# linear
# ALGEBRA

## * $R^n$, subspaces, span, linear transformations

→ $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$

= set of all points in x-y plane

→ $\mathbb{R}^3 = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in R\}$

→ $\mathbb{R}^n = \{(x_1, x_2, .. x_n) \mid x_1, x_2 \ldots, x_n \in \mathbb{R}\}$

examples:

$(1,2,3) \Rightarrow \mathbb{R}^3$

$(1,7,-2,13) \Rightarrow \mathbb{R}^4$

$R^1$ = nothing but a set of all real numbers

## Operations using points :

$(1,2,0) + (-1,3,4) = (0,5,4)$

$3(7,-2,1) = (21,-6,3)$

## Span :

$\text{span}\{\overset{\mathbb{R}^3}{(2,1,3)}, \overset{\mathbb{R}^3}{(0,1,5)}\}$

↳ set of all linear combinations of $(2,1,3)$, $(0,1,5)$

→ linear combination means $c_1(2,1,3) + c_2(0,1,5)$

where $c_1$ and $c_2$ are some real numbers

* $D = \text{span}\{(2,1,3), (0,1,5)\}$

does $3(2,1,3) + (0,1,5) \in D$ ? yes

does $\sqrt{2}(2,1,3) + -4(0,1,5) \in D$ ? yes

→ D is a subset of $\mathbb{R}^3$

it is also a subspace

def : let D be a subset of $\mathbb{R}^n$. D is called a subspace of $\mathbb{R}^n$ if D = span { finite number of points $\mathbb{R}^n$ }

* D = span { (1, 2, 1, 0) } is a subspace of $\mathbb{R}^4$

-1 (1, 2, 1, 0) = (-1, -2, -1, 0) ∈ D

is (0, 0, 0, 0) ∈ D ? yes, multiply by 0

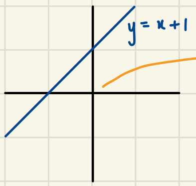
is (1, 4, 2, 0) ∈ D ? no

→ D is a subspace of $\mathbb{R}^4$

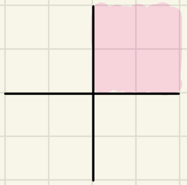
* D = span { (1, 1, 3), (1, -1, 2), (1, 5, 3) }

0 (1, 1, 3) + 0 (1, -1, 2) + 0 (1, 5, 3) = (0, 0, 0) ∈ D

if D is a subspace, (0, 0) ∈ D.



does not go through (0, 0) so D is not a subspace of $\mathbb{R}^2$ but it is a subset.



D = { (x, y) | x ≥ 0, y ≥ 0 }

D is a subset of $\mathbb{R}^2$

D is not a subspace of $\mathbb{R}^2$ because if it was multiplied by a negative number, x and y ≤ 0.

* subspace is always a subset but subset isnt always a subspace

Q. D = { $(x_1, x_2, x_1 + x_2)$ | $x_1, x_2 \in \mathbb{R}$ }

(1, 2, 3) ∈ D, $(\frac{1}{2}, \frac{3}{2}, 2)$ ∈ D, D is infinite

use the concept of span and show that D is a subspace of $\mathbb{R}^3$

→ D = { $x_1 (1, 0, 1) + x_2 (0, 1, 1)$ | $x_1, x_2 \in \mathbb{R}$ }

= span { (1, 0, 1), (0, 1, 1) }

Q. $D = \{(x_1, x_2, -2x_1 + 3x_2, x_4) \mid x_1, x_2, x_4 \in \mathbb{R}\}$

D is an infinite set in $\mathbb{R}^4$

convince that D is a subspace of $\mathbb{R}^4$

→ $D = \{x_1(1,0,-2,0) + x_2(0,1,3,0) + x_4(0,0,0,1) \mid x_1, x_2, x_4 \in \mathbb{R}\}$

$= \text{span}\{(1,0,-2,0), (0,1,3,0), (0,0,0,1)\}$

Q. $D = \{(x_1, x_2, x_1 + 2) \mid x_1, x_2 \in \mathbb{R}\}$

is D a subspace of $\mathbb{R}^3$ ?

→ D is not a subspace because

$D = \{x_1(1,0,1) + x_2(0,1,0) + 2(0,0,1)\}$

$= \{x_1(1,0,1) + x_2(0,1,0) + (0,0,2)\}$

→ this is a fixed point so it cant be multiplied by a number since there isnt a variable before it

$\neq \text{span}\{\text{points}\}$

another method:

→ check if (0,0,0) belongs in D

when $x_1$ and $x_2 = 0$, $(0,0,2) \in D$ so $(0,0,0) \notin D$

Q. $D = \{(x_1, x_1 x_3, x_3) \mid x_1, x_3 \in \mathbb{R}$

→ $D = \{x_1(1, x_3, 0) + x_3(0, x_1, 1) \mid x_1, x_3 \in \mathbb{R}\}$

↳ not specific ↵

D is not a subspace of $\mathbb{R}^3$

Q. $D = \{(x_1, x_3 + 2, x_3, x_4) \mid x_1, x_3, x_4 \in \mathbb{R}\}$

→ D is not a subspace because

$(0,2,0,0) \in D$ so $(0,0,0,0,0) \notin D$

# linear transformations :

→ domain → co-domain

Q. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ (R- homomorpnis)

$T(x_1, x_2) = 2x_1 - 5x_2$

show T is a linear transformation

→ illustrate $T(1,3) = 2(1) - 5(3) = -13$

$T(2,1) = 2(2) - 5(1) = -1$

add the points : $T(3,4) = 2(3) - 5(4) : -14$

*property : $T((1,3) + (2,1)) = T(1,3) + T(2,1)$

$T = \mathbb{R} \longrightarrow \mathbb{R}$

$T(x) = x + 1$

→ illustrate $T(2) = 3$

$T(4) = 5$

$T(6) = 7 \neq T(2) + T(4)$

Def : $T = R^n \longrightarrow \mathbb{R}^m$ is called linear transformation

iff ① $T(Q_1 + Q_2) = T(Q_1) + T(Q_2)$

② $T(cQ) = c\,T(Q)$

Q. $T : \mathbb{R} \rightarrow \mathbb{R}$

$T(x) = 3x$

is T a linear transformation ?

$T(2) = 6$

$T(3) = 9$

$T(5) = 15 = 6 + 9$

Q. $T = \mathbb{R} \rightarrow \mathbb{R}$

$T(x) = x^2$

is T a linear transformation?

$T(1) = 1$

$T(2) = 4$

$T(3) = 9 \neq 1 + 4$ so no

* $T = \mathbb{R} \rightarrow \mathbb{R}$ is linear transformation

iff * $T(x) = mx$ for some real number m

$\Rightarrow$ $T: \mathbb{R} \rightarrow \mathbb{R}$, $T(x) = 3x + 2$ is not linear transformation since it is not of the form $mx$.

* $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation then

$T(\underbrace{0, 0, \dots, 0}_{n\text{-zeros}}) = (\underbrace{0, 0, \dots, 0}_{m\text{-times}})$

$\Rightarrow$ $T: \mathbb{R} \rightarrow \mathbb{R}$, $T(x) = 3x + 2$ is not a linear transformation since $T(0) = 2 \neq 0$

Q. $T = \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$T(x_1, x_2, x_3) = (5x_1, 2x_3, x_1 + x_3)$

prove that it is a linear transformation

* linear combination of $x_1\ x_2\ x_3\ x_4\ x_5$ means $c_1x_1 + c_2x_2 + \dots + c_5x_5$

Q. $T : \mathbb{R}^2 \to \mathbb{R}^4$

is $T(x_1, x_2) = (0, 1, x_1 + x_2, -3x_1)$ a linear transformation?

$0$ is a linear combination because $0 = 0x_1 + 0x_2$

$1$ is not a linear combination because $1 \overset{?}{=} \underbrace{c_1 x_1 + c_2}_{\text{fixed numbers}} x_2$

$x_1 + x_2$ is a linear combination because $x_1 + x_2 = 1x_1 + 1x_2$

$-3x_1$ is a linear combination because $-3x_1 = -3x_1 + 0x_2$

* if $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear transformation,

$T$ (origin of $\mathbb{R}^n$) = origin of $\mathbb{R}^m$

Q. $T : \mathbb{R}^3 \to \mathbb{R}$

is $T(x_1, x_2, x_3) = -10x_3 + x_2$ a linear transformation

$-10x_3 + x_2 = 0x_1 + 1x_2 - 10x_3$ so yes

Q. $T : \mathbb{R}^4 \to \mathbb{R}^5$

$T(x_1, x_2, x_3, x_4) = (-2x_1 + 3x_2, x_3 - x_4, x_1 + 2x_2 - x_3, 0, x_4 + x_1)$ LT?

→ yes because each coordinate is a linear transformation

Q. $T : \mathbb{R}^3 \to \mathbb{R}^4$

$T(x_1, x_2, x_3) = (x_1 x_2, 0, x_3, x_1)$ LT?

→ no because $x_1 x_2$ cannot be formed

Q. $T : \mathbb{R}^2 \to \mathbb{R}$ is a L.T

$T(1,1) = 5$, $T(-1,1) = 7$

find $T(0,2) = ?$

$T(0,2) = T(1,1) + T(-1,1) = 5 + 7 = 12$

$T(-4,4) = 4[T(-1,1)] = 4(7) = 28$

$T(0,0) = 0[T(1,1)] = 0$

$T(0,6) = 3[T(0,2)] = 3(12) = 36$

Week 3

## * linear transformations / Range + kernel

* range is a subset of co-domain
* zeros of T = $z(T)$ = ker (T) = null space

Q. $T: \mathbb{R} \rightarrow \mathbb{R}$

$T(x) = x^2 + 1$

we know T is not L.T.

range : $1 \leqslant y \leqslant \infty$

x-intercept : zeros of T

↳ lives in the domain

Q. $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$

$T(x_1, x_2) = (3x_2, x_1 - x_2, x_1 + 5x_2)$

→ each coordinate is a linear combination so L.T.

① find range of T

range $= \{(3x_2, x_1 - x_2, x_1 + 5x_2) \mid x_1, x_2 \in \mathbb{R}\}$

range $= \{x_1(0,1,1) + x_2(3,-1,5) \mid x_1, x_2 \in \mathbb{R}\}$

$= \text{span}\{(0,1,1), (3,-1,5)\}$

↳ subspace of $\mathbb{R}^3$

* is $(5, 2, -1) \in$ range (T) ?

$(5, 2, -1) = c_1(0,1,1) + c_2(3,-1,5)$

$(5, 2, -1) = (0, c_1, c_1) + (3c_2, -1c_2, 5c_2)$

$(5, 2, -1) = (3c_2, c_1 - c_2, c_1 + 5c_2)$

$3c_2 = 5$ , $c_2 = \frac{5}{3}$

$c_1 - c_2 = 2$ , $c_1 - \frac{5}{3} = 2$ , $c_1 = \frac{11}{3}$

$c_1 + 5c_2 \overset{?}{=} -1$ , $\frac{11}{3} + 5\left(\frac{5}{3}\right) \overset{?}{=} -1$ , $\frac{36}{3} \overset{?}{=} -1$ , $12 \neq -1$

so the point is not in the range.

∴ the range lives in $\mathbb{R}^3$ but is not equal to $\mathbb{R}^3$

② find the zeros of T

$Z(T) = \{ (x_1, x_2) \mid T(x_1, x_2) = (3x_2, x_1 - x_2, x_1 + 5x_2) = (0,0,0) \}$

$3x_2 = 0$ , $x_2 = 0$

$x_1 - x_2 = 0$ , $x_1 = 0$

$Z(T) = \{ (0, 0) \}$

* Z (T) is always a subspace of the domain.

Q. $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$T(x_1, x_2, x_3) = (x_1 + 2x_3, x_2 - 5x_3)$

it is a linear transformation

① find ker (T) = Z(T)

$Z(T) = \{(x_1, x_2, x_3) \mid T(x_1, x_2, x_3) = (0,0) \}$

$(x_1 + 2x_3, x_2 - 5x_3) = (0, 0)$

$$\left.\begin{aligned} x_1 + 2x_3 = 0 &\Rightarrow x_1 = -2x_3 \\ x_2 - 5x_3 = 0 &\Rightarrow x_2 = 5x_3 \end{aligned}\right\} x_3 \in \mathbb{R}$$

$Z(T) = \{ (-2x_3, 5x_3, x_3) \mid x_3 \in \mathbb{R} \}$

$= \{ x_3 (-2, 5, 1) \mid x_3 \in \mathbb{R} \}$

$= \text{span} \{ (-2, 5, 1) \}$

↳ subspace of $\mathbb{R}^3$

② find the range

$\text{Range } (T) = \{ (x_1 + 2x_3, x_2 - 5x_3) \mid x_1, x_2, x_3 \in \mathbb{R} \}$

$= \{ x_1 (1, 0), x_2 (0, 1), x_3 (2, -5) \}$

$= \text{span} \{ (1,0), (0,1), (2,-5) \}$

* if $Q_1, Q_2, \dots Q_k$ in $\mathbb{R}^n$, we say $Q_1, Q_2, \dots, Q_k$ are independent if and only if $c_1 Q_1, c_2 Q_2, \dots c_k Q_k = \underset{n\text{-times}}{(0, 0, \dots 0)}$ then $c_1 = c_2 = \dots = c_k = 0$.

* $Q_1, \dots, Q_k$ are dependent if those exists at least one $c_i \neq 0$ if $c_1 Q_1 + \dots + c_i Q_i + \dots + c_k Q_k = (0, 0, \dots 0)$

↳ equivalent definition (practical)

  * $Q_1, \dots, Q_k$ in $\mathbb{R}^n$ are independent if none of the $Q_i$ is a linear combination of the remaining $Q_i$'s.

  * $Q_1, \dots, Q_k$ are dependent if atleast one of the $Q_i$'s is a linear combination of the remaining $Q_i$s.

Q. $(2, 1, 0), (0, 0, 3), (4, 2, 3) \in \mathbb{R}^3$

are these points dependent or independent?

dependent because

$(4, 2, 3) = 2(2, 1, 0) + (0, 0, 3)$

Q. $(0, 1, 4, 5), (1, 0, 2, 1), (0, 0, 1, 0)$ are indep. in $\mathbb{R}^4$

what do we infer?

* the points are not linear transformations of one another.

* $c_1 Q_1 + c_2 Q_2 + c_3 Q_3 = (0, 0, 0, 0)$

  $\rightarrow c_1 = c_2 = c_3 = 0$

* row-operations allowed:

  → $\alpha R_i$, $\alpha \neq 0$, multiply a row with nonzero number

  → $\alpha R_i + R_k \rightarrow R_k$

  → $R_i \longleftrightarrow R_k$, interchange 2 rows

Q. are $(2, 4, -2)$, $(-1, 2, 3)$, $(0, 6, 4)$ $\in \mathbb{R}^3$ independent?

→ think of each point as a row

① $\begin{bmatrix} 2 & 4 & -2 \\ -1 & 2 & 3 \\ 0 & 6 & 4 \end{bmatrix}$

→ go row by row

→ first row, first nonzero has to be 1

so, multiply by $\frac{1}{2}$

② $\begin{bmatrix} 1 & 2 & -1 \\ -1 & 2 & 3 \\ 0 & 6 & 4 \end{bmatrix}$

→ kill all numbers below the 1

$1R_1 + R_2 \rightarrow R_2$

③ $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 2 \\ 0 & 6 & 4 \end{bmatrix}$

→ repeat everything in row 2

→ divide by 4 to get first nonzero to be 1

④ $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1/2 \\ 0 & 6 & 4 \end{bmatrix}$

→ kill all number below the 1

→ $-6R_2 + R_3 \rightarrow R_3$

⑤ $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$

→ none of the rows $= (0, 0, 0)$

so the points are independent

Q. are $(1, 2, -1, 4)$, $(-2, -3, 4, 6)$, $(-2, -2, 6, 20)$ $\in \mathbb{R}^4$ independent?

$\begin{bmatrix} 1 & 2 & -1 & 4 \\ -2 & -3 & 4 & 6 \\ -2 & -2 & 6 & 20 \end{bmatrix}$

→ $2R_1 + R_2 \rightarrow R_2$

→ $2R_1 + R_3 \rightarrow R_3$

$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 2 & 14 \\ 0 & 2 & 4 & 28 \end{bmatrix}$ → $-2R_2 + R_3 \rightarrow R_3$ $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 2 & 14 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

* since last row $= (0, 0, 0, 0)$ its dependent

* let D be a subspace of $\mathbb{R}^n$, we know $D = \text{span}\{Q_1 \ldots Q_k\}$ for some points in $\mathbb{R}^n$.

→ $\dim(D)$ = max # of independent points in D

(find the independent points out of $Q_1 \ldots Q_k$)

say $P_1, \ldots P_m$ are the max number of independent points in D

then, $D = \text{span}\{P_1, \ldots, P_m\}$

$\dim(D) = m$

Q. $D = \text{span}\{(1,1,0,1), (-2,-2,1,3), (0,0,1,5), (-2,-2,3,13)\}$ is a subspace of $\mathbb{R}^4$.

a) Find $\dim(D)$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ -2 & -2 & 1 & 3 \\ 0 & 0 & 1 & 5 \\ -2 & -2 & 3 & 13 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 3 & 15 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

first 2 are independent

$2R_1 + R_2 = R_2$

$2R_1 + R_4 = R_4$

$-R_2 + R_3 \rightarrow R_3$

$-3R_2 + R_4 \rightarrow R_4$

$\dim(D) = 2$

b) basis for D

$B$ (basis for D) = { all independent points }

$= \{(1,1,0,1), (0,0,1,5)\}$

c) $D = \text{span}\{(1,1,0,1), (0,0,1,5)\}$

d) is $(10, 10, 2, 15) \in D$?

→ Check if its a linear combination of the independent points.

$$(10, 10, 2, 15) \overset{?}{=} c_1(1,1,0,1) + c_2(0,0,1,5)$$

$$= (c_1, c_1, c_2, c_1 + 5c_2)$$

$c_1 = 10$, $c_2 = 2$, $10 + 5(2) \overset{?}{=} 15$

no such $c_1, c_2$ exists hence $(10,10,2,15)$ does not belong to D.

important note:

* assume D is a subspace of $\mathbb{R}^n$ and $\dim(D) = m$ then

$\dim(D) = m \leqslant n$

→ $D = \mathbb{R}^n$ if and only if $n = m$

→ if $k > m$, every $k$ points in D are dependent

* basis for D = { any m independents points in D }

span {basis} = D

Q. Is $\{(2,6), (-3,12)\}$ a basis for $\mathbb{R}^2$?

* If they are independent, it is a basis

$$\begin{bmatrix} 2 & 6 \\ -3 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 \\ -3 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 21 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$\frac{1}{2} R_1$ $\quad 3R_1 + R_2 \rightarrow R_2$ $\quad \frac{1}{21} R_2$ $\quad$ independent

* since they are independent, it is a basis of $\mathbb{R}^2$.

$$\left.\begin{aligned} \mathbb{R}^2 &= \text{span}\,\{(1,3), (0,1)\} \\ \mathbb{R}^2 &= \text{span}\,\{(2,6), (-3,12)\} \end{aligned}\right\} \text{both work}$$

Week 4

## eigen values

→ $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation. A number $\alpha$ is called an eigen value of T if and only if exists a nonzero number point Q in the domain ($\mathbb{R}^n$ in this case).

* $T(x_1, \dots, x_n) = \alpha\,(x_1, \dots, x_n)$

example :

$T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(x_1, x_2, x_3) = (5x_1, 3x_2, -10x_3)$

find all eigen values of T

$T(1,0,0) = (5,0,0) = 5\,(1,0,0)$ ∴ 5 is an eigen value

$T(0,1,0) = (0,3,0) = 3\,(0,1,0)$ ∴ 3 is an eigen value

$T(0,0,1) = (0,0,-10) = -10\,(0,0,1)$ ∴ -10 is an eigen value

→ any point in the span $\{1,0,0\}$ satisfy $T(Q) = 5(Q)$

* span $\{1,0,0\}$ → eigen space correspond to the eigen value 5 .

→ any point in the span $\{0,1,0\}$ satisfy $T(Q) = 5(Q)$

* span $\{0,1,0\}$ → eigen space correspond to the eigen value 3 .

## Matrix Multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} (1\times1)+(2\times6)+(3\times3) \\ (0\times1)+(1\times6)+(1\times3) \\ (2\times1)+(1\times6)+(3\times3) \end{bmatrix} = \begin{bmatrix} 22 \\ 9 \\ 17 \end{bmatrix}$$

$$\underset{2\times 4}{\begin{bmatrix} 1 & 2 & 6 & 8 \\ 0 & 1 & 2 & 3 \end{bmatrix}} \underset{4\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} r1 \cdot c1 & r1 \cdot c2 & r1 \cdot c3 \\ r2 \cdot c1 & r2 \cdot c2 & r2 \cdot c3 \end{bmatrix} = \underset{2\times 3}{\begin{bmatrix} 23 & 4 & 19 \\ 8 & 1 & 6 \end{bmatrix}}$$

should be = to multiply

in multiplying matrices , AB does not have to equal to BA

$$\underset{3\times 4}{\begin{bmatrix} 1 & 0 & 2 & 4 \\ 2 & 1 & 1 & 0 \\ 3 & 1 & 3 & 0 \end{bmatrix}} \underset{4\times 1}{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}} = 1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix}$$

Q. use the concept of LC to find

$$\underset{3\times 4}{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & -2 \end{bmatrix}}\underset{4\times 2}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \\ 3 & 2 \end{bmatrix}} = \begin{bmatrix} 19 & 9 \\ 8 & 5 \\ -4 & -3 \end{bmatrix}$$

first column = $1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 12 \\ 6 \\ -6 \end{bmatrix} = \begin{bmatrix} 19 \\ 8 \\ -4 \end{bmatrix}$

second column = $-1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 8 \\ 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ -3 \end{bmatrix}$

* $\underset{n\times m}{A}\,\underset{m\times n}{B} = \underset{n\times n}{C}$

each column of C is a linear combination of the columns of A.

each row of B is a linear combination of the rows of B.

→ Result

* give me any matrix

$T: \mathbb{R}^m \rightarrow \mathbb{R}^n$ given by $T(x_1 \ldots x_m) = M\begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_m \end{bmatrix}$ is a linear transformation

→ any matrix can be a linear transformation and any linear transformation can be a matrix.

example: $T: \mathbb{R}^2 \rightarrow \mathbb{R}$, $T(x_1, x_2) = \begin{bmatrix} 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$T(1,3) = \begin{bmatrix} 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1[1] + 3[4] = [13] = 13$

Range? $[x_1 + 4x_2] = x_1 + 4x_2 = \{(1)x_1 + (4)x_2\} = \text{span}\{1, 4\} = \text{span}\{1\}$

zeros? $x_1 + 4x_2 = 0$, $x_1 = -4x_2$, $x_2$ = all real numbers $Z(T) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = -4x_2, x_2 \in \mathbb{R}\}$

$= \{(-4x_2, x_2) \mid x_2 \in \mathbb{R}\}$

$= \text{span}\{-4, 1\}$

Q. example from the quiz

$T(a_1, a_2, a_3) = (a_1 - 2a_2 + a_3,\ 4a_1 - 8a_2 + 4a_3) = a_1(1,4) + a_2(-2,-8) + a_3(1,4)$

① find the standard matrix presentation of T

$$M = \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ 1 & -2 & 1 \\ 4 & -8 & 4 \end{array}\right]$$

Standard matrix presentation

Standard basis of the domain $(\mathbb{R}^3)$

$\{(1,0,0), (0,1,0), (0,0,1)\}$

$T(1,0,0) = (1,4)$

$T(0,1,0) = (-2,-8)$

$T(0,0,1) = (1,4)$

Range = span $\{(1,4), (-4,-8), (1,4)\}$

$T(a_1, a_2, a_3) = M \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & \\ 1 & -2 & 1 & 0 \\ 4 & -8 & 4 & 0 \end{array}\right] \xrightarrow{-4R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & \\ 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

augmented matrix

completely reduced

$a_1 - 2a_2 + a_3 = 0$

$a_1 = \underbrace{2a_2 - a_3}_{\text{free variables}}$

$Z(T) = \{(2a_2 - a_3, a_2, a_3) \mid a_2, a_3 \in \mathbb{R}\}$

$Z(T) = \{a_2(2,1,0) + a_3(-1,0,1)\}$

$Z(T) = \text{span}\{(2,1,0), (-1,0,1)\}$

↳ these 2 points must be independent there is no way that they aren't.

* $\dim(Z(T)) = \#$ of free variables when we solve $M \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

② $\dim(\text{Range}(T)) = 1$

$$\begin{bmatrix} 1 & 4 \\ -2 & -8 \\ 1 & 4 \end{bmatrix} \quad \begin{array}{l} 2R_1 + R_2 \to R_2 \\ -1R_1 + R_3 \to R_3 \end{array} \quad \begin{bmatrix} 1 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \to \text{independent}$$

* $\dim(Z(T)) + \dim(\text{Range}(T)) = \dim(\text{domain})$

Week 5

Q. $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$

$T(x_1, x_2, x_3, x_4) = (x_1 - 2x_2 + x_4, 0, 2x_1 - 4x_2 + x_3 + 2x_4)$

find the standard matrix presentation of T

$$M = \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & -4 & 1 & 2 \end{array} \longrightarrow T(x_1, x_2, x_3, x_4) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad T(2,4,0,1) = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ -10 \end{bmatrix}$$

Rank ( any matrix) = # of independent rows of A = # of independent columns of A

Row space of M = Row (M) = Span { independent rows }

Column space M = Col (M) ; independent columns are the ones with the first ones and points chosen should be from original matrix.

$$\text{Rank } (M) : \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & -4 & 1 & 2 \end{bmatrix} \xrightarrow{-2R_1 + R_3 \rightarrow R_3} \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \text{Rank } (M) = 2 \quad = \dim(\text{Row } (M))$$

row (M) = span $\{(1,-2,0,1), (0,0,1,0)\}$ OR span $\{(1,-2,0,1), (2,-4,1,2)\}$

col (M) = span $\{(1,0,2), (0,0,1)\}$ col (M) = Range (T)

dim (Range (T)) = Rank (m) = dim (col(M)) = dim (row(M))

* T is "onto" if and only if Range (T) = co-domain

T is $\underset{\text{one to one}}{1-1}$ if and only if when every $T(Q_1) = T(Q_2)$ then $Q_1 = Q_2$

T is 1-1 if and only if the $Z(T) = \{\text{origin}\}$

$\dim [\text{span } \{\text{origin}\}] = 0$

Q. $T = \mathbb{R}^4 \rightarrow \mathbb{R}^5$

$T(x_1, x_2, x_3, x_4) = (x_2 - x_3 + x_4,\ x_1 + x_2 - x_4,\ x_1 + 2x_2 - x_3,\ x_1 + x_3 + x_4,\ 0)$

Find all points in the domain $(\mathbb{R}^4)$, T(each point) = (1, 4, 5, 6, 0)

$$M = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \end{array}$$

$$M = \left[\begin{array}{cccc} 0 & 1 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 2 & -1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 0 & 1 & -1 & 1 & 1 \\ 1 & 1 & 0 & -1 & 4 \\ 1 & 2 & -1 & 0 & 5 \\ 1 & 0 & 1 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 0 & 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & -2 & 3 \\ 1 & 0 & 1 & -2 & 3 \\ 1 & 0 & 1 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 0 & 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 0 & 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$-R_1 + R_2 \rightarrow R_2$

$-2R_1 + R_3 \rightarrow R_3$

$-R_2 + R_3 \rightarrow R_3$

$-R_2 + R_4 \rightarrow R_4$

$\frac{1}{3}R$

$-R_4 + R_1 \rightarrow R_1$

$2R_4 + R_2 \rightarrow R_2$

completely reduced

$x_2 - x_3 = 0$

$x_1 + x_3 = 5$

$x_4 = 1$

$0 = 0$

$x_1, x_2, x_4$ are leading variables

$x_3$ is a free variable

$x_2 = x_3$

$x_1 = 5 - x_3$

$\{(5 - x_3, x_3, x_3, 1) \mid x_3 \in \mathbb{R}\}$

* when you have a system of linear equations, the completely reduced matrix can have:

① one unique solution ② No solution ③ infinite solutions

→ if ① or ③ is correct, we say the system is consistent

→ if ② is correct, the system is inconsistent

* if there is atleast one free variable, there are infinite solutions.

* if there is no free variable, it can either be unique or no solution

* No solution → if and only if in one of the steps it is observed that 0 = nonzero

# Week 6

Q. $x_1 + 2x_2 - 3x_3 = 4$

$-x_1 + ax_2 + 5x_3 = 10$

$2x_1 + 4x_2 - bx_3 = c$

① for what values of a, b, c does the system have unique solution?

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ -1 & a & 5 & 10 \\ 2 & 4 & b & c \end{array}\right] \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & a+2 & 2 & 14 \\ 0 & 0 & b+6 & c-8 \end{array}\right]$$

$R_1 + R_2 \to R_2$

$-2R_1 + R_3 \to R_3$

$x_1 + 2x_2 - 3x_3 = 4$

$(a+2)x_2 + 2x_3 = 14$

$(b+6)x_3 = c-8$

$$\left[\begin{array}{l} a \neq 2 \\ b \neq -6, \; c \in R \end{array}\right.$$

② for what values of a, b, c will the system be inconsistent?

a system is inconsistent if and only if $0 = \text{nonzero}$ so

$b = -6$ and $c \neq 8$

if $a = -2$, $x_3 = 7$ so $x_3 = \frac{c-8}{b+6} = 7$ when $b \neq -6$ so

$a = -2$ and $\frac{c-8}{b+6} \neq 7$

③ for what values of a, b, c will the system have infinitely many solutions?

a system has infinitely many solutions when there's atleast one free variable

$a = -2$ and $\frac{c-8}{b+6} = 7$ and $b = -6$, $c = 8$, $a \neq -2$

Q. $T : \mathbb{R}^3 \to \mathbb{R}^3$

$T(x_1, x_2, x_3) : (x_1 + 2x_2 - 3x_3, -x_1 + ax_2 + 5x_3, 2x_1 + 4x_2 + bx_3)$

① for what values of a, b there will be a point $(x_1, x_2, x_3)$ in the domain of T s.t. $T(x_1, x_2, x_3) = (4, 10, c)$ $c \in R$?

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ -1 & a & 5 & 10 \\ 2 & 4 & b & c \end{array}\right]$$

Same answer as previous question

Q. $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$

$T(x_1, x_2, x_3) = (4x_1, -2x_2, 3x_3, -x_4)$

$\alpha = 4, \; T(1,0,0,0) = (4,0,0,0) = 4(1,0,0,0)$ $\quad E_4 = \text{span}\{(1,0,0,0)\}$

$\alpha = -2, \; T(0,1,0,0) = (0,-2,0,0) = -2(0,1,0,0)$ $\quad E_{-2} = \text{span}\{(0,1,0,0)\}$

$\alpha = 3, \; T(0,0,1,0) = (0,0,3,0) = 3(0,0,1,0)$ $\quad E_3 = \text{span}\{(0,0,1,0)\}$

$\alpha = -1, \; T(0,0,0,1) = (0,0,0,-1) = -1(0,0,0,1)$ $\quad E_{-1} = \text{span}\{(0,0,0,1)\}$

Q. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 4 \\ 0 & 0 & 6 \end{bmatrix}$

① find all eigen values

② find $E_\alpha$

* tools needed to find eigen-values

① determinant

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 4 & 1 \\ 1 & 2 & 6 \end{bmatrix} \quad 1(4\times6 - 1\times2) - 3(2\times6 - 1\times1) - 1(2\times2 - 4\times1)$$

$$= (24-2) - 3(12-1) - 1(4-4) = 22 - 33 = -11$$

cross product

→ the system has a unique solution if the determinant ≠ 0.

→ if determinant = 0, no solution or infinitely many

## Cramer's Rule

$$\begin{vmatrix} 1 & 2 & -1 \\ 1 & 4 & 10 \\ -3 & 10 & 9 \end{vmatrix} \neq 0$$

$$x_1 = \frac{\begin{vmatrix} 10 & 2 & -1 \\ 11 & 4 & 10 \\ 30 & 10 & 9 \end{vmatrix}}{d} \leftarrow \text{determinant}$$

# Week 7

Q.

upper triangular

$$A = \begin{bmatrix}1&2&3\\-2&2&5\\-1&-2&10\end{bmatrix} \quad \begin{bmatrix}1&2&3\\0&6&11\\0&0&13\end{bmatrix} \quad \begin{bmatrix}1&2&3\\0&1&\frac{11}{6}\\0&0&13\end{bmatrix}$$

$2R_1 + R_2 \to R_2$ $\qquad$ $\frac{1}{6}R_2 \to R_2$

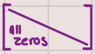
$R_1 + R_3 \to R_3$

$\det(A) = \det(B) = 6\det(C)$

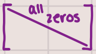
* addition of rows doesn't change the determinant but multiplication of a nonzero does.

* let A be n x n triangular matrix, |A| = multiplication of all numbers, on the main diagonal

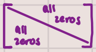
→ A is triangular if it has one of the following forms:



upper triangular



lower triangular



diagonal

Q.

$$A : \begin{bmatrix}0&4&12\\1&0&10\\4&-2&6\end{bmatrix}$$

find |A|

$\frac{1}{4}R_1 \to R_1$

$$B = \begin{bmatrix}0&1&3\\1&0&10\\4&-2&6\end{bmatrix} \quad C = \begin{bmatrix}0&1&3\\1&0&10\\4&0&12\end{bmatrix} \quad D = \begin{bmatrix}0&1&3\\1&0&10\\0&0&-28\end{bmatrix} \quad E : \begin{bmatrix}1&0&10\\0&1&3\\0&0&-28\end{bmatrix}$$

$|B| = \frac{1}{4}|A|$ $\qquad$ $|C| = |B| = \frac{1}{4}|A|$ $\qquad$ $|D| = |C| = |B| = \frac{1}{4}|A|$ $\qquad$ $|E| = -|D| = -|C| = -|B| = -\frac{1}{4}|A|$

$2R_1 + R_3 \to R_3$ $\qquad$ $-4R_2 + R_3 \to R_3$ $\qquad$ $R_1 \leftrightarrow R_2$ $\qquad$ upper triangle

$$\boxed{|A| = -4|E| = -4(-28) = 112}$$

Q.

$$A = \begin{bmatrix} 2 & 4 & 6 & 10 \\ -2 & 5 & 10 & 13 \\ -4 & -8 & 10 & 10 \\ 16 & 32 & 48 & 100 \end{bmatrix} \quad \frac{1}{2}R_1 \rightarrow R_1$$

$$B = \begin{bmatrix} 1 & 2 & 3 & 5 \\ -2 & 5 & 10 & 13 \\ -4 & -8 & 10 & 10 \\ 16 & 32 & 48 & 100 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 9 & 16 & 23 \\ 0 & 0 & 22 & 30 \\ 0 & 0 & 0 & 20 \end{bmatrix}$$

$|A| = 2|C| = 2(9)(22)(20)$

$= 7920$

$|B| = \frac{1}{2}|A|$

$|C| = |B| = \frac{1}{2}|A|$

$2R_1 + R_2 \rightarrow R_2$

$4R_1 + R_3 \rightarrow R_3$

$-16R_1 + R_4 \rightarrow R_4$

Big Result ($A$, $B$ are $n \times n$ matrices)

① $|AB| = |A||B|$

② $|\alpha A| = \alpha^n |A|$

③ $|A^T| = |A|$ * $A^T$ means switching the rows and columns

④ $AB$ doesnt have to equal to $BA$ but $|AB| = |BA|$

⑤ $|A \pm B|$ need not equal to $|A| \pm |B|$

minor result :

$I_n$ = identity matrix

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

whenever multiplication is legal ($BI_n = B$)

$$\underset{3\times5}{A}\ \underset{5\times5}{I_5} = \underset{3\times5}{A} \qquad \underset{3\times3}{I_3}\ \underset{3\times5}{A} = \underset{3\times5}{A}$$

→ lets take $A(n\times n)$, imagine $\alpha$ is an eigen value → there exists a nonzero point $(a_1, \dots, a_n) \in \mathbb{R}^n$

so $A\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \alpha \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \Rightarrow A\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \alpha I_n \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \Rightarrow A\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} - \alpha I_n \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow [\alpha I_n - A]\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

→ the solution of a homogeneous is a subspace so there should be origin solution. $|\alpha I_n - A| = 0$

Q. $A = \underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}}$ find all eigen values of A

set $|\alpha I_2 - A| = 0$ solve for $\alpha$

$$\left|\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}\right| = 0 \qquad \left|\begin{bmatrix} \alpha-1 & -2 \\ 0 & \alpha-4 \end{bmatrix}\right| = 0 \qquad (\alpha-1)(\alpha-4) = 0$$

$$\boxed{\alpha = 1 \quad \alpha = 4}$$

Q. $A = \underset{3\times 3}{\begin{bmatrix} 2 & 1 & 3 \\ -2 & 4 & 5 \\ 2 & -4 & -5 \end{bmatrix}}$ find all eigen values of A

for each eigen value, $\alpha$ find $E_\alpha$

$$|\alpha I_3 - A| = 0 \qquad \alpha I_3 - A = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} - \begin{bmatrix} 2 & 1 & 3 \\ -2 & 4 & 5 \\ 2 & -4 & -5 \end{bmatrix} = \underset{R_2 + R_3 \to R_3}{\begin{bmatrix} \alpha-2 & -1 & -3 \\ 2 & \alpha-4 & -5 \\ -2 & 4 & \alpha+5 \end{bmatrix}}$$

$$\begin{bmatrix} \alpha-2 & -1 & -3 \\ 2 & \alpha-4 & -5 \\ 0 & \alpha & \alpha \end{bmatrix}, \quad \begin{vmatrix} \alpha-2 & -1 & -3 \\ 2 & \alpha-4 & -5 \\ 0 & \alpha & \alpha \end{vmatrix} = 0 ,$$ solve for $\alpha$ (use the column or row with most zeros)

$$\text{determinant} = 0(5+3(\alpha-4)) - \alpha(-5(\alpha-2)+6) + \alpha((\alpha-2)(\alpha-4)+2) = 0$$

$$= 0 - \alpha(-5\alpha+10+6) + \alpha(\alpha^2-4\alpha-2\alpha+8+2) = 0$$

$$= 5\alpha^2 - 16\alpha + \alpha^3 - 6\alpha^2 + 10\alpha = 0$$

$$= \alpha^3 - \alpha^2 - 6\alpha = \alpha(\alpha^2 - \alpha - 6) = \alpha(\alpha-3)(\alpha+2) = 0 \qquad \boxed{\alpha = 0, \ \alpha = 3, \ \alpha = -2}$$

how to get the eigen space :

① take $\alpha$ as 0 in the last matrix

and make it homogeneous

$$\underset{-\frac{1}{2}R_1 \to R_1}{\left[\begin{array}{ccc|c} -2 & -1 & -3 & 0 \\ 2 & -4 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]} \underset{-2R_1 + R_2 \to R_2}{\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 2 & -4 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]} \underset{-\frac{1}{5}R_2 \to R_2}{\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & -5 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]} \underset{-\frac{1}{2}R_2 + R_1 \to R_1}{\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & \frac{8}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]} \rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{10} & 0 \\ 0 & 1 & \frac{8}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$a_1 + \frac{7}{10} a_3 = 0$$

$$a_2 + \frac{8}{5} a_3 = 0$$

$$0 = 0$$

$$E_0 = \left\{ \left( -\frac{7}{10} a_3 , -\frac{8}{5} a_3 , a_3 \right) \mid a_3 \in R \right\}$$

$$= \left\{ a_3 \left( -\frac{7}{10} , -\frac{8}{5} , 1 \right) \mid a_3 \in R \right\}$$

$$= \text{span} \left\{ \left( -\frac{7}{10} , -\frac{8}{5} , 1 \right) \right\}$$

$E_\alpha$ is the set of all points in $R^n$, say $Q = (a_1, \dots, a_n)$,

where $A \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \alpha \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

to find $E_3$ :

① take $\alpha$ as $3$ in the last matrix and make it homogeneous

$$\begin{bmatrix} \alpha-2 & -1 & -3 \\ 2 & \alpha-4 & -5 \\ 0 & \alpha & \alpha \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -3 & 0 \\ 2 & -1 & -5 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & -1 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-2R_1 + R_2 \rightarrow R_2$

$R_2 + R_1 \rightarrow R_1$

$-3R_2 + R_3 \rightarrow R_3$

$$a_1 - 2a_3 = 0$$

$$a_2 + a_3 = 0$$

$$0 = 0$$

$$E_3 = \left\{ ( -2a_3 , -a_3 , a_3 ) \mid a_3 \in R \right\}$$

$$= \left\{ a_3 (-2, -1, 1) \mid a_3 \in R \right\}$$

$$= \text{Span} \left\{ (-2, -1, 1) \right\}$$

Homework :

find $E_{-2}$

# Week 8

* definition : (for n×n matrices)

A, n×n, we say A is nonsingular (invertible) if there exists a matrix, denoted by $A^{-1}$, s.t. $AA^{-1} = I_n$

→ A, n×n, is invertible iff $|A| \neq 0$

Find $A^{-1}$

$$\left[\begin{array}{c|c} A & I_n \end{array}\right] \xrightarrow{\text{row operations}} \underbrace{\left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]}_{\substack{\text{invertible} \\ \text{nonsingular}}} \quad \text{OR} \quad \underbrace{\left[\begin{array}{c|c} \text{not } I_n & A^{-1} \end{array}\right]}_{\substack{\text{non-invertible} \\ \text{singular}}}$$

Q. $A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$ find $A^{-1}$

$$\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 0 \end{array}\right] \xrightarrow{-2R_1 + R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ \underbrace{0 \quad 0}_{\substack{\text{No way to} \\ \text{get } I_n}} & & 1 & -2 \end{array}\right]$$

A is non-invertible (singular)

Q. $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & -1 & 2 \\ 2 & 4 & 5 \end{bmatrix}$ find $A^{-1}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 & 1 & 0 \\ 2 & 4 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_1 + R_3 \to R_3]{R_1 + R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{-2R_2 + R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & -2 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{2R_3 + R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -2 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

↓ $A^{-1}$

$$\boxed{|A^{-1}| = \frac{1}{|A|}}$$

To find A from a given $A^{-1}$ :

$$A^{-1}\left[A\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}\right] \rightarrow A^{-1}A\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=A^{-1}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} \rightarrow I_3\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=A^{-1}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$$

Q. $A^{-1}=\begin{bmatrix}1&2&3\\2&10&1\\-2&6&10\end{bmatrix}$ find A knowing $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\4\\7\end{bmatrix}$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=2\begin{bmatrix}1\\2\\-2\end{bmatrix}+4\begin{bmatrix}2\\10\\6\end{bmatrix}+7\begin{bmatrix}3\\1\\10\end{bmatrix}$$

Definition :

Know : A, B are invertible n x n. then $(AB)^{-1}=B^{-1}A^{-1}$

* $|A^T|=|A|$

Know : C, n x m, and D, m x n, $(CD)^T=D^TC^T$

Special case for 2 x 2 :

$$A=\begin{bmatrix}a&b\\c&d\end{bmatrix} \qquad A^{-1}=\frac{1}{|A|}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$$

Know : $A_{n\times m}$ $B_{n\times m}$ , $(A\pm B)^T=A^T\pm B^T$

Q. $\left(\left(A\begin{bmatrix}1&2\\-1&-1\end{bmatrix}+\begin{bmatrix}1&2\\4&5\end{bmatrix}\right)^T=\begin{bmatrix}1&1\\-1&1\end{bmatrix}\right)^T$ find A

$$A\begin{bmatrix}1&2\\-1&-1\end{bmatrix}+\begin{bmatrix}1&2\\4&5\end{bmatrix}=\begin{bmatrix}1&-1\\1&1\end{bmatrix}$$

$$\underbrace{A\begin{bmatrix}1&2\\-1&-1\end{bmatrix}}_{B}=\begin{bmatrix}1&-1\\1&1\end{bmatrix}-\begin{bmatrix}1&2\\4&5\end{bmatrix}=\begin{bmatrix}0&-3\\-3&-4\end{bmatrix}$$

$$B^{-1}=\frac{1}{1}\begin{bmatrix}-1&-2\\1&1\end{bmatrix} \qquad \left(A\begin{bmatrix}1&2\\-1&-1\end{bmatrix}=\begin{bmatrix}0&-3\\-3&-4\end{bmatrix}\right)\times B^{-1}$$

$$ABB^{-1}=\begin{bmatrix}0&-3\\-3&-4\end{bmatrix}\begin{bmatrix}1&-2\\1&1\end{bmatrix}$$

# week 9

Recall :

$|A| \neq 0 \Rightarrow A^{-1}$ exists

$|A^{-1}| = \frac{1}{|A|}$

Know : $(A^T)^{-1} = (A^{-1})^T$

* $[A \mid \text{constants}]$ has unique solution iff $|A| \neq 0$ iff $A^{-1}$ exists

Know : A, n x n , assume A has atleast 2 identical rows on columns then $|A| = 0$

→ assume $i^{th}$ row and $k^{th}$ row are identical

$A = \begin{array}{c} i \\ k \end{array}\begin{bmatrix} \cdots\cdots\cdots \\ \cdots\cdots\cdots \end{bmatrix} \quad -R_i + R_k \to R_k \quad B = \begin{bmatrix} \cdots\cdots\cdots \\ 0\,0 \cdots 0 \end{bmatrix} \qquad |B| = |A| = 0$

→ if $i^{th}$ column and $k^{th}$ column of A are identical, then $i^{th}$ and $k^{th}$ row of $A^T$ are identical. Since $|A| = |A^T|$ and $|A^T| = 0$, $|A| = 0$.

* $|A| = 0$, either consistent with infinitely many solution OR inconsistent with no solution.

example :

$Q_1 = (1, 2, 3, 4)$

$Q_2 = (-1, 4, 6, 8)$

$Q_3 = (2, 1, 1, 6)$

$Q_4 = (0, 0, 1, 2)$

we want a unique solution $c_1 = c_2 = c_3 = c_4 = 0$

$$\underbrace{\left[\begin{array}{cccc} Q_1 & Q_2 & Q_3 & Q_4 \\ & & & \\ & & & \\ & & & \end{array}\right]}_{A}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

* A has unique solution (0, 0, 0, 0) iff $|A| \neq 0$

Result : assume $Q_1 \ldots Q_n$ are points in $R^5$, then $Q_1, Q_2, \ldots, Q_n$ are independent iff $\left|\begin{bmatrix} Q_1 \\ Q_n \end{bmatrix}\right| \neq 0$.

A, 4 x 4,

$C_A(\alpha) = |\alpha I_4 - A|$

$C_A(\alpha) = (\alpha - 3)^2 (\alpha + 5)^2$

eigen values : $\alpha = 3$ $\alpha = -5$

- Both repeated twice

* $|A|$ = multiplication of eigen values with repetition

$|A| = (3)(3)(-5)(-5)$

* if its given that 0 is not an eigen value, $|A| \neq 0$, meaning A is invertible

$\alpha$ is an eigen value of $A$, $n\times n$, $|A| \neq 0$

→ there exists a nonzero point $(a_1, \dots, a_n)$ such that :

$$A\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \alpha \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \rightarrow AA^{-1}\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \alpha A^{-1}\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \rightarrow I_n\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \alpha A^{-1}\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \rightarrow \frac{1}{\alpha}\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = A^{-1}\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

* $\frac{1}{\alpha}$ is an eigen value of $A^{-1}$

Q. $A$, $3\times 3$,

$C_A(\alpha) = (\alpha-2)^2(\alpha-4)$

① Find $|A|$

② Find eigen values of $A^{-1}$

③ Find $|A^{-1}|$

④ Given $E_2 = \text{Span}\{(1,0,2)\}$, $E_4 = \text{Span}\{(0,2,3)\}$

Find $E_{1/2}$ and $E_{1/4}$

① $|A| = (2)(2)(4) = 16$

② $\frac{1}{2}$ and $\frac{1}{4}$

③ $|A^{-1}| = \frac{1}{|A|} = \frac{1}{16}$

④ $E_{1/2} = E_2$ and $E_{1/4} = E_4$

* $A^{-1}\begin{bmatrix} 0 \\ 6 \\ 9 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{6}{4} \\ \frac{9}{4} \end{bmatrix}$

* $A\begin{bmatrix} 0 \\ 6 \\ 9 \end{bmatrix} = 4\begin{bmatrix} 0 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 24 \\ 36 \end{bmatrix}$

Trace $(A)$ = Sum of the numbers on the main diagonal

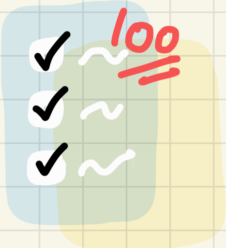
= sum of the eigen values with repetition

Know :

* $\alpha^{-1} = \frac{1}{\alpha}$ assuming $\alpha \neq 0$

* $\alpha^k$ is an eigen value of $A^k$

* $C\alpha$ is an eigen value of $CA$ $C$ = constant



2 + 3 + 1 = 6 ← 6 x 6 matrix

$\deg(C_A(\alpha)) = n$

$C_A(\alpha) = (\alpha+1)^{2}(\alpha-3)^{3}(\alpha+4)^{1}$

① what are the eigen values ?

$\alpha = -1 \qquad \alpha = 3 \qquad \alpha = -4$

② Trace (A) $= -1 + -1 + 3 + 3 + 3 + 4 = 3$

③ $|A| = (-1)(-1)(3)(3)(3)(4) = -108$

④ eigen values of $A^{-1}$

$\alpha = -1 \qquad \alpha = \frac{1}{3} \qquad \alpha = -\frac{1}{4}$

⑤ eigen values of $A^2$

$\alpha = 1 \qquad \alpha = 9 \qquad \alpha = 16$

$AQ^T = \alpha Q^T$ * multiply A on both sides

$A^2Q^T = \alpha \underline{AQ^T}$

$A^2Q^T = \alpha(\alpha Q^T)$

$A^2Q^T = \alpha^2 Q^T$

∴ if α is an eigen value of A, then $\alpha^k$ is an eigen value of $A^k$.

Q. A, 3 x 3,

$C_A(\alpha) = |\alpha I_3 - A| = (\alpha-4)^2(\alpha+4)$

$B = 2A^2 + 5A^{-1} - 4I_3$ find $|B|$ and trace(B)

eigen values = $\alpha = 4 \qquad \alpha = -4$

for $\alpha = 4$

$BQ^T = 2A^2Q^T + 5A^{-1}Q^T - 4I_3Q^T$

$= 2\alpha^2Q^T + 5(\frac{1}{\alpha})Q^T - 4Q^T$

$= 2(4)^2Q^T + 5(\frac{1}{4})Q^T - 4Q^T$

$= 32Q^T + \frac{5}{4}Q^T - 4Q^T$

$= (32 + \frac{5}{4} - 4)Q^T$

$= 29.25\, Q^T$ (repeated twice)

↳ eigen value of B

for $\alpha = -4$

$B = 2(-4)^2 + 5(\frac{1}{-4}) - 4$

$= 32 - \frac{5}{4} - 4$

$= 26.75$

↳ eigen value of B

$|B| = (29.25)(29.25)(26.75)$

trace (B) $= 29.25 + 29.25 + 26.75$

Definition : A, n x n,

we say AB diagnolizable if there exists an invertible matrix Q, and a diagonal matrix D so that $Q^{-1}AQ = D \xrightarrow{\text{solve for A}} A = QDQ^{-1}$, $A^2 = QD^2Q^{-1} = D^2$

$QQ^{-1} = I_n = 1$

# Week 10

## * adjoint method.

entry (i, k)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad \text{adjoint of } A = C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ \vdots & & & \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} \qquad C_{ik} = \frac{(-1)^{i+k} \left| \begin{matrix} A \text{ after deleting} \\ k^{th} \text{ row } \& \ i^{th} \text{ column} \end{matrix} \right|}{|A|}$$

$a_{3,4}$ = third row, fourth column

$(i, k)$ - entry of $C$ = $C_{ik}$

know: $A \cdot \text{adjoint}(A) = |A| I_n$

assume $|A| \neq 0 \Longrightarrow A^{-1}$ exists

$$A \cdot \underbrace{\frac{\text{adjoint}(A)}{|A|}}_{A^{-1}} = I_n$$

**Q**

$$A = \begin{bmatrix} 2 & 3 & 4 \\ -2 & 6 & 1 \\ -2 & -3 & 5 \end{bmatrix}$$

* find the (2, 3) entry of $A^{-1}$

↳ third row, second column

$$A^{ik} = \frac{(-1)^{2+3} \begin{vmatrix} 2 & 4 \\ -2 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 4 \\ -2 & 6 & 1 \\ -2 & -3 & 5 \end{vmatrix}}$$

$$|A| = \overset{66}{2(30+3)} \overset{+}{-} \overset{24}{3(-10+2)} \overset{+}{+} \overset{72}{4(6+12)} = 162$$

↳ you can use the triangle method as well

$$= \frac{(-1)(10)}{162} = \frac{-5}{81}$$

* Assume $C_A(\alpha) = (\alpha - a_1)^{n_1} (\alpha - a_2)^{n_2} (\alpha - a_k)^{n_k}$

$\rightarrow \quad 0 < \dim(E_{a_i}) \leq n_i$

**Result:** A, n×n, is diagnolizable if and only if for every eigen value $(a_i)$, $\dim(E_{a_i}) = n_i$

Q. A, 3x3, $C_A(\alpha) = (\alpha-2)^{2}(\alpha+4)$, $E_2 = \text{Span}\{(1,3,2)\}$, $E_{-4} = \text{Span}\{(0,1,5)\}$

is A diagnolizable ?

$\dim(E_2) = 1$ $\dim(E_{-4}) = 1$

no because the dimension of $E_2 \neq 2 \neq$ the power

Q. A, 5 x 5, $C_A(\alpha) = (\alpha-3)^{2}(\alpha+5)^{2}(\alpha-6)^{\bullet}$, $E_3 = \text{span}\{(1,1,1,1,1),(-1,1,1,1,1)\}$ → independent

$E_{-5} = \text{Span}\{(-1,-1,1,1,1),(-1,-1,-1,1,1)\}$ → independent

$E_6 = \text{span}\{(0,0,0,0,1)\}$

↳ you can select 0,0,0,0,3 (multiply by 3) because dim = 1

$\dim(E_3) = 2 = n_3$

$\dim(E_{-5}) = 2 = n_{-5}$

$\dim(E_6) = 1 = n_6$

$\therefore$ A is diagonalizable

→ Find a diagonal matrix D and an invertible matrix Q such that $Q^{-1}AQ = D$

$$D = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix} \xleftrightarrow{\text{correspond}} Q = \begin{bmatrix} 1 & -1 & -1 & -1 & 0 \\ 1 & 1 & -1 & -1 & 0 \\ 1 & 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ \underbrace{1 & 1 & 1 & 1 & 1}_{\text{cant repeat points}} \end{bmatrix}$$

Q.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

* if A is diagonilizable, find a diagonal matrix D and invertible matrix Q st. $Q^{-1}AQ = D$

$$C_A(\alpha) = |\alpha I_3 - A| = \begin{vmatrix} \alpha-2 & 0 & 0 \\ 0 & \alpha-2 & 0 \\ 1 & -1 & \alpha-3 \end{vmatrix} = (\alpha-2)^2(\alpha-3) = 0$$

$\alpha = 2$ (twice) $\alpha = 3$

$$E_2 = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

$x_1 = x_2 + x_3$

Solution set $= \{(x_2 + x_3, x_2, x_3) \mid x_2, x_3 \in \mathbb{R}\}$

$= \text{Span}\{(1,1,0),(1,0,1)\}$

$\dim(E_2) = 2$

$$E_3 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-R_1 + R_3 \rightarrow R_3$ $R_2 + R_3 \rightarrow R_3$

$x_1 = 0$

$x_2 = 0$

Solution set $= \{(0, 0, x_3) \mid x_3 \in \mathbb{R}\}$

$= \text{span}\{(0,0,1)\}$

$\dim(E_3) = 1$

diagonalizable ✓

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

* let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation

T is invertible iff $n = m$ and T is an isomorprism

↳ 1-1 and onto

→ for it to be 1-1 , $\dim(\text{domain}) = \dim(\text{Range}) + \dim(Z(T))$

where $\dim(Z(T)) = 0$ , $\underset{n}{\dim(\text{domain})} \underset{=}{=} \underset{m}{\dim(\text{Range})}$

# Week 11

$f_1(x) = x^2$ $\qquad f_1(f_2(x)) = f_1 \circ f_2 = f_1(x+3) = (x+3)^2$

$f_2(x) = x+3$ $\qquad f \circ f^{-1} =$ identity $= 1$ → function is invertible

Q. $T : R^2 \to R^2$

$T(a_1, a_2) = (a_1 + 2a_2, -a_1 + a_2)$

1) is T invertible ?

* find the standard matrix presentation $\qquad$ * T is invertible iff $M^{-1}$ exists

$M = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ $\quad |M| = (1)(1) - (2)(-1) = 3$ $\quad |M^{-1}| = \frac{1}{3}$ → $M^{-1}$ exists so T is invertible

2) if yes, find $T^{-1}$

$T^{-1}(a_1, a_2) = M^{-1}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ $\qquad m^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{-2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$

$$T^{-1}(a_1, a_2) = \begin{bmatrix} \frac{1}{3} & \frac{-2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}a_1 & -\frac{2}{8}a_2 \\ \frac{1}{3}a_1 & \frac{1}{3}a_2 \end{bmatrix} = \left(\frac{1}{3}a_1 - \frac{2}{3}a_2, \frac{1}{3}a_1 + \frac{1}{3}a_2\right)$$

Fact: lets say $T : R^n \to R^m$ ($n \neq m$ so isnt invertible)

$T_2 : R^k \to R^n$

$M_1$ → Standard Matrix for $T_1$ ($m \times n$)

$M_2$ → Standard Matrix for $T_2$ ($n \times k$)

Find standard matrix presentation of $T_1 \circ T_2$

→ $M = \underbrace{M_1}_{m \times n} \underbrace{M_2}_{n \times k}$ ($m \times k$)

Q. $T_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\quad$ $T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ $\quad$ Find $T_1 O T_0 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$T_1(a_1, a_2) = (a_1 + a_3, -a_1)$ $\quad$ $T_2(a_1, a_2) = (3a_1 - a_2, a_1 + a_2)$ $\quad$ $T_1 O T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$(T_1 O T_2)(a_1, a_2) = M_1 M_2 \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

example:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad f(x) = 3x^2 - 6x + 7$$

find $f(A)$

$\boxed{A^{-n} = (A^{-1})^n \qquad A^{\frac{1}{2}} = \text{undefined}}$

$$f(A) = \overset{2\times2}{3A^2} - \overset{2\times2}{6A} + \overset{2\times2}{7I_2}$$

$$=$$

Q. $A = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$

$$C_A(\alpha) = |\alpha I_2 - A| = \begin{vmatrix} \alpha & -2 \\ 0 & \alpha - 1 \end{vmatrix} = \alpha(\alpha - 1)$$

$$C_A(A) : A(A - I_2) = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

## Caley's Theorem :

$$A, \; n \times m, \quad C_A(A) = \underset{n \times n}{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & & 0 \end{bmatrix}}$$

$$\boxed{C_A(\alpha) = \alpha^n + a_{n-1}\alpha^{n-1} + \dots + a_1\alpha + a_0}$$

Q.

$$A : \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

$$C_A(\alpha) : |\alpha I_3 - A| = \begin{vmatrix} \alpha - 1 & 0 & -2 \\ 0 & \alpha - 2 & -3 \\ 0 & 0 & \alpha - 4 \end{vmatrix}$$

$$= (\alpha - 1)(\alpha - 2)(\alpha - 4)$$

$$C_A(A) = (A - I_3)(A - 2I_3)(A - 4I_3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Q.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 2 & 5 \\ -1 & -2 & 6 \end{bmatrix}$$

find $(1,3)$ - entry of $A^{-1}$

$$(2\times5) - (4\times2) = 8$$

$$= \frac{(-1)^{1+3} \begin{vmatrix} 2 & 4 \\ 2 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 4 \\ -1 & 2 & 5 \\ -1 & -2 & 6 \end{vmatrix}} = \frac{8}{40} = \frac{1}{20}$$

$$1\left[(2\times6)-(5\times-2)\right] - 2\left[(-1\times6)-(5\times-1)\right] + 4\left[(-2\times-1)-(2\times-1)\right] = 40$$

# Week 12

$R^{2\times 2} = \text{span}\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ → $\dim(R^{2\times 2}) = 4$

$\mathbb{R}^{2\times 3} = \left\{\begin{bmatrix}a_1&a_2&a_3\\a_4&a_5&a_6\end{bmatrix} \,\middle|\, a_1, a_2, a_3, a_4, a_5, a_6 \in R\right\}$

$\dim(\mathbb{R}^{3\times 5}) = 15$

$\dim(\mathbb{R}^{n\times m}) = nm$

$$c_1\begin{bmatrix}1&0\\0&0\end{bmatrix} + c_2\begin{bmatrix}0&1\\0&0\end{bmatrix} + c_3\begin{bmatrix}0&0\\1&0\end{bmatrix} + c_4\begin{bmatrix}0&0\\0&1\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$$

Q. $D = \left\{\begin{bmatrix}a+b&-1\\0&a\end{bmatrix} \,\middle|\, a, b \in \mathbb{R}\right\}$

convince me that $D$ is not a subspace of $\mathbb{R}^{2\times 2}$

① for it to be a subspace, there should be an origin matrix

→ $-1 \neq 0$ so it is not a subspace, $\begin{bmatrix}0&0\\0&0\end{bmatrix} \notin D$

$= \left\{a\begin{bmatrix}1&0\\0&1\end{bmatrix} + b\begin{bmatrix}1&0\\0&0\end{bmatrix} + \begin{bmatrix}0&-1\\0&0\end{bmatrix}\right\}$ → it is fixed so cannot be written as a span of finite numbers of matrices.

$P_n$ = set of all polynomials of degree $< n$

$P_3 = \{a_2x^2 + a_1x + a_0 \mid a_0, a_1, a_2 \in R\}$

$5 \in P_3$? yes  $2x + \sqrt{3} \in P_3$? yes  $6x^2 - \sqrt{2}x + \sqrt{11} \in P_3$? yes  $2x^{\textcircled{3}} + 1 \in P_3$? No

$P_3$ is a subspace because it is a span of finite number of polynomials

$= \text{span}\{1, x, x^2\}$

→ the origin of polynomials is just 0.

$c_0(1) + c_1(x) + c_2(x^2) = 0$  $c_0 = 0$ $c_1 = 0$ $c_2 = 0$ → linearly independent

Q. convince me that $D = \{a_0 + a_2x + a_2x^2 \mid a_0, a_2 \in R\}$ is a subspace

$= \{a_0(1) + a_2(x + x^2)\} = \text{span}\{1, x + x^2\}$

Result :

$$\overbrace{\mathbb{R}^{n\times m} \underset{\text{isomorphic}}{\approx} \mathbb{R}^{nm}}^{\text{same as subspaces}}$$

→ they share the same properties

- if one is independent, so is the other.

$\mathbb{R}^{2\times 2} \approx \mathbb{R}^4$

$\begin{bmatrix}1&2\\3&4\end{bmatrix} \longrightarrow (1, 2, 3, 4)$

Q. $D = \text{Span}\left\{\begin{bmatrix}1&2\\0&1\end{bmatrix}, \begin{bmatrix}-1&-1\\1&1\end{bmatrix}, \begin{bmatrix}1&3\\1&3\end{bmatrix}\right\}$

Find dim (D) and write D as a span of basis

① Do the calculation in the $\overset{\mathbb{R}^4}{\underline{\text{co space}}}$ of $\mathbb{R}^{2\times2}$

$\begin{bmatrix}1&2\\0&1\end{bmatrix} \rightarrow (1,2,0,1)$ $\begin{bmatrix}-1&-1\\1&1\end{bmatrix} \rightarrow (-1,-1,1,1)$ $\begin{bmatrix}1&3\\1&3\end{bmatrix} \rightarrow (1,3,1,3)$

$$\begin{bmatrix}1&2&0&1\\-1&-1&1&1\\1&3&1&3\end{bmatrix} \rightarrow \begin{bmatrix}1&2&0&1\\0&1&1&2\\0&1&1&2\end{bmatrix} \rightarrow \begin{bmatrix}1&2&0&1\\0&1&1&2\\0&0&0&0\end{bmatrix}$$

Dim (D) = 2

$R_1 + R_2 \rightarrow R_2$ $-R_2 + R_3 \rightarrow R_3$

$-R_1 + R_3 \rightarrow R_3$

basis = $\left\{\begin{bmatrix}1&2\\0&1\end{bmatrix}, \begin{bmatrix}-1&-1\\1&1\end{bmatrix}\right\}$ $D = \text{span}\{\text{basis}\}$

Q. find a basis for $\mathbb{R}^{2\times2}$, say B, such that $\begin{bmatrix}1&1\\1&1\end{bmatrix}, \begin{bmatrix}-1&-1\\1&1\end{bmatrix} \in B$.

→ consider the cospace $\mathbb{R}^4$

$\mathbb{R}^{2\times2}$ $\mathbb{R}^4$

$\begin{bmatrix}1&1\\1&1\end{bmatrix} \rightarrow (1,1,1,1)$ $\begin{bmatrix}-1&-1\\1&1\end{bmatrix} \rightarrow (-1,-1,1,1)$

$$\begin{bmatrix}1&1&1&1\\-1&-1&1&1\\&&&\\&&&\end{bmatrix} \begin{bmatrix}1&1&1&1\\0&0&2&2\\&&&\\&&&\end{bmatrix} \begin{bmatrix}1&1&1&1\\0&1&0&0\\0&0&2&2\\0&0&0&1\end{bmatrix}$$

$R_1 + R_2 \rightarrow R_2$

add 2 points to make the matrix ind.

$B = \left\{\begin{bmatrix}1&1\\1&1\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix} \begin{bmatrix}-1&-1\\1&1\end{bmatrix} \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$

Fact: $\underbrace{P_n \approx \mathbb{R}^n}_{\text{as subspaces}}$

$$P_4 \longleftrightarrow \mathbb{R}^4$$

$$a_3x^3 + a_2x^2 + a_1x + a_0 \longleftrightarrow (a_3, a_2, a_1, a_0)$$

$$2x^3 - 10x + 15 \longleftrightarrow (2, 0, -10, 15)$$

$$13x^2 - 10x + x^3 + 2 \longleftrightarrow (1, 13, -10, 2)$$

Q. $D = \{ (a_2 + a_1)x^3 + a_2x^2 - a_1x + a_1 \mid a_1, a_2 \in \mathbb{R} \}$ lives in $P_4$

① convince me that D is a subspace of $P_4$ and write its basis

$D = \{ a_2(x^3 + x^2) + a_1(x^3 - x + 1) \}$

$= \text{span}\{ x^3 + x^2, x^3 - x + 1 \}$

→ to check for independence, use the rowspace

$x^3 + x^2 \to (1,1,0,0)$

$x^3 - x + 1 \to (1,0,-1,1)$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix}$$

$-R_1 + R_2 \to R_2$ ↳ independent

$B = \{ x^3 + x^2, x^3 - x + 1 \}$

Q. $T: \mathbb{R}^{2\times2} \longrightarrow P_3$

$$T\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = (a_1 + a_4)x^2 + a_1x + a_4$$

① convince me that T is a L.T.

② find all matrices in $\mathbb{R}^{2\times2}$ such that $T\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = x^2 - x + 3$

③ find $Z(T)$ such that $T\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = 0x^2 + 0x + 0 = 0$

→ find the co-linear transformation

$\mathbb{R}^4 \longrightarrow \mathbb{R}^3$

$L(a_1, a_2, a_3, a_4) = (a_1 + a_4, a_1, a_4)$

each coordinate is a linear combination

# Week 13

Q. $T : P_3 \longrightarrow \mathbb{R}^3$

$T(a_2x^2 + a_1x + a_0) = (a_2 + a_1 + a_0 , a_1 , a_0)$

① is T LT? yes because linear combination

② Find the co-matrix presentation of T

$\mathbb{R}^3 \longrightarrow R^3$

$L(a_2, a_1, a_0) = (a_2 + a_1 + a_0 , a_1 , a_0)$

$$\begin{array}{ccc} a_2 & a_1 & a_0 \\ \hline 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}$$

③ is T invertible ?

T is invertible if M is invertible if L is invertible

Find $M^{-1}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$-R_2 + R_1 \rightarrow R_1$ $\qquad$ $-R_3 + R_1 \rightarrow R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \longleftarrow \text{L is invertible}$$

④ Find $T^{-1}$

$$L^{-1} = M^{-1}\begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix}$$

$= (a_2 - a_1 - a_0 , a_1 , a_0)$

$T^{-1}(a_2, a_1, a_0) = (a_2 - a_1 - a_0)x^2 + a_1x + a_0$

$T^{-1}(1,1,0) = 0x^2 + x = x$

Recall :

A linear transformation, T, is 1-1 iff $z(T) = \{0\text{ - elements}\}$ so $\dim(z(T)) = 0$

Result :

Assume D is a subspace and $\dim(D) < \infty$ , then the following must hold :

- for every $a, b \in D$ , $a + b \in D$. (closed under addition)
- for every $c \in \mathbb{R}$ and $a \in D$, $ca \in D$. (closed under scalar multiplication)

Q. convince me that $D = \left\{ \begin{bmatrix} a+b & a \\ a & a+1 \end{bmatrix} \middle| \; a, b \in \mathbb{R} \right\}$ is not a subspace

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin D$ because if $a = b = 0$ , $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Q. convince me that $D = \{ A \in \mathbb{R}^{3\times3} \mid |A| = 0 \}$ is not a subspace of $\mathbb{R}^{3\times3}$

↳ all 3x3 matrices where $|A| = 0$, infinitely many.

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in D$ , so we can't use this to convince

if $a, b \in D$ , $a + b \in D$

lets take $a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ , $a+b \in D$ ? $a+b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $a + b \notin D$

det = 0 $a \in D$ det = 0 $b \in D$ det = 1

Q. $D = \{ f(x) \in P_3 \mid f(0) = 0$ OR $f(1) = 0 \}$, Show that D is not a subspace

$f_1(x) = x \in D \rightarrow f(0) = 0$

$f_2(x) = 1 - x \in D \rightarrow f(1) = 0$

$f_3 = f_1 + f_2 = 1 \notin D \rightarrow f(0) \neq 0 \quad f(1) \neq 0$

Q. Show $D = \{ A \in \mathbb{R}^{2\times2} \mid A^T = -A \}$ is a subspace

① $a^T = -a$ , $b^T = -b \rightarrow A^T = -A$

$(a + b)^T = a^T + b^T = -a - b = -(a+b)$

② let $a \in D$ and $c \in \mathbb{R}$ , show $ca \in D$

$(ca)^T = ca^T = -ca \in D$

OR : $D = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 \\ -a_3 & -a_4 \end{bmatrix} \right\}$

$a_1 = -a_1 = 0$
$a_2 = -a_3$
$a_3 = -a_2$
$a_4 = -a_4 = 0$

$D = \left\{ \begin{bmatrix} 0 & a_2 \\ -a_2 & 0 \end{bmatrix} \middle| \; a_2 \in \mathbb{R} \right\} \rightarrow \left\{ a_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \middle| \; a_2 \in \mathbb{R} \right\} \rightarrow D = \text{Span} \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$

Q. $D = \{ f(x) \in P_3 \mid f(0) = 0$ and $f(1) = 0 \}$, Show that D is a subspace and find dim(D)

$D = \{ a_2x^2 + a_1x + a_0 \mid f(0) = a_0 = 0$ and $f(1) = a_2 + a_1 + 0 = 0 \}$

$a_1 = -a_2$

$D = \{ a_2x^2 - a_2x \mid a_2 \in \mathbb{R} \} = \{ a_2(x^2 - x) \mid a_2 \in \mathbb{R} \}$

$= \text{Span} \{ x^2 - x \}$ Dim (D) = 1

Q. $\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}} \underset{\downarrow E_3}{\xrightarrow{2R_1}} \begin{bmatrix} 2 & 4 \\ 3 & 0 \end{bmatrix} \underset{\downarrow E_2}{\xrightarrow{R_1 \leftrightarrow R_2}} \begin{bmatrix} 3 & 0 \\ 2 & 4 \end{bmatrix} \underset{\downarrow E_1}{\xrightarrow{2R_1 + R_2 \to R_2}} \underset{B}{\begin{bmatrix} 3 & 0 \\ 8 & 4 \end{bmatrix}}$

Find 3 elementary matrices $E_1, E_2, E_3$ s.t. $E_1 E_2 E_3 A = B$

apply the row operations on identity matrix

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} A = B$$

Q. $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 3 \end{bmatrix} \underset{\downarrow E_2}{\xrightarrow{R_1 \leftrightarrow R_2}} \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 3 \end{bmatrix} \underset{\downarrow E_1}{\xrightarrow{-2R_3}} \underset{B}{\begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ -2 & -2 & 0 & -6 \end{bmatrix}}$

Find elementary matrices $E_1 E_2 A = B$

↳ $\underline{3} \times 4$

so $I_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = B$$

## Dot Product over $\mathbb{R}^n$

$$(a_1, a_2, \dots, a_n) \cdot (b_1, b_2, \dots, b_n) = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

Def. let $Q_1, Q_2, \dots Q_n \in R^n$

→ we say $Q_1 \dots Q_m$ are orthogonal iff $Q_i \cdot Q_k = 0$
$i \neq k$

Q Convince me $\{(1,2), (0,4)\}$ is a basis for $\mathbb{R}^2$

$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$ → independent ; $\mathbb{R}^2 = \text{span}\{(1,2), (0,4)\}$

# Week 14

Q. $D = \text{span}\{\underset{Q_1}{(1,2,1)}, \underset{Q_2}{(-1,1,1)}\}$, $\dim(D) = 2$, Find an orthogonal basis of D

gram-shmidtt algorithm $O = \{w_1, w_2\}$ where $w_1 = Q_1 = (1,2,1)$

$$w_2 = Q_2 - \frac{Q_2 \cdot Q_1}{|Q_1|^2} . Q_1$$

$$= (-1,1,1) - \frac{2}{6}(1,2,1)$$

$$= (-1,1,1) - \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

$$= \left(-\frac{4}{3}, \frac{1}{3}, \frac{2}{3}\right)$$

to check $\longrightarrow$ $w_1 \cdot w_2 = 0$

## Gram-shmidtt algorithm

$O = \{W_1, \ldots, W_K\}$ where

$$W_m = Q_m - \frac{Q_m W_1}{|W_1|^2} W_1 - \frac{Q_m W_2}{|W_2|^2} W_2 - \cdots\cdots$$

$$W_3 = Q_3 - \frac{Q_3 W_1}{|W_1|^2} W_1 - \frac{Q_3 W_2}{|W_2|^2} W_2$$

Result: if $Q_1, Q_2, \ldots, Q_k$ are non-zero points in $\mathbb{R}^n$ and orthogonal then $Q_1, \ldots, Q_k$ are independent

* independent does not imply orthogonal.

# Week 15

$D = \text{Span}\{1, x^2+1\} \leq P_3$

↳ inner product on polynomials → $\langle f_1, f_2 \rangle = \int_a^b f_1 f_2 \, dx$

* finding orthogonal basis $O = \langle w_1, w_2 \rangle$, $\langle w_1, w_2 \rangle = 0$ where $\int_0^1 w_1 w_2 = 0$

* if $f$ is a polynomial, $|f| = \sqrt{\int_a^b f^2 \, dx}$, where a and b are given

Q. $D = \text{Span}\{1, x^2+1\} \leq P_2$, find the orthogonal basis

$$w_1 = f_1 = 1$$

$$w_2 = f_2 - \frac{\int_0^1 f_1 f_2 \, dx}{|w_1|^2} = (x^2+1) - \frac{\int_0^1 1(x^2+1)\, dx}{\int_0^1 1 \, dx} = x^2 + 1 - \frac{\frac{4}{3}}{1} = x^2 - \frac{1}{3}$$

$$O = \{1, x^2 - \tfrac{1}{3}\} \quad \rightarrow \quad D = \text{span}\{1, x^2 - \tfrac{1}{3}\}$$

Q. $D = \text{span}\{x, x^2, x^4\}$ "lives" in $P_5$, inner product on D is defined $\langle f_1, f_2 \rangle = \int_0^1 f_1 \cdot f_2 \, dx$

find the orthogonal basis, $O$

$O = \{w_1, w_2, w_3\}$ → the inner product will be zero for any two

$$w_1 = f_1 = x$$

$$w_2 = f_2 - \frac{\int_0^1 w_1 f_2}{|w_1|^2} w_1 = x^2 - \frac{\int_0^1 x^3 \, dx}{\int_0^1 x^2 \, dx} \cdot x = x^2 - \frac{\frac{1}{4}}{\frac{1}{3}} \cdot x = x^2 - \frac{3}{4}x$$

$$w_3 = f_3 - \frac{\int_0^1 w_2 f_3}{|w_2|^2} w_2 - \frac{\int_0^1 w_1 f_2}{|w_1|^2} w_1 = x^4 - \frac{\int_0^1 x^6 \, dx}{\int_0^1 x^4 \, dx} \cdot x^2 - \frac{3}{4}x = x^4 - \frac{\frac{1}{7}}{\frac{1}{5}} x^2 - \frac{3}{4}x = x^4 - \frac{5}{7}x^2 - \frac{3}{4}x$$

$$O = \{x, x^2 - \tfrac{3}{4}x, x^4 - \tfrac{5}{7}x^2 - \tfrac{3}{4}x\}$$

set, addition, scalar multiplication

$(V, +, \cdot)$ is called a vectorspace if

1) there exists $x, y \in V$, $x + y \in V$

2) there exists $c \in \mathbb{R}$ and $x \in V$, $cx \in V$

3) zero element in V, call it $0$, $0 + x = x + 0 = x$, there exists $x \in V$

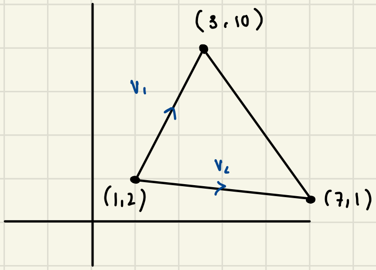
4) there exists $x \in V$, then $-x \in V$.

5) for every $c_1, c_2 \in \mathbb{R}$ and $x \in V$, $(c_1 + c_2)x = c_1 x + c_2 x$

6) for every $c_1, c_2 \in \mathbb{R}$ and $x \in V$, $(c_1 c_2)x = c_1(c_2 x)$

7) for every $c \in \mathbb{R}$, $x, y \in V$, $c(x+y) = cx + cy$

Q.



$*$ find the area

$v_1$ and $v_2$ should have the same initial point

$v_1 = (\Delta x, \Delta y) = (3-1, 10-2) = (2, 8)$

$v_2 = (\Delta x, \Delta y) = (7-1, 1-2) = (6, -1)$

absolute value

$$\text{Area} = \left| \frac{\begin{vmatrix} 2 & 8 \\ 6 & -1 \end{vmatrix}}{2} \right| = \left| \frac{-2-48}{2} \right| = 25$$