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The total graph of a commutative ring

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Abstract

Let R be a commutative ring with $Nil(R)$ its ideal of nilpotent elements, $Z(R)$ its set of zero-divisors, and $Reg(R)$ its set of regular elements. In this paper, we introduce and investigate the *total graph* of R, denoted by $T(\Gamma(R))$. It is the (undirected) graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in Z(R)$. We also study the three (induced) subgraphs $Nil(\Gamma(R))$, $Z(\Gamma(R))$, and $Reg(\Gamma(R))$ of $T(\Gamma(R))$, with vertices $Nil(R)$, $Z(R)$, and $Reg(R)$, respectively.

Keywords: Commutative rings; Zero-divisor graph; Regular elements; Zero-divisors

1. Introduction

We assume throughout that all rings are commutative with $1 \neq 0$. Let R be a commutative ring with $T(R)$ its total quotient ring, $Reg(R)$ its set of regular elements, $Z(R)$ its set of zero-divisors, and $Nil(R)$ its ideal of nilpotent elements. In [3], Anderson and Livingston introduced the *zero-divisor graph* of R, denoted by $\Gamma(R)$, as the (undirected) graph with vertices $Z(R)^* = Z(R) \setminus \{0\}$, the set of nonzero zero-divisors of R, and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$. This concept is due to Beck [7], who let all the elements of R be vertices and was mainly interested in colorings. For some other recent papers on zero-divisor graphs, see [1,2,4–6,12–14].

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In this paper, we introduce the *total graph* of R, denoted by $T(\Gamma(R))$, as the (undirected) graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in Z(R)$. Let $Reg(\Gamma(R))$ be the (induced) subgraph of $T(\Gamma(R))$ with vertices $Reg(R)$, let $Z(\Gamma(R))$ be the (induced) subgraph of $T(\Gamma(R))$ with vertices $Z(R)$, and let $Nil(\Gamma(R))$ be the (induced) subgraph of $T(\Gamma(R))$ (and $Z(\Gamma(R))$) with vertices $Nil(R)$. Note that if A is a subring of a commutative ring B, then $T(\Gamma(A))$ need not be an induced subgraph of $T(\Gamma(B))$. Although $x, y \in A$ are adjacent in $T(\Gamma(B))$ if they are adjacent in $T(\Gamma(A))$ since $Z(A) \subseteq Z(B)$, they may be adjacent in $T(\Gamma(B))$, but not adjacent in $T(\Gamma(A))$. In fact, $T(\Gamma(A))$ is an induced subgraph of $T(\Gamma(B))$ if and only if $Z(B) \cap A = Z(A)$.

The study of $T(\Gamma(R))$ breaks naturally into two cases depending on whether or not $Z(R)$ is an ideal of R. In the second section, we handle the case when $Z(R)$ is an ideal of R; in the third section, we do the case when $Z(R)$ is not an ideal of R. The subgraph $Z(\Gamma(R))$ of $T(\Gamma(R))$ is always connected, and $Z(\Gamma(R))$ is complete if and only if $Z(R)$ is an ideal of R. Moreover, if $Z(R)$ is an ideal of R, then $Z(\Gamma(R))$ and $Reg(\Gamma(R))$ are disjoint subgraphs of $T(\Gamma(R))$, and $Reg(\Gamma(R))$ is the union of disjoint subgraphs, each of which is either a complete graph or a complete bipartite graph. However, if $Z(R)$ is not an ideal of R, then the subgraphs $Z(\Gamma(R))$ and $Reg(\Gamma(R))$ of $T(\Gamma(R))$ are never disjoint, and $T(\Gamma(R))$ is connected if and only if $(Z(R)) = R$.

Let G be a graph. We say that G is *connected* if there is a path between any two distinct vertices of G. At the other extreme, we say that G is *totally disconnected* if no two vertices of G are adjacent. For vertices x and y of G, we define $d(x, y)$ to be the length of a shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path). The *diameter* of G is $\mathrm{diam}(G) = \sup\{d(x, y) \mid x \text{ and } y \text{ are vertices of } G\}$. The *girth* of G, denoted by $\mathrm{gr}(G)$, is the length of a shortest cycle in G ($\mathrm{gr}(G) = \infty$ if G contains no cycles). We denote the complete graph on n vertices by K^n and the complete bipartite graph on m and n vertices by $K^{m,n}$ (we allow m and n to be infinite cardinals). We will sometimes call a $K^{1,n}$ a *star graph*. We say that two (induced) subgraphs G_1 and G_2 of G are *disjoint* if G_1 and G_2 have no common vertices and no vertex of G_1 (respectively, G_2) is adjacent (in G) to any vertex not in G_1 (respectively, G_2). A general reference for graph theory is [8].

As usual, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$, and $\mathbb{F}_q$ will denote the integers, rational numbers, integers modulo n, and the finite field with q elements, respectively. The group of units of a commutative ring R will be denoted by $U(R)$, the nonzero elements of $A \subseteq R$ will be denoted by A^*, and $\subset$ will denote proper inclusion. We say that R is *reduced* if $Nil(R) = \{0\}$. General references for ring theory are [10] and [11].

Throughout this paper, we will use the technique of idealization of a module to construct examples. Recall that for an R-module M, the *idealization of M over R* is the commutative ring formed from $R \times M$ by defining addition and multiplication as $(r, m) + (s, n) = (r + s, m + n)$ and $(r, m)(s, n) = (rs, rn + sm)$, respectively. A standard notation for this "idealized ring" is $R(+)M$; see [10] for basic properties of rings resulting from the idealization construction. The zero-divisor graph $\Gamma(R(+)M)$ has recently been studied in [4] and [6].

2. The case when $Z(R)$ is an ideal of R

In this section, we study the case when $Z(R)$ is an ideal of R (i.e., when $Z(R)$ is closed under addition). Note that since $Z(R)$ is a union of prime ideals of R [11, p. 3], we always have $xy \in Z(R)$ for $x, y \in R \Rightarrow x \in Z(R)$ or $y \in Z(R)$. So if $Z(R)$ is an ideal of R, then $Z(R)$ is actually a prime ideal of R, and hence $R/Z(R)$ is an integral domain. Moreover, if R is a finite commutative ring and $Z(R)$ is an ideal of R, then R is local with $Z(R) = Nil(R)$ its unique maximal ideal. The main goal of this section is a general structure theorem (Theorem 2.2) for $Reg(\Gamma(R))$ when $Z(R)$ is an ideal of R. But first, we record the trivial observation that if $Z(R)$ is an ideal of R, then $Z(\Gamma(R))$ is a complete subgraph of $T(\Gamma(R))$ and is disjoint from $Reg(\Gamma(R))$. Thus we will concentrate on the subgraph $Reg(\Gamma(R))$ throughout this section.

Theorem 2.1. *Let R be a commutative ring such that $Z(R)$ is an ideal of R. Then $Z(\Gamma(R))$ is a complete (induced) subgraph of $T(\Gamma(R))$ and $Z(\Gamma(R))$ is disjoint from $Reg(\Gamma(R))$.*

Proof. This follows directly from the definitions. □

We now give the main result of this section. Since $Z(\Gamma(R))$ is a complete subgraph of $T(\Gamma(R))$ and is disjoint from $Reg(\Gamma(R))$, our next theorem also gives a complete description of $T(\Gamma(R))$. We allow α and β to be infinite cardinals; if β is infinite, then of course $\beta - 1 = (\beta - 1)/2 = \beta$.

Theorem 2.2. *Let R be a commutative ring such that $Z(R)$ is an ideal of R, and let $|Z(R)| = \alpha$ and $|R/Z(R)| = \beta$.*

(1) *If $2 \in Z(R)$, then $Reg(\Gamma(R))$ is the union of $\beta - 1$ disjoint K^{α}'s.*
(2) *If $2 \notin Z(R)$, then $Reg(\Gamma(R))$ is the union of $(\beta - 1)/2$ disjoint $K^{\alpha,\alpha}$'s.*

Proof. (1) Assume that $2 \in Z(R)$, and let $x \in Reg(R)$. Then each coset $x + Z(R)$ is a complete subgraph of $Reg(\Gamma(R))$ since $(x + z_1) + (x + z_2) = 2x + z_1 + z_2 \in Z(R)$ for all $z_1, z_2 \in Z(R)$ since $2 \in Z(R)$ and $Z(R)$ is an ideal of R. Note that distinct cosets form disjoint subgraphs of $Reg(\Gamma(R))$ since if $y + z_1$ and $x + z_2$ are adjacent for some $y \in Reg(R)$ and $z_1, z_2 \in Z(R)$, then $x + y = (x + z_1) + (y + z_2) - (z_1 + z_2) \in Z(R)$, and hence $x - y = (x + y) - 2y \in Z(R)$ since $Z(R)$ is an ideal of R and $2 \in Z(R)$. But then $x + Z(R) = y + Z(R)$. Thus $Reg(\Gamma(R))$ is the union of $\beta - 1$ disjoint (induced) subgraphs $x + Z(R)$, each of which is a K^{α}, where $\alpha = |Z(R)| = |x + Z(R)|$.

(2) Next assume that $2 \notin Z(R)$, and let $x \in Reg(R)$. Then no two distinct elements in $x + Z(R)$ are adjacent since $(x + z_1) + (x + z_2) \in Z(R)$ for $z_1, z_2 \in Z(R)$ implies that $2x \in Z(R)$, and hence $2 \in Z(R)$, a contradiction. Also, the two cosets $x + Z(R)$ and $-x + Z(R)$ are disjoint, and each element of $x + Z(R)$ is adjacent to each element of $-x + Z(R)$. Thus $(x + Z(R)) \cup (-x + Z(R))$ is a complete bipartite (induced) subgraph of $Reg(\Gamma(R))$. Furthermore, if $y + z_1$ is adjacent to $x + z_2$ for some $y \in Reg(R)$ and $z_1, z_2 \in Z(R)$, then $x + y \in Z(R)$, and hence $y + Z(R) = -x + Z(R)$. Thus $Reg(\Gamma(R))$ is the union of $(\beta - 1)/2$ disjoint (induced) subgraphs $(x + Z(R)) \cup (-x + Z(R))$, each of which is a $K^{\alpha,\alpha}$, where $\alpha = |Z(R)| = |x + Z(R)|$. □

Remark 2.3. Note that if $Z(R) = \{0\}$ (i.e., if R is an integral domain), then $2 \in Z(R)$ if and only if $\operatorname{char} R = 2$. This need not hold if R is not an integral domain; for example, consider $R = \mathbb{Z}_4$. If R is an integral domain with $\operatorname{char} R = 2$, then $Reg(\Gamma(R))$ is the union of $\beta - 1$ disjoint K^1's. If R is an integral domain with $\operatorname{char} R \neq 2$, then $Reg(\Gamma(R))$ is the union of $(\beta - 1)/2$ disjoint $K^{1,1}$'s ($= K^2$'s).

From the above theorem, we can easily deduce when $Reg(\Gamma(R))$ is complete or connected, and we can explicitly compute its diameter and girth. We first determine when $Reg(\Gamma(R))$ is either complete or connected.

Theorem 2.4. *Let R be a commutative ring such that $Z(R)$ is an ideal of R. Then*

(1) *$Reg(\Gamma(R))$ is complete if and only if either $R/Z(R) \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_3$.*
(2) *$Reg(\Gamma(R))$ is connected if and only if either $R/Z(R) \cong \mathbb{Z}_2$ or $R/Z(R) \cong \mathbb{Z}_3$.*
(3) *$Reg(\Gamma(R))$ (and hence $Z(\Gamma(R))$ and $T(\Gamma(R))$) is totally disconnected if and only if R is an integral domain with* $\operatorname{char} R = 2$.

Proof. Let $|Z(R)| = \alpha$ and $|R/Z(R)| = \beta$.

(1) By Theorem 2.2, $Reg(\Gamma(R))$ is complete if and only if $Reg(\Gamma(R))$ is a single K^{α} or $K^{1,1}$. If $2 \in Z(R)$, then $\beta - 1 = 1$. Thus $\beta = 2$, and hence $R/Z(R) \cong \mathbb{Z}_2$. If $2 \notin Z(R)$, then $\alpha = 1$ and $(\beta - 1)/2 = 1$. Thus $Z(R) = \{0\}$ and $\beta = 3$; so $R \cong R/Z(R) \cong \mathbb{Z}_3$.

(2) By Theorem 2.2, $Reg(\Gamma(R))$ is connected if and only if $Reg(\Gamma(R))$ is a single K^{α} or $K^{\alpha,\alpha}$. Thus either $\beta - 1 = 1$ if $2 \in Z(R)$ or $(\beta - 1)/2 = 1$ if $2 \notin Z(R)$; so $\beta = 2$ or $\beta = 3$, respectively. Thus $R/Z(R) \cong \mathbb{Z}_2$ or $R/Z(R) \cong \mathbb{Z}_3$, respectively.

(3) $Reg(\Gamma(R))$ is totally disconnected if and only if it is a disjoint union of K^1's. So by Theorem 2.2, R must be an integral domain with $2 \in Z(R)$, i.e., $\operatorname{char} R = 2$. □

It is also easy to compute the diameter and girth of $Reg(\Gamma(R))$ using Theorem 2.2.

Theorem 2.5. *Let R be a commutative ring such that $Z(R)$ is an ideal of R. Then*

(1) $\operatorname{diam}(Reg(\Gamma(R))) = 0, 1, 2,$ *or* ∞. *In particular,* $\operatorname{diam}(Reg(\Gamma(R))) \leqslant 2$ *if $Reg(\Gamma(R))$ is connected.*
(2) $\operatorname{gr}(Reg(\Gamma(R))) = 3, 4,$ *or* ∞. *In particular,* $\operatorname{gr}(Reg(\Gamma(R))) \leqslant 4$ *if $Reg(\Gamma(R))$ contains a cycle.*

Proof. (1) Suppose that $Reg(\Gamma(R))$ is connected. Then $Reg(\Gamma(R))$ is a singleton, a complete graph, or a complete bipartite graph by Theorem 2.2. Thus $\operatorname{diam}(Reg(\Gamma(R))) \leqslant 2$.

(2) Suppose that $Reg(\Gamma(R))$ contains a cycle. Since $Reg(\Gamma(R))$ is a disjoint union of either complete or complete bipartite graphs by Theorem 2.2, it must contain either a 3-cycle or a 4-cycle. Thus $\operatorname{gr}(Reg(\Gamma(R))) \leqslant 4$. □

The next theorem gives a more explicit description of the diameter and girth of $Reg(\Gamma(R))$.

Theorem 2.6. *Let R be a commutative ring such that $Z(R)$ is an ideal of R.*

(1) (a) $\operatorname{diam}(Reg(\Gamma(R))) = 0$ *if and only if $R \cong \mathbb{Z}_2$.*
  (b) $\operatorname{diam}(Reg(\Gamma(R))) = 1$ *if and only if either $R/Z(R) \cong \mathbb{Z}_2$ and $R \ncong \mathbb{Z}_2$ (i.e., $R/Z(R) \cong \mathbb{Z}_2$ and $|Z(R)| \geqslant 2$), or $R \cong \mathbb{Z}_3$.*
  (c) $\operatorname{diam}(Reg(\Gamma(R))) = 2$ *if and only if $R/Z(R) \cong \mathbb{Z}_3$ and $R \ncong \mathbb{Z}_3$ (i.e., $R/Z(R) \cong \mathbb{Z}_3$ and $|Z(R)| \geqslant 2$).*
  (d) *Otherwise,* $\operatorname{diam}(Reg(\Gamma(R))) = \infty$.
(2) (a) $\operatorname{gr}(Reg(\Gamma(R))) = 3$ *if and only if $2 \in Z(R)$ and $|Z(R)| \geqslant 3$.*
  (b) $\operatorname{gr}(Reg(\Gamma(R))) = 4$ *if and only if $2 \notin Z(R)$ and $|Z(R)| \geqslant 2$.*
  (c) *Otherwise,* $\operatorname{gr}(Reg(\Gamma(R))) = \infty$.
(3) (a) $\operatorname{gr}(T(\Gamma(R))) = 3$ *if and only if $|Z(R)| \geqslant 3$.*
  (b) $\operatorname{gr}(T(\Gamma(R))) = 4$ *if and only if $2 \notin Z(R)$ and $|Z(R)| = 2$.*
  (c) *Otherwise,* $\operatorname{gr}(T(\Gamma(R))) = \infty$.

Proof. These results all follow directly from Theorems 2.1 and 2.2. □

The following examples illustrate the previous theorems.

Example 2.7. (a) Let $m \geqslant 2$ be an integer. Then $Z(\mathbb{Z}_m)$ is an ideal of $\mathbb{Z}_m$ if and only if $m = p^n$ for some prime p and integer $n \geqslant 1$. So suppose that $Z(\mathbb{Z}_m)$ is an ideal of $\mathbb{Z}_m$. Thus $Reg(\Gamma(\mathbb{Z}_m))$ is connected if and only if either $m = 2^n$ or $m = 3^n$ for some integer $n \geqslant 1$. Moreover, $Reg(\Gamma(\mathbb{Z}_m))$ is complete if and only if either $m = 2^n$ for some integer $n \geqslant 1$ or $m = 3$. An analogous result holds for any PID.

(b) Let K be a field with $|K| = \alpha$, $n \geqslant 2$ an integer, and $R = K[X]/(X^n)$. Then R is local with maximal ideal $Z(R) = Nil(R) = (X)/(X^n)$, $R/Z(R) \cong K$, $|R| = \alpha^n$, $|Z(R)| = \alpha^{n-1}$, $|Reg(R)| = \alpha^{n-1}(\alpha - 1)$, and $|R/Z(R)| = \alpha$. If $\operatorname{char} K = 2$, then $Reg(\Gamma(R))$ is the union of $\alpha - 1$ disjoint complete graphs K^m, where $m = \alpha^{n-1}$. If $\operatorname{char} K \neq 2$, then $Reg(\Gamma(R))$ is the union of $(\alpha - 1)/2$ disjoint complete bipartite graphs $K^{m,m}$, where $m = \alpha^{n-1}$. Thus $Reg(\Gamma(R))$ is connected if and only if $K \cong \mathbb{Z}_2$ or $K \cong \mathbb{Z}_3$, and $Reg(\Gamma(R))$ is complete if and only if $K \cong \mathbb{Z}_2$.

For the special case when $K = \mathbb{F}_{p^k}$, we have that $Reg(\Gamma(R))$ is the union of $2^k - 1$ disjoint K^m's, where $m = 2^{k(n-1)}$, when $p = 2$; and $Reg(\Gamma(R))$ is the union of $(p^k - 1)/2$ disjoint $K^{m,m}$'s, where $m = p^{k(n-1)}$, when $p \neq 2$.

(c) Let $m \geqslant 2$ be an integer and $R = \mathbb{Z}(+)\mathbb{Z}_m$. Then $Z(R)$ is an ideal of R if and only if $m = p^n$ for some prime p and integer $n \geqslant 1$. Moreover, $Z(R) = p\mathbb{Z}(+)\mathbb{Z}_{p^n}$ and $R/Z(R) \cong \mathbb{Z}_p$ when $m = p^n$ and $n \geqslant 1$. So in this case, $Reg(\Gamma(R))$ is connected if and only if $p = 2$ or 3, and $Reg(\Gamma(R))$ is complete if and only if $p = 2$. For any $0 \neq a \in \mathbb{Z}_m$, the 3-cycle $(1,0) - (-1,0) - (1,a) - (1,0)$ shows that $\operatorname{gr}(Reg(\Gamma(R))) = 3$ when $m = 2^n$; and the 4-cycle $(1,0) - (-1,0) - (1-p,0) - (p-1,0) - (1,0)$ shows that $\operatorname{gr}(Reg(\Gamma(R))) = 4$ when $m = p^n$ and $p \neq 2$.

Specifically, let $R_1 = \mathbb{Z}(+)\mathbb{Z}_2$ and $R_2 = \mathbb{Z}(+)\mathbb{Z}_3$. Then $Reg(\Gamma(R_1))$ is complete with $\operatorname{diam}(Reg(\Gamma(R_1))) = 1$ and $\operatorname{gr}(Reg(\Gamma(R_1))) = 3$, and $Reg(\Gamma(R_2))$ is connected (but not complete) with $\operatorname{diam}(Reg(\Gamma(R_1))) = 2$ and $\operatorname{gr}(Reg(\Gamma(R_2))) = 4$. (See Theorem 3.21 for the case when $Z(R)$ is not an ideal of R.)

Many of the earlier results of this section can also be easily proved directly without recourse to Theorem 2.2. We give two such cases.

Theorem 2.8. *Let R be a commutative ring such that $Z(R)$ is an ideal of R.*

(1) *Let G be an induced subgraph of $Reg(\Gamma(R))$, and let x and y be distinct vertices of G that are connected by a path in G. Then there is a path in G of length at most 2 between x and y. In particular, if $Reg(\Gamma(R))$ is connected, then $\operatorname{diam}(Reg(\Gamma(R))) \leqslant 2$.*
(2) *Let x and y be distinct regular elements of R that are connected by a path. If $x + y \notin Z(R)$ (i.e., if x and y are not adjacent), then $x - (-x) - y$ and $x - (-y) - y$ are paths of length 2 between x and y in $Reg(\Gamma(R))$.*

Proof. (1) It suffices to show that if x_1, x_2, x_3, and x_4 are distinct vertices of G and there is a path $x_1 - x_2 - x_3 - x_4$ from x_1 to x_4, then x_1 and x_4 are adjacent. Now $x_1 + x_2, x_2 + x_3, x_3 + x_4 \in Z(R)$ implies $x_1 + x_4 = (x_1 + x_2) - (x_2 + x_3) + (x_3 + x_4) \in Z(R)$ since $Z(R)$ is an ideal of R. Thus x_1 and x_4 are adjacent.

(2) Suppose that $x + y \notin Z(R)$. Then there is a $z \in Reg(R)$ such that $x - z - y$ is a path of length 2 by part (1) above (note that necessarily $z \in Reg(R)$ since $x, y \in Reg(R)$). Thus $x + z$,

$z + y \in Z(R)$, and hence $x - y = (x + z) - (z + y) \in Z(R)$ since $Z(R)$ is an ideal of R. Also, $x \neq -x$ and $y \neq -x$ since $x + y \notin Z(R)$. Thus $x - (-x) - y$ and $x - (-y) - y$ are paths of length 2 between x and y in $Reg(\Gamma(R))$. □

We have already observed that $Z(\Gamma(R))$ is always connected and $T(\Gamma(R))$ is never connected when $Z(R)$ is an ideal of R. We next give several new criteria for when $Reg(\Gamma(R))$ is connected.

Theorem 2.9. *Let R be a commutative ring such that $Z(R)$ is an ideal of R. Then the following statements are equivalent.*

(1) *$Reg(\Gamma(R))$ is connected.*
(2) *Either $x + y \in Z(R)$ or $x - y \in Z(R)$ for all $x, y \in Reg(R)$.*
(3) *Either $x + y \in Z(R)$ or $x + 2y \in Z(R)$ for all $x, y \in Reg(R)$. In particular, either $2x \in Z(R)$ or $3x \in Z(R)$ (but not both) for all $x \in Reg(R)$.*
(4) *Either $R/Z(R) \cong \mathbb{Z}_2$ or $R/Z(R) \cong \mathbb{Z}_3$.*

Proof. (1) ⇒ (2) Suppose that $Reg(\Gamma(R))$ is connected, and let $x, y \in Reg(R)$. If $x = y$, then $x - y \in Z(R)$. Hence assume that $x \neq y$. If $x + y \notin Z(R)$, then $x - (-y) - (y)$ is a path from x to y by Theorem 2.8(2), and thus $x - y \in Z(R)$.

(2) ⇒ (3) Let $x, y \in Reg(R)$, and suppose that $x + y \notin Z(R)$. Since $(x + y) - y = x \notin Z(R)$, thus $x + 2y = (x + y) + y \in Z(R)$ by hypothesis. In particular, if $x \in Reg(R)$, then either $2x \in Z(R)$ or $3x \in Z(R)$. Both $2x$ and $3x$ cannot be in $Z(R)$ since then $x = 3x - 2x \in Z(R)$, a contradiction.

(3) ⇒ (1) Let $x, y \in Reg(R)$ be distinct elements of R such that $x + y \notin Z(R)$. Then $x + 2y \in Z(R)$ by hypothesis. Since $Z(R)$ is an ideal of R and $x + 2y \in Z(R)$, we conclude that $2y \notin Z(R)$. Thus $3y \in Z(R)$ by hypothesis. Since $x + y \notin Z(R)$ and $3y \in Z(R)$, we conclude that $x \neq 2y$, and hence $x - 2y - y$ is a path from x to y in $Reg(\Gamma(R))$. Thus $Reg(\Gamma(R))$ is connected.

(2) ⇒ (4) Let $x \in Reg(R)$. Then either $x - 1 \in Z(R)$ or $x + 1 \in Z(R)$ by hypothesis, and thus either $x + Z(R) = 1 + Z(R)$ or $x + Z(R) = -1 + Z(R)$. If $2 \in Z(R)$, then $R/Z(R) \cong \mathbb{Z}_2$; otherwise, $R/Z(R) \cong \mathbb{Z}_3$.

(4) ⇒ (2) This is clear. □

One can also can consider the two (induced) subgraphs $Nil(\Gamma(R))$ and $U(\Gamma(R))$ of $T(\Gamma(R))$ (and $Z(\Gamma(R))$ and $Reg(\Gamma(R))$, respectively) with vertices $Nil(R) \subseteq Z(R)$ and $U(R) \subseteq Reg(R)$, respectively. The basic properties of $Nil(\Gamma(R))$ are given below and show that $Nil(\Gamma(R))$ has a very simple structure, independent of whether or not $Z(R)$ is an ideal of R (cf. Theorems 2.1 and 3.1). Basic properties of $U(\Gamma(R))$ are left to the reader.

Theorem 2.10. *Let R be a commutative ring.*

(1) *$Nil(\Gamma(R))$ is a complete (induced) subgraph of $Z(\Gamma(R))$.*
(2) *Each vertex of $Nil(\Gamma(R))$ is adjacent to each distinct vertex of $Z(\Gamma(R))$.*
(3) *$Nil(\Gamma(R))$ is disjoint from $Reg(\Gamma(R))$.*
(4) *If $\{0\} \neq Nil(R) \subset Z(R)$, then $\mathrm{gr}(Z(\Gamma(R))) = 3$.*

Proof. Part (1) follows since $Nil(R) \subseteq Z(R)$ is an ideal of R. Parts (2) and (3) follow from the facts that $Nil(R) + Z(R) \subseteq Z(R)$ and $Nil(R) + Reg(R) \subseteq Reg(R)$ for any commutative ring R, respectively.

(4) Let $x \in Nil(R)^*$ and $y \in Z(R) \setminus Nil(R)$. Then $0 - x - y - 0$ is a 3-cycle in $Z(\Gamma(R))$ by part (2) above; so $\mathrm{gr}(Z(\Gamma(R))) = 3$. □

3. The case when $Z(R)$ is not an ideal of R

In this section, we consider the remaining case when $Z(R)$ is not an ideal of R. Since $Z(R)$ is always closed under multiplication by elements of R, this just means that there are distinct $x, y \in Z(R)^*$ such that $x + y \in Reg(R)$. In this case, $Z(\Gamma(R))$ is always connected (but never complete), $Z(\Gamma(R))$ and $Reg(\Gamma(R))$ are never disjoint subgraphs of $T(\Gamma(R))$, and $|Z(R)| \geqslant 3$. We first show that $T(\Gamma(R))$ is connected when $Reg(\Gamma(R))$ is connected. However, we give an example to show that the converse fails.

Theorem 3.1. *Let R be a commutative ring such that $Z(R)$ is not an ideal of R.*

(1) $Z(\Gamma(R))$ *is connected with* $\mathrm{diam}(Z(\Gamma(R))) = 2$.
(2) *Some vertex of $Z(\Gamma(R))$ is adjacent to a vertex of $Reg(\Gamma(R))$. In particular, the subgraphs $Z(\Gamma(R))$ and $Reg(\Gamma(R))$ of $T(\Gamma(R))$ are not disjoint.*
(3) *If $Reg(\Gamma(R))$ is connected, then $T(\Gamma(R))$ is connected.*

Proof. (1) Each $x \in Z(R)^*$ is adjacent to 0. Thus $x - 0 - y$ is a path in $Z(\Gamma(R))$ of length two between any two distinct $x, y \in Z(R)^*$. Moreover, there are nonadjacent $x, y \in Z(R)^*$ since $Z(R)$ is not an ideal of R; so $\mathrm{diam}(Z(\Gamma(R))) = 2$.

(2) Since $Z(R)$ is not an ideal of R, there are distinct $x, y \in Z(R)^*$ such that $x + y \in Reg(R)$. Then $-x \in Z(R)$ and $x + y \in Reg(R)$ are adjacent vertices in $T(\Gamma(R))$ since $-x + (x + y) = y \in Z(R)$. The "in particular" statement is clear.

(3) Suppose that $Reg(\Gamma(R))$ is connected. Since $Z(\Gamma(R))$ is also connected by part (1) above, it is sufficient to show that there is a path from x to y in $T(\Gamma(R))$ for any $x \in Z(R)$ and $y \in Reg(R)$. By part (2) above, there are adjacent vertices z and w in $Z(\Gamma(R))$ and $Reg(\Gamma(R))$, respectively. Since $Z(\Gamma(R))$ is connected, there is a path from x to z in $Z(\Gamma(R))$; and since $Reg(\Gamma(R))$ is connected, there is a path from w to y in $Reg(\Gamma(R))$. As z and w are adjacent in $T(\Gamma(R))$, there is a path from x to y in $T(\Gamma(R))$. Thus $T(\Gamma(R))$ is connected. □

Example 3.2. Let $R = \mathbb{Q}[X](+)(\mathbb{Q}(X)/\mathbb{Q}[X])$. Then one can easily show that $Z(R) = (\mathbb{Q}[X] \setminus \mathbb{Q}^*)(+)(\mathbb{Q}(X)/\mathbb{Q}[X])$ is not an ideal of R and $Reg(R) = U(R) = \mathbb{Q}^*(+)(\mathbb{Q}(X)/\mathbb{Q}[X])$. Thus $T(\Gamma(R))$ is connected with $\mathrm{diam}(T(\Gamma(R))) = 2$ (by Theorems 3.3 and 3.4 below) since $R = ((X, 0), (X + 1, 0))$ with $(X, 0), (X + 1, 0) \in Z(R)$. However, $Reg(\Gamma(R))$ is not connected since there is no path from $(1, 0)$ to $(2, 0)$ in $Reg(\Gamma(R))$. We have already observed that $Z(\Gamma(R))$ is connected with $\mathrm{diam}(Z(\Gamma(R))) = 2$.

We next determine when $T(\Gamma(R))$ is connected and compute $\mathrm{diam}(T(\Gamma(R)))$. In particular, $T(\Gamma(R))$ is connected if and only if $\mathrm{diam}(T(\Gamma(R))) < \infty$.

Theorem 3.3. *Let R be a commutative ring such that $Z(R)$ is not an ideal of R. Then $T(\Gamma(R))$ is connected if and only if $(Z(R)) = R$ (i.e., $R = (z_1, \ldots, z_n)$ for some $z_1, \ldots, z_n \in Z(R)$). In* *particular, if R is a finite commutative ring and $Z(R)$ is not an ideal of R, then $T(\Gamma(R))$ is connected.*

Proof. Suppose that $T(\Gamma(R))$ is connected. Then there is a path $0 - b_1 - b_2 - \cdots - b_n - 1$ from 0 to 1 in $T(\Gamma(R))$. Thus $b_1, b_1 + b_2, \ldots, b_{n-1} + b_n, b_n + 1 \in Z(R)$. Hence $1 \in (b_1, b_1 + b_2, \ldots, b_{n-1} + b_n, b_n + 1) \subseteq (Z(R))$; so $R = (Z(R))$.

Conversely, suppose that $(Z(R)) = R$. We first show that there is a path from 0 to x in $T(\Gamma(R))$ for any $0 \neq x \in R$. By hypothesis, $x = a_1 + \cdots + a_n$ for some $a_1, \ldots, a_n \in Z(R)$. Let $b_0 = 0$ and $b_k = (-1)^{n+k}(a_1 + \cdots + a_k)$ for each integer k with $1 \leqslant k \leqslant n$. Then $b_k + b_{k+1} = (-1)^{n+k+1}a_{k+1} \in Z(R)$ for each integer k with $0 \leqslant k \leqslant n-1$, and thus $0 - b_1 - b_2 - \cdots - b_{n-1} - b_n = x$ is a path from 0 to x in $T(\Gamma(R))$ of length at most n. Now let $0 \neq z, w \in R$. Then by the preceding argument, there are paths from z to 0 and 0 to w in $T(\Gamma(R))$. Hence there is a path from z to w in $T(\Gamma(R))$; so $T(\Gamma(R))$ is connected.

The "in particular" statement is clear. □

Theorem 3.4. *Let R be a commutative ring such that $Z(R)$ is not an ideal of R and $(Z(R)) = R$ (i.e., $T(\Gamma(R))$ is connected). Let $n \geqslant 2$ be the least integer such that $R = (z_1, \ldots, z_n)$ for some $z_1, \ldots, z_n \in Z(R)$. Then* $\operatorname{diam}(T(\Gamma(R))) = n$. *In particular, if R is a finite commutative ring and $Z(R)$ is not an ideal of R, then* $\operatorname{diam}(T(\Gamma(R))) = 2$.

Proof. We first show that any path from 0 to 1 in $T(\Gamma(R))$ has length $\geqslant n$. Suppose that $0 - b_1 - b_2 - \cdots - b_{m-1} - 1$ is a path from 0 to 1 in $T(\Gamma(R))$ of length m. Thus $b_1, b_1 + b_2, \ldots, b_{m-2} + b_{m-1}, b_{m-1} + 1 \in Z(R)$, and hence $1 \in (b_1, b_1 + b_2, \ldots, b_{m-2} + b_{m-1}, b_{m-1} + 1) \subseteq (Z(R))$. Thus $m \geqslant n$.

Now, let x and y be distinct elements in R. We show that there is a path from x to y in $T(\Gamma(R))$ with length $\leqslant n$. Let $1 = z_1 + \cdots + z_n$ for some $z_1, \ldots, z_n \in Z(R)$, and let $z = y + (-1)^{n+1}x$. Define $d_0 = x$ and $d_k = (-1)^{n+k}z(z_1 + \cdots + z_k) + (-1)^k x$ for each integer k with $1 \leqslant k \leqslant n$. Then $d_k + d_{k+1} = (-1)^{n+k+1}zz_{k+1} \in Z(R)$ for each integer k with $0 \leqslant k \leqslant n-1$ and $d_n = z + (-1)^n x = y$. Thus $x - d_1 - \cdots - d_{n-1} - y$ is a path from x to y in $T(\Gamma(R))$ with length at most n. In particular, we conclude that a shortest path between 0 and 1 in $T(\Gamma(R))$ has length n, and thus $\operatorname{diam}(T(\Gamma(R))) = n$.

For the "in particular" statement, suppose that R is finite and $Z(R)$ is not an ideal of R. Then $x + y \in Reg(\Gamma(R))$ for some $x, y \in Z(R)$. Since every regular element of a finite commutative ring is a unit, we conclude that $R = (x, y)$, and thus $\operatorname{diam}(T(\Gamma(R))) = 2$. □

Corollary 3.5. *Let R be a commutative ring such that $Z(R)$ is not an ideal of R, and suppose that $T(\Gamma(R))$ is connected.*

(1) $\operatorname{diam}(T(\Gamma(R))) = d(0, 1)$.
(2) *If* $\operatorname{diam}(T(\Gamma(R))) = n$, *then* $\operatorname{diam}(Reg(\Gamma(R))) \geqslant n - 2$.

Proof. (1) This is clear from the proof of Theorem 3.4.

(2) Since $n = \operatorname{diam}(T(\Gamma(R))) = d(0, 1)$ by part (1) above, let $0 - s_1 - \cdots - s_{n-1} - 1$ be a shortest path from 0 to 1 in $T(\Gamma(R))$. Clearly $s_1 \in Z(R)$. If $s_i \in Z(R)$ for some integer i with $2 \leqslant i \leqslant n-1$, then we can construct the path $0 - s_i - \cdots - s_{n-1} - 1$ from 0 to 1 which has length less than n, a contradiction. Thus $s_i \in Reg(R)$ for each integer i with $2 \leqslant i \leqslant n-1$.

Hence $s_2 - \cdots - s_{n-1} - 1$ is a shortest path from s_2 to 1 in $Reg(\Gamma(R))$, and it has length $n-2$. Thus $\operatorname{diam}(Reg(\Gamma(R))) \geqslant n-2$. □

Corollary 3.6. *Let R be a commutative ring. If R has a nontrivial idempotent, then $T(\Gamma(R))$ is connected with* $\operatorname{diam}(T(\Gamma(R))) = 2$.

Proof. Let $e \in R \setminus \{0, 1\}$ be idempotent. Then $R = (e, 1-e)$ with $e, 1-e \in Z(R)$; so the claim is clear by Theorems 3.3 and 3.4, respectively. □

Corollary 3.7. *Let $\{R_\alpha\}_{\alpha \in \Lambda}$ be a family of commutative rings with $|\Lambda| \geqslant 2$, and let $R = \prod_{\alpha \in \Lambda} R_\alpha$. Then $T(\Gamma(R))$ is connected with* $\operatorname{diam}(T(\Gamma(R))) = 2$.

Proof. This follows directly from Corollary 3.6 since in this case R has a nontrivial idempotent. □

If $Z(R)$ is not an ideal of R, then $\operatorname{diam}(Z(\Gamma(R))) = 2$. Moreover, we have $2 \leqslant \operatorname{diam}(T(\Gamma(R))) < \infty$ when $T(\Gamma(R))$ is connected. In the following example, for each integer $n \geqslant 2$, we construct a commutative ring R_n such that $Z(R_n)$ is not an ideal of R_n and $T(\Gamma(R_n))$ is connected with $\operatorname{diam}(T(\Gamma(R_n))) = n$.

Example 3.8. Let $n \geqslant 2$ be an integer, $D = \mathbb{Z}[X_1, X_2, \ldots, X_{n-1}]$, K be the quotient field of D, $P_0 = (X_1 + X_2 + \cdots + X_{n-1})$, $P_i = (X_i)$ for each integer i with $1 \leqslant i \leqslant n-2$, and $P_{n-1} = (X_{n-1} + 1)$. Then $P_0, P_1, \ldots, P_{n-1}$ are distinct prime ideals of D. Let $F = P_0 \cup P_1 \cup \cdots \cup P_{n-1}$; then $S = D \setminus F$ is a multiplicative subset of D. Set $R_n = D(+)(K/D_S)$. Then $Z(R_n) = F(+)(K/D_S)$. Since $(1,0) = (-X_1 - X_2 - \cdots - X_{n-1}, 0) + (X_1, 0) + (X_2, 0) + (X_3, 0) + \cdots + (X_{n-1} + 1, 0)$ is the sum of n zero-divisors of R_n, by construction we conclude that n is the least integer $m \geqslant 2$ such that R_n is generated by m zero-divisors of R_n. Hence $T(\Gamma(R_n))$ is connected with $\operatorname{diam}(T(\Gamma(R_n))) = n$ by Theorems 3.3 and 3.4, respectively.

Example 3.2 shows that we may have $\operatorname{diam}(T(\Gamma(R))) < \infty$ and $\operatorname{diam}(Reg(\Gamma(R))) = \infty$. The next example shows that we may also have either $\operatorname{diam}(T(\Gamma(R))) = \operatorname{diam}(Reg(\Gamma(R)))$ or $\operatorname{diam}(T(\Gamma(R))) > \operatorname{diam}(Reg(\Gamma(R)))$ when $Z(R)$ is not an ideal of R.

Example 3.9. (a) Let $R = \mathbb{Z}_5 \times \mathbb{Z}_5$. Then $\operatorname{diam}(T(\Gamma(R))) = 2$ by Theorem 3.4 (or Corollary 3.7), and it is easy to check that $\operatorname{diam}(Reg(\Gamma(R))) = 2$. Thus $\operatorname{diam}(T(\Gamma(R))) = \operatorname{diam}(Reg(\Gamma(R)))$.

(b) Let $R = \mathbb{Z}_2 \times \mathbb{Z}_3$. Then $\operatorname{diam}(T(\Gamma(R))) = 2$ by Theorem 3.4 (or Corollary 3.7), and it is easy to check that $\operatorname{diam}(Reg(\Gamma(R))) = 1$. Thus $\operatorname{diam}(T(\Gamma(R))) > \operatorname{diam}(Reg(\Gamma(R)))$.

We next briefly discuss the diameter of $Reg(\Gamma(R \times S))$ for commutative rings R and S. Note that $Reg(R \times S) = Reg(R) \times Reg(S)$. So for distinct $(a,b), (c,d) \in Reg(R \times S)$, $(a,b) - (-a,-d) - (c,d)$ is a path of length at most two in $Reg(\Gamma(R \times S))$. Thus $Reg(\Gamma(R \times S))$ is connected with $\operatorname{diam}(Reg(\Gamma(R \times S))) \leqslant 2$. In particular, if $Z(\mathbb{Z}_m)$ is not an ideal of $\mathbb{Z}_m$, then $Reg(\Gamma(\mathbb{Z}_m))$ is always connected (cf. Example 2.7(a)). For example, $Reg(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2))$, $Reg(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3))$, and $Reg(\Gamma(\mathbb{Z}_5 \times \mathbb{Z}_5))$ have diameters 0, 1, and 2, respectively.

Theorem 3.10. *Let R be a commutative ring such that $Z(R)$ is not an ideal of R. Then $T(\Gamma(T(R)))$ is connected with* $\operatorname{diam}(T(\Gamma(T(R)))) = 2$. *In particular, if R is a finite commutative ring and $Z(R)$ is not an ideal of R, then $T(\Gamma(R))$ is connected with* $\operatorname{diam}(T(\Gamma(R))) = 2$.

Proof. Let $T = T(R)$. Since $Z(R)$ is not an ideal of R, there are $z_1, z_2 \in Z(R)$ such that $s = z_1 + z_2 \in Reg(R)$. Thus $z_1/s + z_2/s = 1$ in T; so $Z(T)$ is not an ideal of T. Hence $T = (z_1/s, z_2/s)T$, and thus $T(\Gamma(T))$ is connected with $\operatorname{diam}(T(\Gamma(R))) = 2$ by Theorems 3.3 and 3.4, respectively. The "in particular" statement is clear (and has already been observed in Theorem 3.4) since $T(R) = R$ when R is finite. □

The following result is related to the previous theorem.

Theorem 3.11. *Let P_1 and P_2 be prime ideals of a commutative ring R such that $xy = 0$ for some $x \in P_1 \setminus P_2$ and $y \in P_2 \setminus P_1$, and let $S = R \setminus (P_1 \cup P_2)$. Then $T(\Gamma(R_S))$ is connected with* $\operatorname{diam}(T(\Gamma(R_S))) = 2$.

Proof. Since $x \notin P_2$, $y \notin P_1$, and $s \notin P_1 \cup P_2$ for all $s \in S$, we have $sx \neq 0$ and $sy \neq 0$ for all $s \in S$. Thus x/s and y/s are nonzero zero-divisors in R_S for all $s \in S$. Note that $t = x + y \in S$, and hence is a unit in R_S, since $t \notin P_1 \cup P_2$. Thus $R_S = (x/t, y/t)R_S$, and hence $T(\Gamma(R_S))$ is connected with $\operatorname{diam}(T(\Gamma(R_S))) = 2$ by Theorems 3.3 and 3.4, respectively. □

The following is an example of a commutative ring R such that neither $Reg(\Gamma(R))$ nor $T(\Gamma(R))$ is connected, but $T(\Gamma(R_S))$ is connected for some multiplicative subset S of R with $S \neq R \setminus Z(R)$.

Example 3.12. Let $R = \mathbb{Z}[X_1, X_2, X_3]/(X_1X_2X_3) = \mathbb{Z}[x_1, x_2, x_3]$, let $P_1 = (x_1)$ and $P_2 = (x_2)$ be prime ideals of R, and let $x = x_1$ and $y = x_2x_3$. Then $xy = 0$ and $x \in P_1 \setminus P_2$ and $y \in P_2 \setminus P_1$. Let $S = R \setminus (P_1 \cup P_2) \supset R \setminus Z(R)$. Then $T(\Gamma(R_S))$ is connected with $\operatorname{diam}(T(\Gamma(R_S))) = 2$ by Theorem 3.11. By Theorem 3.3, $T(\Gamma(R))$ is not connected since $(Z(R)) \subset R$ and $Z(R)$ is not an ideal of R, and $Reg(\Gamma(R))$ is not connected since there is no path from 1 to 2 in $Reg(\Gamma(R))$ (or use Theorem 3.1(3)).

We next investigate the girth of $Z(\Gamma(R))$, $Reg(\Gamma(R))$, and $T(\Gamma(R))$ when $Z(R)$ is not an ideal of R. Recall that $|Z(R)| \geqslant 3$ if $Z(R)$ is not an ideal of R. We start with a lemma.

Lemma 3.13. *Let R be a commutative ring such that $Z(R)$ is not an ideal of R. Then* $\operatorname{char} R = 2$ *if and only if* $2Z(R) = \{0\}$.

Proof. If $\operatorname{char} R = 2$, then clearly $2Z(R) = \{0\}$. Conversely, suppose that $2z = 0$ for all $z \in Z(R)$. Since $Z(R)$ is not an ideal of R, there are distinct $x, y \in Z(R)$ such that $z = x + y \in Reg(R)$. Then $2z = 2x + 2y = 0$; so $2 = 0$ since $z \in Reg(R)$, i.e., $\operatorname{char} R = 2$. □

Theorem 3.14. *Let R be a commutative ring such that $Z(R)$ is not an ideal of R.*

(1) *Either* $\operatorname{gr}(Z(\Gamma(R))) = 3$ *or* $\operatorname{gr}(Z(\Gamma(R))) = \infty$. *Moreover, if* $\operatorname{gr}(Z(\Gamma(R))) = \infty$, *then* $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$; *so $Z(\Gamma(R))$ is a $K^{1,2}$ star graph with center* 0.
(2) $\operatorname{gr}(T(\Gamma(R))) = 3$ *if and only if* $\operatorname{gr}(Z(\Gamma(R))) = 3$ (*if and only if* $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$).
(3) $\operatorname{gr}(T(\Gamma(R))) = 4$ *if and only if* $\operatorname{gr}(Z(\Gamma(R))) = \infty$ (*if and only if* $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$).
(4) *If* $\operatorname{char} R = 2$, *then* $\operatorname{gr}(Reg(\Gamma(R))) = 3$ *or* ∞. *In particular,* $\operatorname{gr}(Reg(\Gamma(R))) = 3$ *if* $\operatorname{char} R = 2$ *and $Reg(\Gamma(R))$ contains a cycle.*

(5) $\mathrm{gr}(Reg(\Gamma(R))) = 3, 4$, *or* ∞. *In particular,* $\mathrm{gr}(Reg(\Gamma(R))) \leqslant 4$ *if* $Reg(\Gamma(R))$ *contains a cycle.*

Proof. (1) If $x + y \in Z(R)$ for some distinct $x, y \in Z(R)^*$, then $0 - x - y - 0$ is a 3-cycle in $Z(\Gamma(R))$; so $\mathrm{gr}(Z(\Gamma(R))) = 3$. Otherwise, $x + y \in Reg(R)$ for all distinct $x, y \in Z(R)^*$. So in this case, each $x \in Z(R)^*$ is adjacent to 0, and no two distinct $x, y \in Z(R)^*$ are adjacent. Thus $Z(\Gamma(R))$ is a star graph with center 0; so $\mathrm{gr}(Z(\Gamma(R))) = \infty$.

Let $Z(R) = \bigcup_{\alpha \in \Lambda} P_\alpha$, where each P_α is a prime ideal of R [11, p. 3]. Then $|\Lambda| \geqslant 2$ since $Z(R)$ is not an ideal of R. Assume that $\mathrm{gr}(Z(\Gamma(R))) = \infty$. Then $x + y \in Reg(R)$ for all distinct $x, y \in Z(R)^*$, and thus each $|P_\alpha| = 2$. Hence the intersection of any two distinct P_α's is $\{0\}$, and thus $|\Lambda| = 2$. So let $Z(R) = P_1 \cup P_2$ for prime ideals P_1, P_2 of R with $P_1 \cap P_2 = \{0\}$ and $|P_1| = |P_2| = 2$. Hence $|Z(R)| = 3$, and thus R is also finite [9, Theorem 1]. So P_1 and P_2 are the only prime (maximal) ideals of R. By the Chinese Remainder Theorem, we have $R \cong R/P_1 \times R/P_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(2) We need only show that $\mathrm{gr}(Z(\Gamma(R))) = 3$ when $\mathrm{gr}(T(\Gamma(R))) = 3$. If $2z \neq 0$ for some $z \in Z(R)^*$, then $0 - z - (-z) - 0$ is a 3-cycle in $Z(\Gamma(R))$. Thus we may assume that $2z = 0$ for all $z \in Z(R)$, and hence $\mathrm{char}\, R = 2$ by Lemma 3.13. Let $a - b - c - a$ be a 3-cycle in $T(\Gamma(R))$. Then $z = a + b$, $w = a + c$, $b + c \in Z(R)^*$. Moreover, $z + w = (a + b) + (a + c) = 2a + (b + c) = b + c \in Z(R)$. Thus $0 - z - w - 0$ is a 3-cycle in $Z(\Gamma(R))$; so $\mathrm{gr}(Z(\Gamma(R))) = 3$.

(3) Suppose that $\mathrm{gr}(Z(\Gamma(R))) = \infty$. Then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ by part (1) above; so $\mathrm{gr}(T(\Gamma(R))) = 4$. Conversely, suppose that $\mathrm{gr}(T(\Gamma(R))) = 4$. Then $\mathrm{gr}(Z(\Gamma(R))) = \infty$ by parts (1) and (2) above.

(4) Suppose that $\mathrm{char}\, R = 2$ and $Reg(\Gamma(R))$ contains a cycle C. Then C contains two distinct vertices $x, y \in Reg(R)$ such that $x \neq 1$, $y \neq 1$, and $x + y \in Z(R)$. Suppose that R contains a $0 \neq w \in Nil(R)$. If $w = wx = wy$, then $x + 1$ and $y + 1$ are nonzero zero-divisors of R, and thus $1 - x - y - 1$ is a 3-cycle in $Reg(\Gamma(R))$. If either $wx \neq w$ or $wy \neq w$, then either $1 - (w + 1) - (wx + 1) - 1$ or $1 - (w + 1) - (wy + 1) - 1$ is a 3-cycle in $Reg(\Gamma(R))$. If R is reduced, then $x^2 + y^2 = (x + y)^2 \neq 0$. Hence $x^2 \neq y^2$, and thus $x^2 - xy - y^2 - x^2$ is a 3-cycle in $Reg(\Gamma(R))$. Hence $\mathrm{gr}(Reg(\Gamma(R))) = 3$.

(5) By part (4) above, we may assume that $\mathrm{char}\, R \neq 2$. Suppose that $Reg(\Gamma(R))$ contains a cycle C. Then C contains two distinct vertices $x, y \in Reg(R)$ such that $y \neq -x$ and $x + y \in Z(R)$. Thus $x - y - (-y) - (-x) - x$ is a 4-cycle in $Reg(\Gamma(R))$; so $\mathrm{gr}(Reg(\Gamma(R))) \leqslant 4$. □

The next example shows that the 3 possibilities for $\mathrm{gr}(Reg(\Gamma(R)))$ when $Z(R)$ is not an ideal of R from Theorem 3.14(5) above may occur when $\mathrm{gr}(Z(\Gamma(R))) = \mathrm{gr}(T(\Gamma(R))) = 3$. However, if $\mathrm{gr}(Z(\Gamma(R))) = \infty$ and $Z(R)$ is not an ideal of R, then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ by Theorem 3.14(1), and thus $\mathrm{gr}(Reg(\Gamma(R))) = \infty$ and $\mathrm{gr}(T(\Gamma(R))) = 4$. In particular, $\mathrm{gr}(Z(\Gamma(R))) = 3$ when R is not reduced and $Z(R)$ is not an ideal of R (this observation also follows from Theorem 2.10(4)).

Example 3.15. (a) Let $R = \mathbb{Z}_2 \times \mathbb{Z}_3$. Then it is easy to check that $\mathrm{gr}(Z(\Gamma(R))) = \mathrm{gr}(T(\Gamma(R))) = 3$ and $\mathrm{gr}(Reg(\Gamma(R))) = \infty$.

(b) Let $R = \mathbb{Z}_3 \times \mathbb{Z}_4$. Then it is easy to check that $\mathrm{gr}(Reg(\Gamma(R))) = \mathrm{gr}(Z(\Gamma(R))) = \mathrm{gr}(T(\Gamma(R))) = 3$.

(c) Let $R = \mathbb{Z}_3 \times \mathbb{F}_4$. Then it is easy to check that $Reg(\Gamma(R))$ is a $K^{3,3}$. Thus $Reg(\Gamma(R))$ (and hence $T(\Gamma(R))$) is connected with $\mathrm{gr}(Reg(\Gamma(R))) = 4$. It is also easy to check that $\mathrm{gr}(T(\Gamma(R))) = \mathrm{gr}(Z(\Gamma(R))) = 3$.

Let M be an R-module. We conclude this paper with some results about the graphs of the idealization $R(+)M$. In the first result, we assume that $Z(R)(+)M = Z(R(+)M)$. Note that $Z(R)(+)M \subseteq Z(R(+)M)$ always holds, but the inclusion may be proper since $Z(\mathbb{Z}(+)\mathbb{Z}_2) = 2\mathbb{Z}(+)\mathbb{Z}_2$. However, equality holds if either M is an ideal of R or R is an integral domain and M is torsionfree.

Theorem 3.16. *Let R be a commutative ring such that $Z(R)$ is not an ideal of R, and let M be an R-module such that $Z(R(+)M) = Z(R)(+)M$.*

(1) *$T(\Gamma(R(+)M))$ is connected if and only if $T(\Gamma(R))$ is connected.*
(2) $\operatorname{diam}(T(\Gamma(R(+)M))) = \operatorname{diam}(T(\Gamma(R)))$.

Proof. (1) Suppose that $T(\Gamma(R(+)M))$ is connected. Let $x, y \in R$ be distinct. Then $(x,0)$, $(y,0) \in R(+)M$; so there is a path $(x,0) - (s_1,t_1) - \cdots - (s_n,t_n) - (y,0)$ from $(x,0)$ to $(y,0)$ in $T(\Gamma(R(+)M))$. Since $Z(R(+)M) = Z(R)(+)M$, we conclude that $x - s_1 - \cdots - s_n - y$ is a path from x to y in $T(\Gamma(R))$. Thus $T(\Gamma(R))$ is connected and $\operatorname{diam}(T(\Gamma(R(+)M))) \geqslant \operatorname{diam}(T(\Gamma(R)))$.

Conversely, suppose that $T(\Gamma(R))$ is connected (so $\operatorname{diam}(T(\Gamma(R))) \geqslant 2$ by Theorem 3.4). Let $(x,a), (y,b) \in R(+)M$ be distinct. Then there is a path $x - s_1 - \cdots - s_n - y$ from x to y in $T(\Gamma(R))$. Since $Z(R)(+)M \subseteq Z(R(+)M)$, we have that $(x,a) - (s_1,0) - \cdots - (s_n,0) - (y,b)$ is a path from (x,a) to (y,b) in $T(\Gamma(R(+)M))$ (if $x = y$, then use the path $(x,a) - (-x,0) - (y,b)$). Thus $T(\Gamma(R(+)M))$ is connected and $\operatorname{diam}(T(\Gamma(R(+)M))) \leqslant \operatorname{diam}(T(\Gamma(R)))$. (Observe that the hypothesis that $Z(R(+)M) = Z(R)(+)M$ is not needed in this direction.)

(2) This follows directly from the proof of part (1) above. □

In view of the (proof of the) above theorem, we have the following corollary.

Corollary 3.17. *Let R be a commutative ring such that $Z(R)$ is not an ideal of R, and let M be an R-module. If $T(\Gamma(R))$ is connected, then $T(\Gamma(R(+)M))$ is connected with* $\operatorname{diam}(T(\Gamma(R(+)M))) \leqslant \operatorname{diam}(T(\Gamma(R)))$.

The following is an example of a commutative ring R such that $Z(R)$ is not an ideal of R, both $T(\Gamma(R))$ and $T(\Gamma(R(+)M))$ are connected, but $\operatorname{diam}(T(\Gamma(R))) < \operatorname{diam}(T(R(+)M))$. Thus the hypothesis that $Z(R(+)M) = Z(R)(+)M$ is needed in Theorem 3.16(2) and the inequality in Corollary 3.17 may be strict.

Example 3.18. Let $R = R_3$ be the ring constructed in Example 3.8, and let $M = T(R)/R$. Since $(X_2,0), (X_2+1,0) \in Z(R(+)M)$, we have $R(+)M = ((X_2,0),(X_2+1,0))$, and hence $\operatorname{diam}(T(\Gamma(R(+)M))) = 2$ by Theorem 3.4. However, $\operatorname{diam}(T(\Gamma(R))) = 3$ as in Example 3.8, and thus $\operatorname{diam}(T(\Gamma(R(+)M))) < \operatorname{diam}(T(\Gamma(R)))$.

We next investigate the girth of $T(\Gamma(R(+)M))$ and its subgraphs $Z(\Gamma(R(+)M))$ and $Reg(\Gamma(R(+)M))$. Note that in Theorem 3.19 we do not assume that $Z(R)$ is not an ideal of R. In fact, $Z(R)$ is an ideal of R in parts of Example 3.20.

Theorem 3.19. *Let R be a commutative ring, and let M be a nonzero R-module.*

(1) $\operatorname{gr}(Reg(\Gamma(R(+)M))) = 3, 4,$ *or* ∞. *In particular,* $\operatorname{gr}(Reg(\Gamma(R(+)M))) \leqslant 4$ *if $Reg(\Gamma(R(+)M))$ contains a cycle.*

(2) *If* $|M| \geqslant 3$, *then* $\mathrm{gr}(Z(\Gamma(R(+)M))) = \mathrm{gr}(T(\Gamma(R(+)M))) = 3$ *and* $\mathrm{gr}(Reg(\Gamma(R(+)M))) \leqslant 4$.

(3) *If* $|M| = |R| = 2$ (*i.e., if* R *and* M *are both isomorphic to* $\mathbb{Z}_2$), *then* $\mathrm{gr}(Z(\Gamma(R(+)M))) = \mathrm{gr}(Reg(\Gamma(R(+)M))) = \mathrm{gr}(T(\Gamma(R(+)M))) = \infty$.

(4) *If* $2 = |M| < |R|$ (*i.e., if* $R \ncong \mathbb{Z}_2$ *and* $M \cong \mathbb{Z}_2$), *then* $\mathrm{gr}(Z(\Gamma(R(+)M))) = \mathrm{gr}(T(\Gamma(R(+)M))) = 3$ *and* $\mathrm{gr}(Reg(\Gamma(R(+)M))) = 3$ *or* ∞. *In particular,* $\mathrm{gr}(Reg(\Gamma(R(+)M))) = 3$ *if* $Reg(\Gamma(R(+)M))$ *contains a cycle.*

Proof. (1) This follows directly from Theorems 2.5 and 3.14(4).

(2) Let a and b be distinct nonzero elements of M. Then the 3-cycle $(0,0) - (0,a) - (0,b) - (0,0)$ shows that $\mathrm{gr}(Z(\Gamma(R(+)M))) = \mathrm{gr}(T(\Gamma(R(+)M))) = 3$. If $\mathrm{char}\, R = 2$, then $(1,0) - (1,a) - (1,b) - (1,0)$ is a 3-cycle; so $\mathrm{gr}(Reg(\Gamma(R(+)M))) = 3$. If $\mathrm{char}\, R \neq 2$, then $(1,0) - (-1,0) - (1,a) - (-1,a) - (1,0)$ is a 4-cycle; so $\mathrm{gr}(Reg(\Gamma(R))) \leqslant 4$.

(3) This is easy to check.

(4) Let $M = \{0, m\}$ and $I = ann_R(m)$. Then $R/I \cong M \cong \mathbb{Z}_2$. Note that I is a maximal ideal of R and $I(+)M \subseteq Z(R(+)M)$. Let $0 \neq r \in I$. Then the 3-cycle $(0,0) - (r,0) - (0,m) - (0,0)$ shows that $\mathrm{gr}(Z(\Gamma(R(+)M))) = \mathrm{gr}(T(\Gamma(R(+)M))) = 3$. Suppose that $Reg(\Gamma(R(+)M))$ contains a cycle C. Then C contains three distinct vertices $x = (r_1, a), y = (r_2, b), z = (r_3, c) \in Reg(R(+)M)$. Since $r_1, r_2, r_3 \notin I$, we have $r_1 + I = r_2 + I = r_3 + I = 1 + I$, and thus $r_1 + r_2, r_2 + r_3, r_3 + r_1 \in I \subseteq Z(R)$. Hence $x + y, y + z, z + x \in Z(R)(+)M \subseteq Z(R(+)M)$; so $x - y - z - x$ is a 3-cycle in $Reg(\Gamma(R(+)M))$. Thus $\mathrm{gr}(Reg(\Gamma(R(+)M))) = 3$. □

The following example shows that, unlike the case for the diameter in Theorem 3.16, we can have both $T(\Gamma(R))$ and $T(\Gamma(R(+)M))$ connected and $Z(R(+)M) = Z(R)(+)M$, but $\mathrm{gr}(T(\Gamma(R))) \neq \mathrm{gr}(T(\Gamma(R(+)M)))$ (the inequality $\mathrm{gr}(T(\Gamma(R))) \geqslant \mathrm{gr}(T(\Gamma(R(+)M)))$ always holds). We also give examples to illustrate the possible values for $\mathrm{gr}(Reg(\Gamma(R(+)M)))$ in parts (2) and (4) of Theorem 3.19.

Example 3.20. (a) Let $R = M = \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $Z(R)$ is not an ideal of R, $T(\Gamma(R))$ and $T(\Gamma(R(+)M))$ are both connected, and $Z(R(+)M) = Z(R)(+)M$. However, $\mathrm{gr}(T(\Gamma(R))) \neq \mathrm{gr}(T(\Gamma(R(+)M)))$ since $\mathrm{gr}(T(\Gamma(R(+)M))) = 3$ by Theorem 3.19(2) and $\mathrm{gr}(T(\Gamma(R))) = 4$.

(b) It is clear that $\mathrm{gr}(Z(\Gamma(R(+)M))) \leqslant \mathrm{gr}(Z(\Gamma(R)))$ and $\mathrm{gr}(Reg(\Gamma(R(+)M))) \leqslant \mathrm{gr}(Reg(\Gamma(R)))$. However, both inequalities may be strict, even if $Z(R(+)M) = Z(R)(+)M$. For example, let $R = M = \mathbb{Z}_3$; then $\mathrm{gr}(Z(\Gamma(R))) = \mathrm{gr}(Reg(\Gamma(R))) = \infty$, $\mathrm{gr}(Z(\Gamma(R(+)M))) = 3$, and $\mathrm{gr}(Reg(\Gamma(R(+)M))) = 4$.

(c) If $|M| \geqslant 3$, then $\mathrm{gr}(Reg(\Gamma(R(+)M))) = 3$ or 4 by Theorem 3.19(2). Both values are possible. For example, we have $\mathrm{gr}(Reg(\Gamma(\mathbb{Z}_3(+)\mathbb{Z}_3))) = 4$ and $\mathrm{gr}(Reg(\Gamma(\mathbb{Z}_2(+)\mathbb{F}_4))) = 3$.

(d) If $2 = |M| < |R|$, then $\mathrm{gr}(Reg(\Gamma(T(+)M))) = 3$ or ∞ by Theorem 3.19(4). Both values are possible. For example, we have $\mathrm{gr}(Reg(\Gamma((\mathbb{Z}_2 \times \mathbb{Z}_2)(+)\mathbb{Z}_2))) = \infty$ and $\mathrm{gr}(Reg(\Gamma(\mathbb{Z}_4(+)\mathbb{Z}_2))) = 3$.

Let D be a PID and $m \in D$ a nonzero nonunit, and let $R = D(+)(D/mD)$. Then $Z(R)$ is not an ideal of R if and only if $m = q_1^{m_1} \cdots q_n^{m_n}$, where the q_i's are distinct nonassociate primes of D, $n \geqslant 2$, and each $m_i \geqslant 1$ (cf. Example 2.7(c)).

Theorem 3.21. *Let* D *be a PID* (*e.g.,* $\mathbb{Z}$) *and* $m = q_1^{m_1} \cdots q_n^{m_n}$, *where the* q_i*'s are distinct nonassociate primes of* D, $n \geqslant 2$, *and each* $m_i \geqslant 1$, *and let* $R = D(+)/(D/mD)$.

(1) *$Z(R) = ((q_1) \cup \cdots \cup (q_n))(+)(D/mD)$ is not an ideal of R.*
(2) *$Reg(\Gamma(R))$ is connected with* $\mathrm{diam}(Reg(\Gamma(R))) = 2$ *and* $\mathrm{gr}(Reg(\Gamma(R))) = 3$.
(3) *$Z(\Gamma(R))$ is connected with* $\mathrm{diam}(Z(\Gamma(R))) = 2$ *and* $\mathrm{gr}(Z(\Gamma(R))) = 3$.
(4) *$T(\Gamma(R))$ is connected with* $\mathrm{diam}(T(\Gamma(R))) = 2$ *and* $\mathrm{gr}(T(\Gamma(R))) = 3$.

Proof. (1) This is clear.

(2) Let $x = (r_1, a)$, $y = (r_2, b) \in Reg(R)$ such that $x + y \notin Z(R)$. Then $r_1, r_2 \notin (q_i)$ for each integer i with $1 \leqslant i \leqslant n$ by part (1) above. By the Chinese Remainder Theorem, there is a $z \in D$ such that $z + r_1 \in (q_1), z + r_2 \in (q_2), \ldots, z + r_2 \in (q_n)$. By construction, $z \notin (q_i)$ for each integer i with $1 \leqslant i \leqslant n$, and hence $(z, a) \in Reg(R)$ by part (1) above. Thus $x - (z, a) - y$ is a path from x to y in $Reg(\Gamma(R))$ of length 2; so $\mathrm{diam}(Reg(\Gamma(R))) = 2$. Now, let $d = (m + 1, 0)$, $c = (-1, 0) \in R$. Clearly $d, c \in Reg(R)$ and $c + d \in Z(R)$ by part (1) above. Again, by the Chinese Remainder Theorem, there is a $w \in D$ such that $w + (m + 1) \in (q_1), w - 1 \in (q_2), \ldots, w - 1 \in (q_n)$. By construction $w \notin (q_i)$ for each integer i with $1 \leqslant i \leqslant n$. Thus $(w, 0) \in Reg(R)$ by part (1) above, and hence $d - (w, 0) - c - d$ is a 3-cycle in $Reg(\Gamma(R))$; so $\mathrm{gr}(Reg(\Gamma(R))) = 3$.

(3) $Z(\Gamma(R))$ is connected with $\mathrm{diam}(Z(\Gamma(R))) = 2$ by Theorem 3.1(1), and $\mathrm{gr}(Z(\Gamma(R))) = 3$ by Theorem 3.19(2) since $|D/mD| \geqslant 3$.

(4) Since $(q_1, 0), (q_2, 0) \in Z(R)$ and $R = ((q_1, 0), (q_2, 0))$, we have that $T(\Gamma(R))$ is connected with $\mathrm{diam}(T(\Gamma(R))) = 2$ by Theorems 3.3 and 3.4, respectively. Also, $\mathrm{gr}(T(\Gamma(R))) = 3$ since $\mathrm{gr}(Z(\Gamma(R))) = 3$ by part (3) above. □

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