} + X\mathbb{C}[[X]]$ is a PVD with maximal ideal $N = \{x \in K \mid x^4 \in M\} = X\mathbb{C}[[X]]$, where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Note that $\overline{R}$ is not a valuation domain; in fact, $\overline{R}$ is not an n-VD for any positive integer n, and R is not an n-PVD for $n = 1, 2$, or 3.

We now give yet another "n" generalization of PVDs.

Definition 5.2. Let R an integral domain with quotient field K. A prime ideal P of R is a *pseudo n-strongly prime ideal* of R if whenever $xyP \subseteq P$ for $x, y \in K$, then $x^n \in R$ or $y^n P \subseteq P$. If every prime ideal of R is a pseudo n-strongly prime ideal of R, then R is a *pseudo n-valuation domain* (PnVD).

A P1VD is just a PVD [24, Proposition 1.2], an n-VD is a PnVD, a PnVD is a PAVD, and a PnVD is also a P(mn)VD for every positive integer m. Moreover, from Theorem 5.4 and Remark 5.6, it follows that a PnVD R is quasilocal, the prime ideals of R are linearly ordered by inclusion, and $dim(R) \leq 1$ when R is Noetherian.

The following is an example of a PAVD which is not a PnVD for any positive integer n.

Example 5.3. Let p be a positive prime integer and $F = \overline{\mathbb{Z}_p}$ the algebraic closure of $\mathbb{Z}_p$. Then $R = \mathbb{Z}_p + \mathbb{Z}_pX + X^2F[[X]]$ is quasilocal with maximal ideal $M = \mathbb{Z}_pX + X^2F[[X]]$ and quotient field $K = F[[X]][1/X]$. Let $y \in K$ with $y^n \notin R$ for every positive integer n. Then $y = z/X^m$, where $z \in U(F[[X]])$ and $m \geq 0$. If $m > 0$, then $y^{-2}M \subseteq M$. If $m = 0$, then there is a positive integer n such that $z(0)^n = 1$; so $y^{-n}M \subseteq M$. Thus R is a PAVD by [16, Lemma 2.1 and Theorem 2.5]. We now show that R is not a PnVD for any positive integer n. For n a positive integer, there is a $b \in F$ with $b^n \notin \mathbb{Z}_p$ and $b^{-n} \notin \mathbb{Z}_p$. Hence $b^n \notin R$ and $b^{-n}X \notin M$; so R is not a PnVD by Theorem 5.4(a)(b) below. However, R is an n-PVD for every integer $n \geq 2$ by Example 4.34(c).

The proofs of the following results are similar to the proofs given in [16], and thus the details are left to the reader.

Theorem 5.4. *Let R an integral domain with quotient field K.*

(a) *Let P be a prime ideal of R. Then P is a pseudo n-strongly prime ideal of R if and only if $x^{-n}P \subseteq P$ for every $x \in E_n(R)$* (*see* [16, Lemma 2.1]).

(b) *R is a PnVD if and only if R is quasilocal with pseudo n-strongly prime maximal ideal* (*see* [16, Theorem 2.5]).

(c) *R is a PnVD if and only if for every $a, b \in R$, we have $a^n \mid b^n$ in R or $b^n \mid a^nc$ in R for every nonunit c of R* (*see* [16, Proposition 2.9]).

(d) *Let P be a prime ideal of R. If R is a PnVD, then R/P is a PnVD* (*see* [16, Proposition 2.14]).

(d) *An n-root closed PnVD is a PVD* (*see* [16, Theorem 2.13]).

The next example gives some more n-PVDs that are not PnVDs.

Example 5.5. Let $m \geq 2$ be an integer. Then $R = \mathbb{R} + \mathbb{R}X^{m-1} + X^m\mathbb{C}[[X]]$ is quasilocal with maximal ideal $M = \mathbb{R}X^{m-1} + X^m\mathbb{C}[[X]]$, quotient field $K = \mathbb{C}[[X]][1/X]$, and integral closure $\overline{R} = \mathbb{C}[[X]]$. By Theorem 4.29, R is an n-PVD for every integer $n \geq m$. For a positive integer k, let $y = e^{-i\pi/2k}$. Then $y^k = -i \notin R$ and $y^{-k}X^{m-1} = iX^{m-1} \notin R$; so R is not a PkVD for any positive integer k by Theorem 5.4(a).

Remark 5.6. Let R an integral domain with quotient field K. Since $A_n(P) \subseteq P$ for every prime ideal P of R, every pseudo n-strongly prime ideal of R is also an n-powerful semiprimary ideal of R by Corollary 4.15 and Theorem 5.4(a), and thus a PnVD is an n-PVD. Hence, we have the following implications

$$n{-}VD \Rightarrow PnVD \Rightarrow n{-}PVD.$$

Neither of the above two implications is reversible. A PnVD need not be an n-VD by Theorem 5.13, and an n-PVD need not be a PnVD by Examples 5.3 and 5.5. Also, note that the ring in Example 5.1 is a 4-PVD, but not a P4VD.

The next theorem gives a case where an n-PVD is a PnVD. Note that the $n = 1$ case is just [11, Proposition 2.5]. We may have $M \neq (A_n(M))$ for every integer $n \geq 2$ (see Example 4.34(a)(b)). Note that in the next two theorems, we need the extra condition $(*)_n$ (cf. Example 4.34(d)(e), and recall that if R is not an n-PVD, then R is not a PnVD by Remark 5.6).

Theorem 5.7. *Let R be a quasilocal integral domain with maximal ideal $M = (A_n(M))$ and quotient field K. Then the following statements are equivalent.*

(1) *R is a PnVD.*
(2) *R is an n-PVD.*
(3) *$V = (M : M)$ is an n-VD with maximal ideal $\sqrt{MV} = \{x \in V \mid x^n \in M\}$, and if $x \in K$ is a nonunit of $\overline{R}$, then $x^n \in M$.*

Proof. (1) $\Rightarrow$ (2) A PnVD is an n-PVD by Remark 5.6.

(2) $\Rightarrow$ (1) Let $x \in E_n(R)$; so $x \in E_n(M)$. Then $x^{-n}A_n(M) \subseteq M$ by Corollary 4.16, and thus $x^{-n}M \subseteq M$ since $M = (A_n(M))$ by hypothesis. Hence R is a PnVD by Theorem 5.4(a)(b).

(2) $\Leftrightarrow$ (3) This is clear by Theorem 4.33. □

The following result recovers that a quasilocal integral domain R with maximal ideal M is a PVD if and only if $(M : M)$ is a valuation domain with maximal ideal M [11, Proposition 2.5]; its proof is an analog of the proof of [16, Theorem 2.15].

Theorem 5.8. *Let R be a quasilocal integral domain with maximal ideal M and quotient field K. Then the following statements are equivalent.*

(1) *R is a PnVD.*
(2) *$V = (M : M)$ is an n-VD with maximal ideal $\sqrt{MV} = \{x \in V \mid x^n \in M\}$, and if $x \in K$ is a nonunit of $\overline{R}$, then $x^n \in M$.*

Proof. (1) $\Rightarrow$ (2) Let R be a PnVD. Let $x \in E_n(V)$; so $x \in E_n(R)$. Then $x^{-n}M \subseteq M$ by Theorem 5.4(a); so $x^{-n} \in V$. Thus V is an n-VD with maximal ideal M_V. Let $x \in M_V$. If $x^n \in R$, then $x^n \in M$. Otherwise, $x \in E_n(R)$. Hence $x^{-n}M \subseteq M$ by Theorem 5.4(a) again; so $x^{-n} \in V$. Thus $x \in U(V)$, a contradiction. Hence $M_V \subseteq \{x \in V \mid x^n \in M\} \subseteq \sqrt{MV}$, and $\sqrt{MV} \subseteq M_V$ since $MV = M \subsetneq V$. Thus $M_V = \sqrt{MV} = \{x \in V \mid x^n \in M\}$. If $x \in K$ is a nonunit of $\overline{R}$, then $x^n \in M$ by Corollary 4.27 since a PnVD is an n-PVD by Remark 5.6.

(2) $\Rightarrow$ (1) Let $V = (M : M)$ be an n-VD with maximal ideal $\sqrt{MV} = \{x \in V \mid x^n \in M\}$. Suppose that $x \in E_n(R)$; so $x^n \notin M$. If $x^n \in V$ and $x^n \notin U(V)$, then $x^{n^2} = (x^n)^n \in M \subseteq R$; so $x \in \overline{R}$. Thus $x^n \in M$ by hypothesis, a contradiction. Hence $x^n \in U(V)$; so $x^{-n}M \subseteq M$. If $x^n \notin V$, then $x^{-n} \in V$ since V is an n-VD. Thus $x^{-n}M \subseteq M$ in either case; so R is a PnVD by Theorem 5.4(a)(b). □

Corollary 5.9. *Let R be a PnVD with maximal ideal M. If P is a prime ideal of R, then $W_P = (P : P)$ is an n-VD. Moreover, if $P \subseteq Q$ are prime ideals of R, then $W_Q = (Q : Q) \subseteq (P : P) = W_P$.*

Proof. We have $V = (M : M) \subseteq (P : P) = W_P$ by [6, Lemma 2.2] since $P \subseteq M$. Thus W_P is an n-VD since V is an n-VD by Theorem 5.8. The "moreover" statement is clear since $(Q : Q) \subseteq (P : P)$ by [6, Lemma 2.2] again. □

Let T be an overring of an integral domain R and n a positive integer. Then T is an *n-root extension* of R if $x^n \in R$ for every $x \in T$, and T is a *root extension* of R if for every $x \in T$, there is a positive integer m such that $x^m \in R$.

Theorem 5.10. *Let R be a quasilocal integral domain with maximal ideal M and quotient field K, n a positive integer, and V a valuation overring of R with maximal ideal $N = \{x \in V \mid x^n \in M\}$. Then R is an n-VD if and only if V is an n-root extension of R.*

Proof. We may assume that $R \subsetneq V$. Suppose that R is an n-VD. Let $x \in V \setminus R$. If $x \in N$, then $x^n \in M \subseteq R$. Thus, assume that $x \notin N$. Since N is the maximal ideal of V, we have $x \in U(V)$. Thus $x^n \notin M$ and $x^{-n} \notin M$. Since R is an n-VD, we have $x^n \in U(R) \subseteq R$. Hence V is an n-root extension of R.

Conversely, suppose that V is an n-root extension of R. Let $x \in K$ with $x^n \notin R$. Then $x \notin V$ since V is an n-root extension of R, and thus $x^{-1} \in V$ since V is a valuation domain. Hence $x^{-n} \in R$ since V is an n-root extension of R, and thus R is an n-VD. □

Lemma 5.11. *Let R be a quasilocal integral domain with maximal ideal M and quotient field K. If R is an n-VD, then $\overline{R}$ is a valuation domain with maximal ideal $\sqrt{M\overline{R}} = \{x \in K \mid x^n \in M\}$ and $R \subseteq \overline{R}$ is an n-root extension.*

Proof. Let R be an n-VD. Then R is an almost valuation domain; so $\overline{R}$ is a valuation domain and $R \subseteq \overline{R}$ is a root extension by [4, Theorem 5.6]. Thus $\sqrt{M\overline{R}} = \{x \in K \mid x^n \in M\}$ is the maximal ideal of $\overline{R}$ by Theorem 4.29 since an n-VD is an n-PVD. Hence $\overline{R}$ is an n-root extension of R by Theorem 5.10. □

Theorem 5.12. *Let R be a quasilocal integral domain with maximal ideal M and quotient field K, and let V be an n-VD overring of R with maximal ideal $N = \{x \in V \mid x^n \in M\}$. Then R is an n-VD if and only if $\overline{V} = \overline{R} = \{x \in K \mid x^n \in R\}$.*

Proof. We may assume that $R \subsetneq V$. Suppose that R is an n-VD. Then $\overline{R}$ is a valuation domain with maximal ideal $W = \{x \in K \mid x^n \in M\}$ and $R \subseteq \overline{R}$ is an n-root extension by Lemma 5.11. Similarly, since V is an n-VD, $\overline{V}$ is a valuation domain with maximal ideal $T = \{x \in K \mid x^n \in N\}$ and $V \subseteq \overline{V}$ is an n-root extension by Lemma 5.11. First, we show that $R \subsetneq V$ is an n-root extension. Let $x \in V \setminus R$. If $x \in N$, then $x^n \in M \subseteq R$. Hence, assume that $x \notin N$. Since N is the maximal ideal of V, we have $x \in U(V)$. Since $x \in U(V)$, neither $x^n \in M$ nor $x^{-n} \in M$. Since R is an n-VD, $x^n \in U(R) \subseteq R$. Thus V is an n-root extension of R. Since V is an integral overring of R, we have that $\overline{V}$ is integral over R, and thus $\overline{R} = \overline{V} = \{x \in K \mid x^n \in R\}$.

Conversely, suppose that $\overline{R} = \overline{V} = \{x \in K \mid x^n \in R\}$, and let $x \in K$ with $x^n \notin R$. Then $x \notin \overline{V}$, and thus $x^{-1} \in \overline{V}$ since $\overline{V}$ is a valuation domain by Lemma 5.11. Hence $x^{-n} \in R$; so R is an n-VD. □

Let V be a valuation domain with maximal ideal M, residue field $F = V/M$, and $\pi : V \longrightarrow F$ the canonical epimorphism. If k is a subfield of F, then $R = \pi^{-1}(k)$ is a PVD with maximal ideal M [11, Proposition 2.6]. Moreover, every PVD arises in this way. Let R be a PVD with maximal ideal M. Then $V = (M : M)$ is a valuation domain with maximal ideal M [11, Proposition 2.5]; so $R = \pi^{-1}(R/M)$. A similar result holds for PnVDs and n-VDs.

Theorem 5.13. *Let V be an n-VD with nonzero maximal ideal M, residue field $F = V/M$, $\pi : V \longrightarrow F$ the canonical epimorphism, k a subfield of F, and $R = \pi^{-1}(k)$. Then the pullback $R = V \times_F k$ is a PnVD with maximal ideal M. In particular, if k is properly contained in F and V is not an n-root extension of R, then R is a PnVD which is not an n-VD.*

Proof. In view of the construction stated in the hypothesis, it is well known that M is a maximal ideal of R for any integral domain V. Also, it is clear that R and V have the same quotient field K by [11, Lemma 3.1]. Let $x \in E_n(R)$. Then

$x^n \in V$ or $x^{-n} \in V$ since V is an n-VD. Suppose that $x^n \in V$. Since $x \in E_n(R)$ and M is the maximal ideal of R, we have $x^n \notin M$. Thus $x^n \in U(V)$, and hence $x^{-n} \in V$; so $x^{-n}M \subseteq M$ since M is an ideal of V. Now suppose that $x^{-n} \in V$. Then $x^{-n}M \subseteq M$ since M is an ideal of V. Thus M is a pseudo n-strongly prime ideal of R by Theorem 5.4(a), and hence R is a PnVD by Theorem 5.4(b). The remaining part is clear from Theorem 5.12. □

The final example illustrates the previous theorem.

Example 5.14. (a) Let $V = \mathbb{Z}_p(t)[[X]]$. Then V is a valuation domain; so $R = \mathbb{Z}_p + X\mathbb{Z}_p(t)[[X]]$ is a PnVD for every positive integer n, but not an n-VD for any positive integer n, by Theorem 5.13 since V is not an n-root extension of R. Note that R is actually a PVD.

(b) Let $T = K + M$ be a quasilocal integral domain with maximal ideal M and K a subfield of T. Let k be a subfield of K and $R = k + M$. Then R is also quasilocal with maximal ideal M. Thus R is an n-PVD (resp., PnVD) if and only if T is an n-PVD (resp., PnVD) by Corollary 4.8 (resp., Theorem 5.4(b)).

For example, $T = \mathbb{R}[[X^2, X^3]] = \mathbb{R} + X^2\mathbb{R}[[X]]$ is an n-PVD $\Leftrightarrow n \geq 2$ (Example 4.34(b)), and thus $R = \mathbb{Q} + X^2\mathbb{R}[[X]]$ is an n-PVD $\Leftrightarrow n \geq 2$.

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References

[1] D. D. Anderson and D. F. Anderson, Multiplicatively closed subsets of fields, Houston J. Math. 13(1987), 1–11.
[2] D. D. Anderson and M. Winders, Idealization of a module, J. Commut. Algebra 1(2009), 3–56.
[3] D. D. Anderson, K. R. Knopp. and R. L. Lewin, Almost Bezout domains, II, J. Algebra 167(1994), 547–556.
[4] D. D. Anderson and M. Zafrullah, Almost Bezout domains, J. Algebra 142(1991), 285–309.
[5] D. D. Anderson and M. Zafrullah, Almost Bezout domains, III, Bull. Math. Soc. Sci. Math. Roumanie 51(2008), 3–9.
[6] D. F. Anderson, Comparability of ideals and valuation overrings, Houston J. Math. 5(1979), 451–463.
[7] D. F. Anderson, Root closure in integral domains, J. Algebra 79(1982), 51–59.
[8] D. F. Anderson, When the dual of an ideal is a ring, Houston J. Math. 9(1983), 325–332.
[9] D. F. Anderson and A. Badawi, On n-absorbing ideals of commutative rings, Comm. Algebra 39(2011), 1646–1672.
[10] D. F. Anderson and A. Badawi, On (m,n)-closed ideals of commutative rings, J. Algebra Appl. 16(2017)(21 pages).
[11] D. F. Anderson and D. E. Dobbs, Pairs of rings with the same prime ideals, Canad. J. Math. 32(1980), 362–384.
[12] D. F. Anderson and D. E. Dobbs, On the product of ideals, Canad. Math. Bull. 26(1983), 238–246.
[13] A. Badawi, On abelian π-regular rings, Comm. Algebra 25(1997), 1009–1021.
[14] A. Badawi, On divided commutative rings, Comm. Algebra 27(1999), 1465–1474.
[15] A. Badawi, On 2-absorbing ideals of commutative rings, Bull. Austral. Math. Soc. 75 (2007), 417–429.
[16] A. Badawi, On pseudo-almost valuation domains, Comm. Algebra 35(2007), 1167–1181.
[17] A. Badawi and E. Houston, Powerful ideals, strongly primary ideals, pseudo-valuation domains, and conducive domains, Comm. Algebra 30(2002), 1591–1606.
[18] H. S. Choi and A. Walker, The radical of an n-absorbing ideal, arXiv:1610.10077 [math.AC], J. Commut. Algebra, to appear.

[19] J. A. Cox and A. J. Hetzel, Uniformly primary ideals, J. Pure. Appl. Algebra 212(2008), 1–8.
[20] D. E. Dobbs, Divided rings and going-down, Pacific J. Math 67(1976), 353–363.
[21] D. E. Dobbs, Coherence, ascent of going-down, and pseudo-valuation domains, Houston J. Math. 4(1978), 551–576.
[22] G. Donadze, A proof of the Anderson–Badawi $rad(I)^n \subseteq I$ formula for n-absorbing ideals, Proc. Indian Acad. Sci. (Math. Sci.) 128(2018), 6–11.
[23] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, Inc., New York, 1972.
[24] J. R. Hedstrom and E. G. Houston, Pseudo-valuation domains, Pacific J. Math. 75(1978), 137–147.
[25] J. R. Hedstrom and E. G. Houston, Pseudo-valuation domains, II, Houston J. Math. 4(1978), 199–207.
[26] J. Huckaba, *Rings with Zero Divisors*, Marcel Dekker, Inc., New York, 1988.
[27] I. Kaplansky, *Commutative Rings, Revised Edition*, The University of Chicago Press, Chicago and London, 1974.
[28] M. D. Larson and P. J. McCarthy, *Multiplicative Theory of Ideals*, Academic Press, New York/London, 1971.
[29] N. Mahdou, A. Mimouni, and M. Moutui, On almost valuation and almost Bezout rings, Comm. Algebra 43(2015), 297–308.

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