Linear Algebra MTH221 | 17 Jan 2022

Monday

R^n

$R^2 = \{ (x, y)$ a set of all points in xy-plane

$R^3 = \{ (x_1, x_2, x_3)$ a set of all points in x_1, x_2, x_3-plane

$R^n = \{ (x_1, x_2, \dots, x_n)$ a set of all points in $x_1, x_2, \dots, x_3$-plane

where they are all real numbers.

→ $(5, 10) \in R^2$ → $(1, 7, -2, 13) \in R^4$ → $(1, 2, 3) \in R^3$

$R^1 = \mathbb{R}$ = set of all real numbers where $n = 1$

- $(2, 3) + (5, 7) = (7, 10)$ ✓
- $-4(5, 10) = (-20, -40)$ ✓
- $(1, 2, 0) + (-1, 3, 4) = (0, 5, 4)$ ✓

{ }	( )
order not important	order important
ex: sets	ex: points on a plane

$(R^n, +)$ is closed means $(x, x_2, \dots, x_n) + (y, y_2, \dots, y_n)$ will equal a sum of $(x+y, x_2+y_2, x_3+y_3+ \dots, x_n+y_n)$ which is in R^n. $(\in R^n)$

$(R^n, \cdot)$ is closed in scalar multiplication too!

$\alpha(x_1, x_2, \dots, x_n) = (\alpha x_1, \alpha x_2, \alpha x_3, \dots, \alpha x_n) \in R^n$

Span $\{(2,1,3), (0,1,5)\} \in R^3$

the set of all linear combinations of $(2,1,3), (0,1,5)$ is called a span.

Linear combination of $(2,1,3), (0,1,5)$ means $c_1(2,1,3) + c_2(0,1,5)$ where c_1 & c_2 are some real numbers that can change makes it an infinite set = span ex: $D = \text{span}\{(2,1,3), (0,1,5)\}$ this is a **Subspace** of R^3

Does $3(2,1,3) + (0,1,5) \in D$? yes

Does $\sqrt{2}(2,1,3) + -4(0,1,5) \in D$? yes

Does D lives inside R^3 ? yes

- D is a **subset** of R^3.

Def: D is a subspace of R^n iff $D = \text{span}\{$ finite number of points in $R^n \}$

ex: $D = \text{span}\{(1,2,1,0)\}$ is a subspace of R^4.

it is infinite! (set of all linear combinations)

$-1(1,2,1,0) = (-1,-2,-1,0) \in D$ lives in D

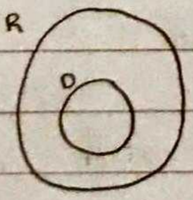
$(0,0,0,0) \in D$ if constant is 0

Is $(1,4,2,0) \in D$?

can we find a c where $c(1,2,1,0) = (1,4,2,0)$?

$= (1c, 2c, 1c, 0c)$

$$\left[\begin{array}{ll} c = 1 & \\ 2c = 4 & c = 2 \end{array}\right]$$ impossible!

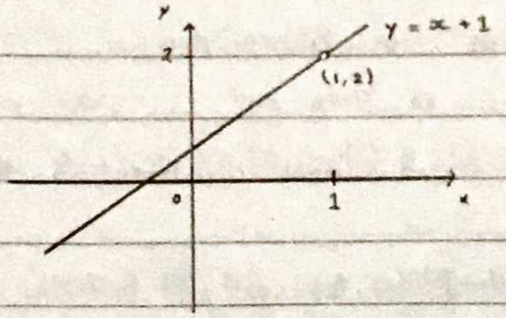


D is a subspace of R^4 but not $= R^4$

instead $D \in R^4$ lives in R^4.

$D = \text{span}\{(1,1,3), (1,-1,2), (1,5,3)\}$

if $c=0$ $0(1,1,3), 0(1,-1,2), 0(1,5,3) = (0,0,0), (0,0,0), (0,0,0) \in D$



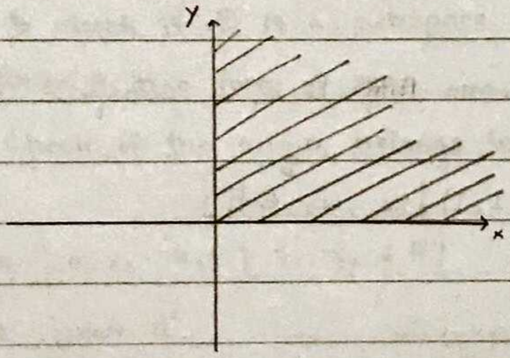
$D = \{ (x, x+1) \mid x \in R \}$ is the set of all points on the line $y = x+1$.

D is a "subset" of R^2

D is not a "subspace" of R^2 because the origin (0,0) is not there.

If D is a subspace then by our def: D = span {some points} :

then $(0,0) \in D$ but $(0,0) \notin D$. Hence D cannot be a subspace.



$D = \{ (x,y) \mid x \geq 0 , y \geq 0 \}$

D is a "subset" of R^2 but D is not a "subspace" of R^2 because

Assume D is a subspace, by def $D = \text{span} \{ (x_1, y_1), (x_2, y_2) \dots (x_n, y_n) \}$

$x_1, x_2, \dots, x_n \geq 0$ and $y_1, y_2, \dots, y_n \geq 0$

which means if $c_1 = -1$ and $c_2 = 0$ and $c_n = 0$

$-1(x_1, y_1) + 0(x_2, y_2) + \dots 0(x_n, y_n) \in D$ Impossible

$(-x_1, -y_1)$ is not in D.

- $D = \text{span} \{ (1,0), (0,1) \}$ then D is a subspace of R^2 & a subset of R^2.

if its a subspace its a subset but if its a subset it is not necessarily a subspace.

$D = R^2$ we know D "lives" in R^2

$D = \text{span}\{(1,0), (0,1)\}$

Every point in R^2 can be written as a linear combination of (1,0), (0,1).

$$(\sqrt{2}, \sqrt{5}) = \sqrt{2}(1,0) + \sqrt{5}(0,1)$$

which means (a, b) where a, b are any R it can be written as

$$a(1,0) + b(0,1) = (a,0) + (0,b) = (a,b)$$

19 Jan 2022

Subspaces

Wednesday

Q: $D = \{(x_1, x_2, x_1 + x_2) \mid x_1, x_2 \in R\}$

- D "lives" inside R^3
- D is infinite

Use the concept of span and show that D is a subspace of R^3.

A: $D = \{x_1(1,0,1) + x_2(0,1,1) \mid x_1, x_2 \in R\}$

$\text{span}\{(1,0,1), (0,1,1)\}$

Q: $D = \{(x_1, x_2, -2x_1 + 3x_2, x_4) \mid x_1, x_2, x_4 \in R\}$

- D "lives" in R^4
- D is infinite set

Use the concept of span and show that D is a subspace of R^4.

A: $D = \{x_1(1,0,-2,0) + x_2(0,1,3,0) + x_4(0,0,0,1) \mid x_1, x_2, x_4 \in R\}$

$\text{span}\{(1,0,-2,0), (0,1,3,0), (0,0,0,1)\}$

Q: $D = \{(x_1, x_2, x_1 + 2) \mid x_1, x_2 \in R\}$

- D "lives inside R^3

A: Is D a subspace of R^3? no, D is not a subspace, D is a subset of R^3.

$x(1,1,3) + 4$ is undefined !

if you want to add 4 to this point do $x(1,1,3) + (4,4,4)$.

$D = \{ x_1(1,0,1) + x_2(0,1,0) + \underline{\underline{1}}(0,0,2) \}$

this will be satisfied iff the last constant is 1, which means its not a span {points} and is not a subspace because for it to be so, c_1, c_2 and c_3 can be any numbers.

Another method for checking if D is a subspace, check if: $(0,0,0) \in D$.

$(0,0,2) \in D$ BUT $(0,0,0) \notin D$

$\therefore$ D is not a subspace.

2 Methods to check if D is a subspace

① Write it in a form of span

② Check if the origin belongs in D

Q $D = \{ (x_1, x_1 x_3, x_3) \mid x_1, x_3 \in R \}$

- D "lives" inside R^3
- D is infinite

A: $D = \{ x_1(1, x_3, 0) + x_3(0, x_1, 1) \mid x_1, x_3 \in R \}$

the points are not specific like the previous examples, when factored, the points still include x_1 & x_3 which are not fixed & can change.

$\therefore$ not a span because the points are infinite when the defenition of span is span = {finite number of points}

$\therefore$ D is not span.

Q: $D = \{(x_1, x_3 + 2, x_3, x_4) \mid x_1, x_3, x_4 \in R\}$

A: $D = \{x_1(1,0,0,0) + x_3(0,1,1,0) + x_4(0,0,0,1) + \underset{\text{specific}}{\underline{1}}(0,2,0,0)\}$

$\therefore$ D is not a subspace, D cannot be written as span

another method : $(0,0,0,0)$

$(0,2,0,0) \in D$ but $(0,0,0,0) \notin D$

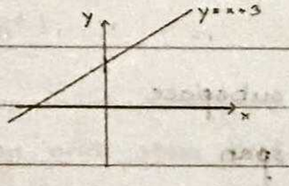
$\therefore$ D is not subspace of R^4

Linear Transformation (R- homomorphism)

$f: \underset{\text{x-axis}}{R} \longrightarrow \underset{\text{y-axis}}{\overset{\text{co-domain}}{R}}$

$f(x) = x + 3$

$f(1) = 4$



Q: $T: \underset{\text{domain}}{R^2} \longrightarrow \underset{\text{co-domain}}{R}$

$T(x_1, x_2) = 2x_1 - 5x_2$

Show T is a linear transformation.

A: Illustrate $T(1,3) = 2(1) - 5(3) = -13$

$T(2,1) = 2(2) - 5(1) = -1$

$T((1,3)+(2,1)) = T(1,3) + T(2,1) = T(3,4) = 2(3) - 5(4) = -14$

Q: $T: R \longrightarrow R$

$T(x) = x + 1$

$T(2) = 3$ $T(4) = 5$

$T(6) = 7$ which is $\neq$ to $T(2) + T(4) = 3 + 5 = 8$

Every linear transformation is a function but not every function is a linear transformation.

Def: $\underset{\text{domain}}{T : R^n} \longrightarrow \underset{\text{co-domain}}{R^n}$ is called linear transformation iff :

① $T(Q_1 + Q_2) = T(Q_1) + T(Q_2)$ for every points $Q_1, Q_2 \in R^n$

the image of the sum = sum of the image

② $T(CQ) = \underset{\text{constant}}{C}\, T(Q)$ for every real number C and every point $Q \in R^n$

ex: $T : R \rightarrow R$

$T(x) = 3x$

Is T a linear transformation ? yes

① $T(1) = 3$ $T(5) = 15$ $T(1+5) = T(6) = 18 = 15+3$ ∴ yes $T(Q_1+Q_2) = T(Q_1) + T(Q_2)$

② $C\,T(Q) = T(CQ)$ $4T(1) = 12$ and $T(4) = 12$ ∴ yes

In general using a_1 & $a_2 \in R$

$T(a_1 + a_2) = T(a_1) + T(a_2)$

- $T(a_1 + a_2) = 3(a_1 + a_2) = 3a_1 + 3a_2$
- $T(a_1) = 3a_1$ and $T(a_2) = 3a_2$

∴ $T(a_1 + a_2) = T(a_1) + T(a_2)$ ✓

second one $C \in R$, $a_1 \in R$

$T(ca_1) = c\,T(a_1)$

$3ca = 3ca$ ✓

24 Jan 2022

Monday

Q: $T : R \to R$

$T(x) = x^2$

Is T a linear transformation? No

A: $T(1) = 1^2 = 1$

$T(2) = 2^2 = 4$

$T(1+2) = T(3) = 3^2 = 9$

$T(1) + T(2) = 5$

$T(1) + T(2) \neq T(3)$

∴ not a linear transformation

Fact: $T : R \to R$ is a linear transformation

iff $T(x) = mx$ for some real number m.

Q: $T : R \to R$ $T(x) = 3x + 2$ is not a linear transformation

A:
- $3x + 2$ is not in the form of mx for some fixed real # m.
- $T(1) = 3(1) + 2 = 5$

$T(-1) = 3(-1) + 2 = -1$

$T(0) = 3(0) + 2 = 2$

$T(1) + T(-1) = 4$ $T(1) + T(-1) \neq T(0)$

Fact: $T : R^n \longrightarrow R^m$ is a linear transformation

then $\underbrace{T(0,0,0,\ldots)}_{n \text{ zeros}} = \underbrace{(0,0,0,\ldots)}_{m \text{ times}}$

- $T(0) = 2$ not 0 ∴ not a L.T.

Q $T: R^3 \longrightarrow R^3$

$T(x_1, x_2, x_3) = (5x_1, 2x_3, x_1 + x_3)$

Convince me this is a L.T.

Fact : $T: R^n \longrightarrow R^m$ is a linear transformation <u>iff</u>

$T(x_1, x_2, x_3) = ($ Linear combination of x_is , Linear comb. of x_is , Linear comb of x_is $)$

Linear Combination of x_1, x_2, x_3, x_4, x_5 means $c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5$

where c_1, c_2, c_3, c_4, c_5 are some real numbers.

Q: $T: R^2 \longrightarrow R^4$

Is $T(x_1, x_2) = (0, 1, x_1 + x_2, -3x_1)$ a linear transformation? No

A:
- 0 is a linear combination of x_1, x_2

  $0 = c_1x_1 + c_2x_2$ when $c_1 = c_2 = 0$
- 1 is not a linear combination of x_1, x_2

  $1 = \underbrace{c_1x_1 + c_2x_2}_{\text{fixed } c_1 \,\&\, c_2}$
- $x_1 + x_2$ is a linear combination $c_1 = 1$ & $c_2 = 1$
- $-3x_1$ is a linear combination $c_1 = -3$ & $c_2 = 0$

Another way $T(0,0) = (0,0,0,0)$ but in this question

$T(0,0) = (0,1,0,0)$

$\therefore$ not a linear transformation

Fact : If $T: R^n \rightarrow R^m$ is L.T.

then T (origin of R^n) = origin of R^m

- origin of $R = 0$
- origin of $R^2 = (0,0)$
- origin of $R^5 = (0,0,0,0,0)$

Q: $T: R^3 \longrightarrow R$

Is $T(x_1, x_2, x_3) = -10x_3 + x_2$ a linear combination? yes

A: $-10x_3 + x_2$ is a l.c. of x_1, x_2, x_3 because

$$-10x_3 + x_2 = \underline{0}x_1 + \underline{1}x_2 \underline{-10}x_3$$

b) Is it true that $T(1,0,2) + (2,5,7) = T(1,0,2) + T(2,5,7)$?

A: yes! because it is a linear transformation.

Q: Is $T: R^4 \longrightarrow R^5$

$$T(x_1, x_2, x_3, x_4) = (-2x_1 + 3x_2,\ x_3 - x_4,\ x_1 + 2x_2 - x_3,\ 0,\ x_4 + x_1)$$

A: Is T a linear transformation? yes

All the ... are linear combinations of $x_1, \dots, x_4$.

Q: Is $T: R^3 \longrightarrow R^4$

$T(x_1, x_2, x_3) = (x_1x_2, 0, x_3, x_1)$ a L.T.? No

A: $x_1x_2 \neq$ fixed c_1x_1 + fixed c_2x_2 + fixed c_3x_3

Hence T is not a LT.

$T(0,0,0) = (0,0,0,0)$

Mental Math Qs

$T: R^2 \longrightarrow R$ is a L.T.

$T(1,1) = 5$ and $T(-1,1) = 7$

- Find $T(0,2) = 5 + 7 = 12$ ✓
- Find $T(-4,4) = 4\,T(-1,1) = 4 \times 7 = 28$ ✓
- Find $T(0,0) = 0$ ✓
- Find $T(0,6) = T(3(0,2)) = 3\,T(0,2) = 3 \times 12 = 36$ ✓

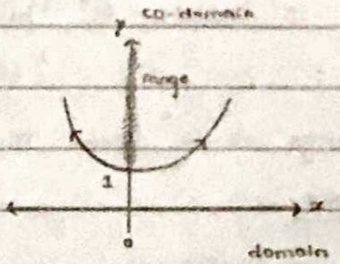
26 Jan 2022

Wednesday

Q $T: R \rightarrow R$

$T(x) = x^2 + 1$

we know T is not linear transformation.



- Range ? $1 \leq y < \infty$

Range is subset of co-domain

- x-intercept = zeros of T (when $y = 0$)

  "live" in domain

Linear transformations are functions

Q: $T: R^2 \longrightarrow R^3$

$T(x_1, x_2) = (3x_2, x_1 - x_2, x_1 + 5x_2)$

T is a linear transformation

- Range of T :

  range "lives" in R^3, the co-domain.

  $\text{Range} = \{ (3x_2, x_1 - x_2, x_1 + 5x_2) \mid x_1, x_2 \in R \}$ can be written as span

  $= \{ x_1 (0,1,1), x_2 (3,-1,5) \mid x_1, x_2 \in R \}$

  $= \text{span} \{ (0,1,1), (3,-1,5) \}$ if it can be written as span, its a subspace.

Q: Is $(5, 2, -1) \in \text{Range}(T)$?

A: to solve this, can we find c_1, c_2 such that

$(5, 2, -1) = c_1 (0,1,1) + c_2 (3,-1,5)$?

$(5, 2, -1) = (0, c_1, c_1) + (3c_2, -c_2, 5c_2)$

$(5, 2, -1) = (3c_2, c_1 - c_2, c_1 + 5c_2)$

$5 = 3c_2$

$c_2 = 5/3$

$c_1 - c_2 = 2$

$c_1 = 11/3$

$11/3 + 5(5/3) = 12$

$-1 \neq 12$

∴ this point is not in Range (T)

Range (T) is inside R^3 but not equal to it.

Fact : Range of a linear transformation is always a subspace of the co-domain.

Fact : zeros of T = Z(T) = Ker(T) = null of T

$$= \{ (x_1, x_2, \dots, x_n) \mid T(x_1, x_2, \dots x_n) = \underbrace{(0, 0, \dots, 0)}_{m \text{ times}} \}$$

· Find Z(T)

$$\{ (x_1, x_2) \mid T(x_1, x_2) = (3x_2, x_1 - x_2, x_1 + 5x_2) = (0,0,0) \}$$

$3x_2 = 0$ | $x_1 - x_2 = 0$ | $x_1 + 5x_2 = 0$

$x_2 = 0$ | $x_1 = 0$

$x_2 = 0$

$$Z(T) = \{ (0,0) \}$$

Q: $T: R^3 \longrightarrow R^2$

$T(x_1, x_2, x_3) = (x_1 + 2x_3, x_2 - 5x_3)$

T is a L.T.

(a) Find **Ker(T) = Z(t)**

$$Z(T) = \{ (x_1, x_2, x_3) \mid T(x_1, x_2, x_3) = (0,0) \}$$

$$(x_1 + 2x_3, x_2 - 5x_3) = (0,0)$$

$x_1 + 2x_3 = 0$ | $x_2 - 5x_3 = 0$

$x_1 = -2x_3$ | $x_2 = 5x_3$ $x_3 \in R$

$$\therefore Z(T) = \{ (-2x_3, 5x_3, x_3) \mid x_3 \in R \} = \{ x_3(-2, 5, 1) \mid x_3 \in R) \}$$

$= \text{span}\{(-2, 5, 1)\}$ subspace of R^3

Fact : Z(T) is always a subspace of the domain.

$\text{Range}(T) = \{ (x_1 + 2x_3, \ x_2 - 5x_3) \mid x_1, x_2, x_3 \in R \}$

$= \{ x_1(1,0) + x_2(0,1) + x_3(2,-5) \}$

$= \text{span} \{ (1,0), (0,1), (2,-5) \}$

In this example Range $= R^2$

Take any point (a,b) in R^2

$(a,b) = a(1,0) + b(0,1) + 0(2,-5) = (a,b)$

31 Jan 2022

Monday

Q. $D = \text{span} \{ (1,0), (0,1), (1,1) \}$

- is a subspace of R^2

$= \text{span} \{ (1,0), (0,1) \}$

Def: if we have $Q_1, Q_2, \dots, Q_K$ in R^n we say $Q_1, Q_2, \dots, Q_K$ are **independent**

iff whenever $c_1 Q_1 + c_2 Q_2 + \dots + c_K Q_K = \underbrace{(0,0,\dots,0)}_{n\text{-times}}$

then $c_1 = c_2 = \dots = c_K = 0$.

$Q_1, Q_2, \dots, Q_K$ are **dependent** if there exists atleast one $c_1 \neq 0$ such that

$c_1 Q_1 + c_2 Q_2 + \dots + c_K Q_K = \underbrace{(0,0,\dots,0)}_{n\text{-times}}$,

equivalent def: (practical)

$Q_1, Q_2, \dots, Q_K$ in R^n are independent if none of the Q_is is a linear combination of the remaining Q_is.

$Q_1, Q_2, \dots, Q_K$ are dependent iff atleast one of the Q_is is a linear combination of the remaining Q_is

Ex: $(2,1,0), (0,0,3), (4,2,3) \in R^3$ is it dependent or independent ?

$2(2,1,0) + 1(0,0,3) = (4,2,3)$

A: are dependent cuz atleast one of them is a linear combination of the other two.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 4 & 5 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{size}(B) = 3 \times 4$$

size (A) = 2×3

↙ #rows ↘ #columns

(0,1,4,5), (1,0,2,1), (0,0,1,0) are independent points in R^4.

- $c_1 Q_1 + c_2 Q_2 + c_3 Q_3 = (0,0,0,0)$

$\therefore c_1 = c_2 = c_3 = 0$

system of linear equations

$$2x_1 + 3x_2 - x_3 = 1$$
$$x_1 + 2x_2 - x_4 = 10$$
$$-2x_1 - 2x_2 + x_3 - x_4 = 100$$

Row-operations allowed

αR_i, $\alpha \neq 0$ multiply a row with a non-zero number.

$\alpha R_i + R_k \longrightarrow R_k$

$R_i \longleftrightarrow R_k$ you can interchange two rows

Q: Are (2, 4, -2), (-1, 2, 3), (0, 6, 4) $\in R^3$ independent ? yes

A: <u>method</u>

① go row by row

1^{st} row, 1^{st} non-zero number need be "1"

$$\begin{bmatrix} \textcircled{2} & 4 & -2 \\ -1 & 2 & 3 \\ 0 & 6 & 4 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} \textcircled{1} & 2 & -1 \\ -1 & 2 & 3 \\ 0 & 6 & 4 \end{bmatrix}$$

equivalent

② use "1" in row one & kill all numbers exactly below "1" (we use row operation #2)

$1R_1 + R_2 \longrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & \textcircled{4} & 2 \\ 0 & 6 & 4 \end{bmatrix}$$

③ Go to the second row R_2 and repeat.

$$\Big\downarrow \tfrac{1}{4}R_2$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & \textcircled{1} & 1/2 \\ 0 & 6 & 4 \end{bmatrix} \xrightarrow{-6R_2 + R_3 \to R_3} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

stare ↑↑ none of the rows = (0, 0, 0)

∴ yes, independent

Q: Are (1, 2, -1, 4), (-2, -3, 4, 6), (-2, -2, 6, 20) $\in R^4$ independent? no

A: $$\begin{bmatrix} \textcircled{1} & 2 & -1 & 4 \\ -2 & -3 & 4 & 6 \\ -2 & -2 & 6 & 20 \end{bmatrix} \xrightarrow[\text{and } 2R_1 + R_2 \to R_3]{2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & \textcircled{1} & 2 & 14 \\ 0 & 2 & 4 & 28 \end{bmatrix}$$

$$\xrightarrow{-2R_2 + R_3 \to R_3} \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 2 & 14 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

stare ↑↑ we see (0.0.0.0) a zero-row

which implies : dependent.

we killed (-2, -2, 6, 20) it is the point that is a linear combination of the others.

2 Feb 2022

Wednesday

Def : Let D be a subspace of $\mathbb{R}^n$, so we know $D = \text{span}\{Q_1, \dots, Q_k\}$ for some points in $\mathbb{R}^n$.

in old exams IN(D) means dim(D)

dim(D) = max # of independent points (i.e. find the independent points of $Q_1, \dots Q_k$)

Say $P_1, \dots P_m$ are the max. number of independent points in D.
Then $D = \text{span}\{P_1, \dots, P_m\}$

$\hookrightarrow$ dim(D) = m

Q: $D = \text{span}\{(1,1,0,1), (-2,-2,1,3), (0,0,1,5), (-2,-2,3,13)\}$

· D is a subspace of $\mathbb{R}^4$

(a) Find a basis for D.

(b) Find dim(D).

(c) Use (a) and rewrite D as span of independent points.

$$A: \begin{array}{l} \text{indep.}\{ \\ \\ \\ \\ \end{array} \begin{bmatrix} \textcircled{1} & 1 & 0 & 1 \\ -2 & -2 & 1 & 3 \\ 0 & 0 & 1 & 5 \\ -2 & -2 & 3 & 13 \end{bmatrix} \xrightarrow[2R_1 + R_4 \to R_4]{2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & \textcircled{1} & 5 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 3 & 15 \end{bmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \end{array}\right\} \text{independent} \\ \\ \\ \end{array}$$

$$\xrightarrow[-3R_2 + R_4 \to R_4]{-R_2 + R_3 \to R_3} \begin{array}{r} \text{indep.}\left\{\begin{array}{l} \\ \\ \end{array}\right. \\ \text{dependent}\left\{\begin{array}{l} \\ \\ \end{array}\right. \end{array} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) B (basis of D) = {all independent points}

$B = \{(1,1,0,1), (0,0,1,5)\}$

(b) dim(D) = 2

(c) $D = \text{span}\{(1,1,0,1), (0,0,1,5)\}$

(d) Is $(10,10,2,15) \in D$? no

$$(10,10,2,15) = c_1(1,1,0,1) + c_2(0,0,1,5)$$

Find c_1 & c_2

$$(10,10,2,15) = (c_1, c_1, 0, c_1) + (0,0,c_2,5c_2)$$

$$(10,10,2,15) = (c_1, c_1, c_2, c_1+5c_2)$$

$$c_1 = 10 \quad \& \quad c_2 = 2$$

$$10 + 5(2) \overset{?}{=} 15 \ ?$$

$$10+10 \neq 15$$

$$20 \neq 15$$

$\therefore$ No such c_1 & c_2 exist.

Hence, $(10,10,2,15) \notin D$

Math (Know)

1) R^n is a subspace of itself (R^n, we call it vector space)

all the points are independent

$$R^n = \text{span}\{(1,0,0,\dots,0), \underbrace{(0,1,0,\dots,0)}_{Q_2 \,:\, 2^{\text{nd}} \text{ coordinate} = 1 \ \& \text{ all others} = 0}, \dots, (0,0,0,\dots,1)\}$$

$$(a_1, a_2, \dots, a_n) = a_1 Q_1 + a_2 Q_2 + \dots + a_n Q_n$$

$$\underset{n \times n}{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & 1\cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}} = I_n = \text{Identity matrix}$$

- $\dim(R^n) = n$
- $B = \{(1,0,\dots,0), (0,1,\dots,0), \dots, (0,0,\dots,1)\}$
  ↳ is the standard basis of R^n

2) Assume D is a subspace of R^n and $\dim(D) = m$. Then

i) $\dim(D) = m \leq n$ # of indep points should be ≤ total # of points

ii) $D = R^n$ iff $n = m$

iii) If $k > m$ then ~~any~~ every k points in D are dependent.

continuation of ②

iv) **Basis** for $D = \left\{ \text{any } \underline{\underline{m}} \text{ independent points in } D \right\}$

basis is a combination of all the linearly independent points.

span { basis } = D

v) span { any L independent points in R^n , $L < m$ } $\neq D$

vi) $D = \text{span} \left\{ \text{any } \underline{\underline{m}} \text{ independent points} \right\}$

Q: Is $\{ (2,6), (-3,12) \}$ a basis for R^2 ?

A:

$$\begin{bmatrix} 2 & 6 \\ -3 & 12 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} \textcircled{1} & 3 \\ -3 & 12 \end{bmatrix} \xrightarrow{3R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 3 \\ 0 & \textcircled{21} \end{bmatrix}$$

$$\Big\downarrow \tfrac{1}{21}R_2$$

$$\text{indep.} \left\{ \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \right.$$

$R_2 = \text{span} \{ (1,0), (0,1) \}$ ✓

$R^2 = \text{span} \{ (2,6), (-3,12) \}$ ✓

R^2 is a span of any 2 independent points

7 Feb 2022

Monday

Eigen values

$T : R^n \to \overset{\text{co-domain}}{R^n}$ is a linear transformation.

A number $\underline{\alpha}$ is called an eigen-value of T iff $\overset{\text{exists}}{\exists}$ a non-zero point Q in the domain (here R^n) such that $T(x_1, \dots, x_n) = \alpha(x_1, \dots, x_n)$

$Q = (x_1, \dots, x_n)$

and $Q \neq (0, \dots, 0)$

$T : R^3 \to R^3$

$(x_1, x_2, x_3) = (5x_1, 3x_2, -10x_3)$

Find all eigenvalues of T.

$T(1,0,0) = (5,0,0) = 5(1,0,0)$

$\therefore$ 5 is an eigen value of T.

$T(0,1,0) = (0,3,0) = 3(0,1,0)$

$\therefore$ 3 is an eigen value of T.

$T(0,0,1) = (0,0,-10) = -10(0,0,1)$

$\therefore$ -10 is an eigen value of T.

$T(1,0,2) = (5,0,-20) \neq 5(1,0,2)$

Note : Any point, say Q, in the span $\{(1,0,0)\}$ satisfy $T(Q) = 5Q$

span $\{(1,0,0)\}$ = eigenspace corresponds to the eigen-value 5.

span $\{(0,1,0)\}$ = " " " " " " 3.

span $\{(0,0,1)\}$ = " " " " " " -10.

Matrix multiplication

Q.

$$\underset{3\times\underline{3}}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}} \underset{\underline{3}\times 1}{\begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}} = \underset{3\times 1}{\begin{bmatrix} 22 \\ 9 \\ 17 \end{bmatrix}}$$

Q.

$$\underset{2\times 4}{\begin{bmatrix} 1 & 2 & 6 & 8 \\ 0 & 1 & 2 & 3 \end{bmatrix}} \underset{4\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}} = \underset{2\times 3}{\begin{bmatrix} 23 & 4 & 19 \\ 8 & 1 & 6 \end{bmatrix}}$$

$$\underset{n\times m}{A}\,\underset{m\times r}{B} = \underset{n\times r}{C}$$

$\boxed{AB \text{ need not } = BA}$

Q.

$$\underset{3\times 4}{\begin{bmatrix} 1 & 0 & 2 & 4 \\ 2 & 1 & 1 & 0 \\ 3 & 1 & 3 & 0 \end{bmatrix}} \underset{4\times 1}{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}} = 1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

$$= \underset{3\times 1}{\begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix}}$$

Q. Use the concept of Linear combination of columns and Find :

$$\underset{3\times 4}{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -2 \end{bmatrix}} \underset{4\times 2}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \\ 3 & 2 \end{bmatrix}} = \underset{3\times 2}{\begin{bmatrix} C \end{bmatrix}}$$

First column of C = $1\begin{bmatrix}1\\0\\0\end{bmatrix} + 0\begin{bmatrix}2\\1\\0\end{bmatrix} + 2\begin{bmatrix}3\\1\\1\end{bmatrix} + 3\begin{bmatrix}4\\2\\-2\end{bmatrix}$

$$= \begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}6\\2\\2\end{bmatrix} + \begin{bmatrix}12\\6\\-6\end{bmatrix} = \begin{bmatrix}19\\8\\-4\end{bmatrix}$$

Second column of C $= -1\begin{bmatrix}1\\0\\0\end{bmatrix} + 1\begin{bmatrix}2\\1\\0\end{bmatrix} + 0\begin{bmatrix}3\\1\\1\end{bmatrix} + 2\begin{bmatrix}4\\2\\-2\end{bmatrix}$

$$= \begin{bmatrix}-1\\0\\0\end{bmatrix} + \begin{bmatrix}2\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}8\\4\\-4\end{bmatrix}$$

$$= \begin{bmatrix}9\\5\\-4\end{bmatrix}$$

$$\therefore\ C = \begin{bmatrix}19 & 9\\8 & 5\\-4 & -4\end{bmatrix}$$

$$\underset{n\times m}{A}\,\underset{m\times r}{B} = \underset{n\times r}{C}$$

Each column of C is a linear combination of the colums of A.

Result (know) : Give me any matrix M, n×m. Then $T: R^m \longrightarrow R^n$ given by $T(x_1, \dots, x_m) = \underset{n\times m}{M}\underset{m\times 1}{\begin{bmatrix}x_1\\\vdots\\x_m\end{bmatrix}}$ is a L.T.

Q: $T: R^2 \rightarrow R$

$$T(x_1, x_2) = \begin{bmatrix} 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$T(1,3) = \begin{bmatrix} 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1\begin{bmatrix} 1 \end{bmatrix} + 3\begin{bmatrix} 4 \end{bmatrix} = \begin{bmatrix} 13 \end{bmatrix}$$

$$\left. \begin{aligned} T(x_1, x_2) &= x_1 + 4x_2 \leftarrow \text{formula for range} \\ &= \left\{ x_1(1) + x_2(4) \right\} \\ &= \text{span}\left\{ 1, 4 \right\} \\ &= \text{span}\{4\} = \text{span}\{1\} = \text{span}\{e^{\pi}\} = \text{span}\{\sqrt[3]{1.5}\} \end{aligned} \right\} \text{Range in } R$$

$$\dim(T) = 1$$

$$\left. \begin{aligned} Z(T): \ & \text{set } x_1 + 4x_2 = 0 \\ & x_1 = -4x_2 \quad x_2 \in R \\ Z(T) &= \left\{ (x_1, x_2) \in R^2 \mid x_1 = -4x_2 , \ x_2 \in R \right\} \\ &= \left\{ (-4x_2, x_2) \mid x_2 \in R \right\} \\ &= \left\{ x_2(-4,1) \mid x_2 \in R \right\} \\ &= \text{span}\left\{ (-4,1) \right\} \end{aligned} \right\} Z(T) \text{ in } R^2$$

9 Feb 2022

Wednesday

Q) $T: R^3 \to R^2$

$T(a_1, a_2, a_3) = (a_1 - 2a_2 + a_3,\ 4a_1 - 8a_2 + 4a_3) = \left(a_1(1,4) + a_2(-2,-8) + a_3(1,4) \right)$

① Find the standard matrix presentation of T.

$$M = \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{bmatrix} 1 & -2 & 1 \\ 4 & -8 & 4 \end{bmatrix} \end{array}$$

standard matrix representation.

every linear transformation can be represented as matrix.

standard basis of the domain (R^3)

$\{ (1,\overset{e_1}{0},0), (0,\overset{e_2}{1},0), (0,\overset{e_3}{0},1) \}$

$T(1,0,0)$ = first column of M

$T(0,1,0)$ = second column of M

$T(0,0,1)$ = third column of M

Range = span $\{ (1,4), (-2,-8), (1,4) \}$

I claim that $T(a_1, a_2, a_3) = M \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

a_1 is the image of $T_1 \longleftrightarrow a_1 = T(e_1)$

Range = span { Columns of A }

$Z(T) = \{ (a_1, a_2, a_3) \in \text{domain} \mid T(a_1, a_2, a_3) = (0,0) \}$

$M \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ same same $\begin{array}{l} a_1 - 2a_2 + a_3 = 0 \\ 4a_1 - 8a_2 + 4a_3 = 0 \end{array}$

$$\left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & \\ 1 & -2 & 1 & 0 \\ 4 & -8 & 4 & 0 \end{array}\right] \overset{-4R_1 + R_2 \to R_2}{=} \left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & \\ 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

augmented matrix

completely reduced

$a_1 - 2a_2 + a_3 = 0$

$a_1 = 2a_2 - a_3$

$\underbrace{a_2, a_3}_{\text{Free variables}} \in R$

$Z(T) = \{(2a_2 - a_3, a_2, a_3) \mid a_2, a_3 \in R\}$ The zeros of T are always independent.

$= \{a_2(2,1,0) + a_3(-1,0,1)\}$

$= \text{span}\{(2,1,0),(-1,0,1)\}$

Fact : The $\dim(Z(T))$ = number of free variable when we solve $M\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$\text{Range}(T) = \text{span}\{(1,4)\}$

$\dim(\text{Range}(T)) = 1$

Know : $\dim(Z(T)) + \dim(\text{Range}(T)) = \dim(\text{domain})$

$2 + 1 = 3 \leftarrow R^3$

14 Feb 2022

Monday

$T: R^4 \rightarrow R^3$

$T(x_1, x_2, x_3, x_4) = (x_1 - 2x_2 + x_4,\ 0,\ 2x_1 - 4x_2 + x_3 + 2x_4)$

Find the standard matrix presentation of T.

M

dim(co-domain) × dim(domain) ← each coordinate is a row.

$$M = \left[\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & -4 & 1 & 2 \end{array}\right].$$

$$T(x_1, x_2, x_3, x_4) = M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$T(2,4,0,1) = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 0 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 4\begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ 0 \\ \ \end{bmatrix}.$$

Rank of any matrix = # of independent rows of A.

↳ which is also = # of independent columns of A.

$$\text{Rank}(M) \rightarrow \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & -4 & 1 & 2 \end{bmatrix} \xrightarrow{-2R_1 + R_3 \rightarrow R_3} \begin{bmatrix} ① & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & ① & 0 \end{bmatrix}$$

← we called the matrix K

$\therefore \text{Rank}(M) = 2$

Row space of M = Row(M) = span { independent rows }

= span { (1, -2, 0, 1) , (0, 0, 1, 0) }

= span { (1, -2, 0, 1), (2, -4, 1, 2) }

Note : Rank(M) = dim (row (M))

Column space of M = Col(M) = span { (1, 0, 2), (0, 0, 1) }

- You choose the columns that have the circled ones ①
- You write it as a span of the original matrix. (not K) because we use row operation & not column operation so we do not guarantee that the new column lives in R^3.

Note : Col (M) = Range (T)

Range (T) = { (1, 0, 2) , (0, 0, 1) }

dim (Range(T)) = Rank (M) = dim (col(M)) = dim (row(M))

Note : • T is **onto** iff Range (T) = co-domain dim(Range(T)) = dim (co-domain)

$\text{no of indep.} \rightarrow 2 \neq 3 \leftarrow R^3$

- T is **1 to 1** iff when every $T(Q_1) = T(Q_2)$ then $Q_1 = Q_2$

2 points cannot share the same image.

- T is 1 to 1 iff $Z(T) = \{\overset{\text{of domain}}{\text{origin}}\}$

↳ in our T, the origin of our domain is (0, 0, 0, 0) dim(Z(T)) = 0

- T is **isomorphism** iff it is both onto and 1 to 1.

- dim { span {origin} } = 0 **Span { (0, 0, 0, 0) } = { 0, 0, 0, 0 }**

∴ dim (span (origin)) = 0

because origin is dependent.

we Know that

$$\dim(\text{Range}(T)) + \dim(Z(T)) = \dim(\text{domain})$$

$$\therefore \quad 2 + \underset{\neq 0}{\underset{\uparrow}{2}} = 4$$

$\neq 0$ ∴ T is not 1-1

Q: $T : R^4 \rightarrow R^5$

$$T(x_1, x_2, x_3, x_4) = (x_2 - x_3 + x_4,\ x_1 + x_2 - x_4,\ x_1 + 2x_2 - x_3,\ x_1 + x_3 + x_4,\ 0)$$

Find all the points in the domain R^4 such that

T(each point) = (1, 4, 2, 6, 0)

$$M = \left[\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ 0 & 1 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 2 & -1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

we have to form an augmented matrix

$$M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \\ 6 \\ 0 \end{bmatrix}$$

Augmented matrix

$$\left[\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \text{constants} \\ 0 & \textcircled{1} & -1 & 1 & 1 \\ 1 & 1 & 0 & -1 & 4 \\ 1 & 2 & -1 & 0 & 2 \\ 1 & 0 & 1 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[-2R_1 + R_3 \rightarrow R_3]{-R_1 + R_2 \rightarrow R_2} \left[\begin{array}{cccc|c} 0 & 1 & -1 & 1 & 1 \\ \textcircled{1} & 0 & 1 & -2 & 3 \\ 1 & 0 & 1 & -2 & 0 \\ 1 & 0 & 1 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[-R_2 + R_4 \rightarrow R_4]{-R_2 + R_3 \rightarrow R_3} \left[\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ 0 & 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & -3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}\right]$$

there is no such point $\longrightarrow$

where $0x_1 + 0x_2 + 0x_3 + 0x_4 = -3$

prof changed the question to :

Q) $T : R^4 \longrightarrow R^5$

$$T(x_1, x_2, x_3, x_4) = (x_2 - x_3 + x_4 \ , \ x_1 + x_2 - x_4 \ , \ x_1 + 2x_2 - x_3 \ , \ x_1 + x_3 + x_4 \ , 0)$$

Find all the points in the domain such that :

T (each point) = (1, 4, **5**, 6, 0)

$$M = \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline 0 & 1 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 2 & -1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}$$

we have to form an augmented matrix

$$M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \\ 6 \\ 0 \end{bmatrix}$$

Augmented matrix

$$\left[\begin{array}{cccc|c} 0 & \textcircled{1} & -1 & 1 & 1 \\ 1 & 1 & 0 & -1 & 4 \\ 1 & 2 & -1 & 0 & 5 \\ 1 & 0 & 1 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[-2R_1 + R_3 \to R_3]{-R_1 + R_2 \to R_2} \left[\begin{array}{cccc|c} 0 & 1 & -1 & 1 & 1 \\ \textcircled{1} & 0 & +1 & -2 & 3 \\ 1 & 0 & 1 & -2 & 3 \\ 1 & 0 & 1 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$-R_2 + R_3 \to R_3$
$-R_2 + R_4 \to R_4$

$$\left[\begin{array}{cccc|c} 0 & 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xleftarrow{\frac{1}{3} R_4} \left[\begin{array}{cccc|c} 0 & 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \textcircled{3} & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$-R_4 + R_1 \to R_1$
$+2R_4 + R_2 \to R_2$

$$\left[\begin{array}{cccc|c} 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\longleftarrow x_2 - x_3 = 0 \longrightarrow x_2 = x_3 \longleftarrow$ free variable

leading variable

$\longleftarrow x_1 + x_3 = 5 \longrightarrow x_1 = 5 - x_3$ ✓

$x_4 = 1$

$\frac{1}{3}$ 0=0

completely reduced ∇

16 Feb 2022

Wednesday

System of linear equations

$$\left[\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & C \\ 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

completely reduced

we have a system of linear equations

(if this is the last step)

$\Updownarrow$

original equation $(\underset{\text{equations}}{3} \times \underset{\text{unknown variable}}{4})$

$n \times m$ system of linear equation

3 possibilities :

① Unique solution (no free variables, all leading)

② No solution

③ Infinetely many solutions (we must have atleast one free variable)

If the system has ① or ③, we say the system is consistent. ← has to have atleast one solution.

If the system has ②, we say it is inconsistent.

Reading the matrix above :

$$\left.\begin{aligned} x_1 + 3x_2 &= 1 \\ x_3 &= 2 \\ x_4 &= 3 \end{aligned}\right\} x_1, x_3 \text{ \& } x_4 \text{ are leading variables}$$

$x_2 \in R$ is a free variable.

Write each leading variable interms of free variable

$x_1 = 1 - 3x_2$, $x_3 = 2$, $x_4 = 3$

$$\overset{F}{\text{solution set}} = \left\{ (1-3x_2 , x_2 , 2 , 3) \mid x_2 \in R \right\}$$

change x_2 & you will get different values :

$(1,0,2,3) \in F$

$(7,-2,2,3) \in F$

$(-8,3,2,3) \in F$

* If a system has no free variables it does not conclude that it has a unique solution. It has to be consistent.

$$\left[\begin{array}{ccc|c} x_1 & x_3 & x_3 & \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right] \longleftarrow x_1 + x_2 + x_3 = 1$$

$0 = 1$

CANNOT

∴ Although it has no free variables (all leading) it is not unique. Infact, its no solution.

System of L.E. has <u>no solution</u> <u>iff</u> in one of the steps you observe that one of the equations become 0 = non-zero number

$\overset{\text{inconsistent}}{\text{no solution}}$

Q $x_1 + x_2 - x_3 = 1$

$-x_1 + 2x_3 = 2$

$2x_1 + 3x_2 - 2x_3 = 10$

System of L.E [3×3]

$$\left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & C \\ 1 & 1 & -1 & 1 \\ -1 & 0 & 2 & 2 \\ 2 & 3 & -2 & 10 \end{array}\right] \quad \begin{array}{c} \\ R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{array} \quad \begin{array}{cccc} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 8 \end{array}$$

$-R_2 + R_1 \to R_1$

$-R_2 + R_3 \to R_3$

$$\begin{array}{cccc} 1 & 0 & 0 & -12 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -5 \end{array} \quad \begin{array}{c} \\ -2R_3 + R_1 \to R_1 \\ -R_3 + R_2 \to R_2 \end{array} \quad \begin{array}{cccc} 1 & 0 & -2 & -2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -5 \end{array} \quad \begin{array}{c} \\ -R_3 \\ \\ \end{array} \quad \begin{array}{cccc} 1 & 0 & -2 & -2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & 5 \end{array}$$

The solution set = $\{(-12, 8, -5)\}$

21 Feb 2022

Monday

Q1) $x_1 + 2x_2 - 3x_3 = 4$

$-x_1 + ax_2 + 5x_3 = 10$

$2x_1 + 4x_2 - bx_3 = c$

(a) For what values a, b, c does the system have unique solution?

(b) " " " " will the system be inconsistent?

(c) " " " " will the system have infinitely many solutions?

Form augmented matrix

$$\left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & c \\ \textcircled{1} & 2 & -3 & 4 \\ -1 & a & 5 & 10 \\ 2 & 4 & +b & c \end{array}\right] \quad \begin{array}{l} R_1 + R_2 \rightarrow R_2 \\ -2R_1 + R_3 \rightarrow R_3 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & a+2 & 2 & 14 \\ 0 & 0 & b+6 & c-8 \end{array}\right]$$

$$\left.\begin{aligned} x_1 + 2x_2 - 3x_3 &= 4 \\ (a+2)x_2 + 2x_3 &= 14 \\ (b+6)x_3 &= c-8 \end{aligned}\right\}$$

for it to be a unique solution, all values should be leading

∴ (a) For it to be unique $a \neq -2$, $b \neq -6$ and $c \in \mathbb{R}$.

(b) For it to be inconsistent $b = -6$ and $c \neq 8$

and also if

$a = -2$ and, $\frac{c-8}{b+6} \neq 7$

(c) For it to have infinitely many solutions: (it has to be consistent)

(i) $a = -2$, $\frac{c-8}{b+6} = 7$

(ii) $b = -6$, $c = 8$ & $a \neq -2$

same question but using linear transformation:

Q) $T: R^3 \longrightarrow R^3$

$T(x_1, x_2, x_3) = (x_1 + 2x_2 - 3x_3,\ -x_1 + ax_2 + 5x_3,\ 2x_1 + 4x_2 + bx_3)$

(a) For what values of a, b there will be a point (x_1, x_2, x_3) in the domain of T s.t. $T(x_1, x_2, x_3) = (4, 10, c)$ where $c \in R$?

(b) For what values of a, b there will be no point (x_1, x_2, x_3) in the domain of T s.t. $T(x_1, x_2, x_3) = (4, 10, c)$ where $c \in R$?

(c) For what values of a, b there will be infinitely many points (x_1, x_2, x_3) in the domain of T s.t. $T(x_1, x_2, x_3) = (4/10, c)$ where $c \in R$?

[Q] $T: R^4 \longrightarrow R^4$

$T(x_1, x_2, x_3, x_4) = (4x_1, -2x_2, 3x_3, -x_4)$

$\underbrace{\alpha = 4}_{\text{Eigen value}}$, $T(1,0,0,0) = (4,0,0,0) = 4(1,0,0,0)$

$E_4 =$ **eigen** $\underbrace{\text{eigen space}}_{\text{subspace of the domain (here } R^4)}$ correspond to the eigen value 4.

$E_4 = \text{span}\{(1,0,0,0)\}$

also, other eigen values are $\alpha = -2$, $\alpha = 3$ and $\alpha = -1$.

Only for $n \times n$ matrices

[Q]

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 4 \\ 0 & 0 & 6 \end{bmatrix}$$

Find all eigen values of A.

For each eigen value of A, say α

Find E_α.

y3ni Find real number say α such that there exist atleast one point in R^3 say $Q = (x_1, x_2, x_3) \neq (0, 0, 0)$

such that

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Tools needed to find eigen values :

Determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 4 & 1 \\ 1 & 2 & 6 \end{bmatrix}$$

Find $|A|$.

Choose any row or any column (recommended, we chose the one that has more zeros)

we chose 1st column :

$$(-1)^{1+1} 1 \cdot \begin{vmatrix} 4 & 1 \\ 2 & 6 \end{vmatrix} + (-1)^{2+1} \cdot 2 \begin{vmatrix} 3 & -1 \\ 2 & 6 \end{vmatrix} + (-1)^{3+1} \cdot 1 \begin{vmatrix} 3 & -1 \\ 4 & 1 \end{vmatrix}$$

$$= \left[(4)(6) - (2)(1)\right] + 2\left[(3)(6) - (-1)(2)\right] + 1\left[3 + 4\right]$$

$$= 22 + -40 + 7$$

$$= -11$$

Ⓠ

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 4 & 3 \\ 0 & 6 & 10 \end{bmatrix} \qquad |A| = (-1)^{3+2} \cdot 6 \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} + (-1)^{3+3} \cdot 10 \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}$$

$$= -6 \, [3-0] + 10 \, [4-2]$$

$$= -18 + 20$$

$$= 2$$

Facts about determinant | |

⓪ $n \times n$ system of linear equation :

$$\left[\begin{array}{c|c} \underbrace{x_1, \cdots x_n}_{\text{coeff. matrix}} & c \end{array}\right] = \left[\begin{array}{c|c} C & c \end{array}\right]$$

will have a unique solution <u>iff</u> $|C| \neq 0$

② If $|C| = 0$ then $\rightarrow$ no solution

$\searrow$ infinitely many

Cramer Rule (explained by example) (only for $n \times n$)

$$\left\{ \begin{array}{l} x_1 + 2x_2 - x_3 = 10 \\ x_1 + 4x_2 + 10x_3 = 11 \\ -3x_1 + 10x_2 + 9x_3 = 30 \end{array} \right\} \; 3 \times 3$$

Assume $|C| \neq 0$

$$|C| = \begin{vmatrix} 1 & 2 & -1 \\ 1 & 4 & 10 \\ -3 & 10 & 9 \end{vmatrix}$$

$$x_1 = \frac{\begin{vmatrix} 10 & 2 & -1 \\ 11 & 4 & 10 \\ 30 & 10 & 9 \end{vmatrix}}{|C|}$$

$$x_2 = \frac{\begin{vmatrix} 1 & 10 & -1 \\ 1 & 11 & 10 \\ -3 & 30 & 9 \end{vmatrix}}{|C|}$$

$$x_3 = \frac{\begin{vmatrix} 1 & 2 & 10 \\ 1 & 4 & 11 \\ -3 & 10 & 30 \end{vmatrix}}{|C|}$$

28 Feb 2022

Monday

The effect of row operation on |A| (determinant)

Explain by doing an example :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 2 & 5 \\ -1 & -2 & 10 \end{bmatrix} \quad \begin{matrix} 2R_1 + R_2 \rightarrow R_2 \\ R_1 + R_3 \rightarrow R_3 \end{matrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 11 \\ 0 & 0 & 13 \end{bmatrix}$$

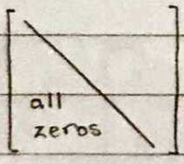
$$|B| = \det(B) = |A| = \det(A)$$
$$= (1)(6)(13) = 78$$

Result: Let A be $\underline{n \times n}$ Triangular Matrix.

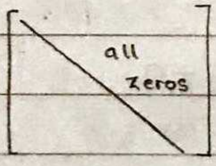
Then |A| = multiplication of $\underline{\underline{\text{all}}}$ numbers on the main diagonal.

Def : A is a **triangular** matrix if it has one of the following forms :

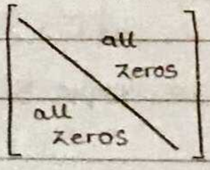
① Upper triangle



② Lower triangle



③ Diagonal



$$C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 11/6 \\ 0 & 0 & 13 \end{bmatrix} \qquad |C| = \frac{1}{6}|B| = \frac{1}{6}|A| \Rightarrow |A| = 6|C|$$

$$|A| = 6 \times (1)(1)(13)$$
$$= 78$$

Note : If you multiply a row with a number (non-zero constant)

$\alpha \neq 0 \quad A \xrightarrow{\alpha R_i} B, \ |B| = \alpha |A|$

Q.

$$A = \begin{bmatrix} 0 & 4 & 12 \\ 1 & 0 & 10 \\ 4 & -2 & 6 \end{bmatrix}$$

Find |A|.

$$\frac{1}{4}R_1 \quad \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 10 \\ 4 & -2 & 6 \end{bmatrix} \quad \xrightarrow{2R_1 + R_3 \to R_3} \quad \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 10 \\ 4 & 0 & 12 \end{bmatrix} \qquad |C| = |B| = \frac{1}{4}|A|$$

$$|B| = \frac{1}{4}|A|$$

$$-4R_2 + R_3 \to R_3$$

$$\begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 3 \\ 0 & 0 & -28 \end{bmatrix} \quad \xleftarrow{R_1 \leftrightarrow R_2} \quad \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 10 \\ 0 & 0 & -28 \end{bmatrix}$$

$$|E| = -|D| = -\frac{1}{4}|A| \qquad |D| = |C| = \frac{1}{4}|A|$$

$$\therefore \ |E| = (1)(1)(-28) = -28$$

$$|A| = -4|E| = -4 \times -28 = 112$$

Q.

$$\begin{matrix} 2 & 4 & 6 & 10 \\ -2 & 5 & 10 & 13 \\ -4 & -8 & 10 & 10 \\ 16 & 32 & 48 & 100 \end{matrix} \quad \xrightarrow{1/2 R_1} \quad \begin{matrix} 1 & 2 & 3 & 5 \\ -2 & 5 & 10 & 13 \\ -4 & -8 & 10 & 10 \\ 16 & 32 & 48 & 100 \end{matrix}$$

$$2R_1 + R_2 \to R_2$$
$$4R_1 + R_3 \to R_3$$
$$-16R_1 + R_4 \to R_4$$

$$\begin{matrix} 1 & 2 & 3 & 5 \\ 0 & 9 & - & - \\ 0 & 0 & 22 & - \\ 0 & 0 & 0 & 20 \end{matrix}$$

upper triangle

$$|C| = |B| = |A| \times \frac{1}{2}$$

$$(1)(9)(22)(20) = \det|C|$$

$$|A| = 2 \times 3960$$

$$= 7920$$

Big Result : A and B are $n \times n$ matrices

① $|AB| = |A||B| \Rightarrow$ In particular $|\underbrace{A^m}_{\substack{A \times A \times A \times \cdots \times A \\ m \text{ times}}}| = [|A|]^m$

② $|\alpha A| = \alpha^n |A|$

③ $|A^T| = |A|$ **transpose**

$$A^T = \begin{bmatrix} 1^{st} \text{ column of } A \\ 2^{nd} \text{ column of } A \\ \vdots \\ m^{th} \text{ column of } A \end{bmatrix}$$

$$\underset{2 \times 3}{A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}} \Rightarrow \underset{3 \times 2}{A^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 4 \end{bmatrix}}$$

④ $|AB| = |BA|$

eventhough, in general AB need not equal to BA. Their determinants are numbers and they are equal.

⑤ In general $|A \pm B|$ need not equal $|A| \pm |B|$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \qquad A+B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$|A+B| = 3$

$|A| = 0 \qquad |B| = 0$

$|A| + |B| = 0 \qquad \therefore |A| + |B| \neq |A+B|$

Small result but useful :

* I_n = **identity** matrix $n \times m$

$$I_3 = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad I_4 = \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

whenever multiplication is <u>legal</u> then $I_nB = B$ and $BI_n = B$

$$\underbrace{\underset{3\times5}{A}\ \underset{5\times5}{I_5}}_{\text{legal}} = A \qquad \underbrace{\underset{3\times3}{I_3}\ \underset{3\times5}{A}}_{\text{legal}} = A$$

* A, $n \times n$

Imagine α is an eigen value of A.

$\exists$ non zero point $(a_1, \dots, a_n) \in R^n$

exist

such that

$$A \begin{bmatrix} a_1 \\ \vdots \\ \vdots \\ a_n \end{bmatrix} = \alpha \begin{bmatrix} a_1 \\ \vdots \\ \vdots \\ a_n \end{bmatrix}$$

$$A = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \alpha I_n \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$\alpha I_n \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} - A \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\left[\alpha I_n - A\right] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\therefore$ we conclude $|\alpha I_n - A| = 0$

Q. $A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}_{2\times 2}$

Find all eigen values of A.

A: set $|\alpha I_2 - A| = 0$. Solve for α.

$$\text{Char}(A) = |\alpha I_2 - A| = \left| \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \right|$$

$$= \begin{vmatrix} \alpha - 1 & -2 \\ 0 & \alpha - 4 \end{vmatrix} = 0$$

$$= (\alpha - 1)(\alpha - 4) = 0$$

$$\Rightarrow \alpha = 1, \alpha = 4$$

2 March 2022

Wednesday

Recall : α is an eigen value of A.

we know 2 things

① $|\alpha I_n - A| = 0$

② $\exists$ a non-zero point Q in R^n, $(a_1, \dots, a_n)$ such that

$$\left[\alpha I_n - A\right]\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Q.

$$A = \begin{matrix} 2 & 1 & 3 \\ -2 & 4 & 5 \\ 2 & -4 & -5 \end{matrix}$$

Find all eigen values of A.

For each eigenvalue α, Find E_α (eigen space).

A: For (1) set $\underbrace{|\alpha I_3 - A|}_{\substack{\shortparallel \\ \text{Char}(A)}} = 0$. Find α

$$\alpha I_3 - A = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} - \begin{bmatrix} 2 & 1 & 3 \\ -2 & 4 & 5 \\ 2 & -4 & -5 \end{bmatrix} = \begin{bmatrix} \alpha-2 & -1 & -3 \\ 2 & \alpha-4 & -5 \\ -2 & 4 & \alpha+5 \end{bmatrix}$$

$$\xrightarrow{R_2 + R_3 \to R_3} \begin{bmatrix} \alpha-2 & -1 & -3 \\ 2 & \alpha-4 & -5 \\ 0 & \alpha & \alpha \end{bmatrix}$$

now

$$\begin{vmatrix} \alpha-2 & -1 & -3 \\ 2 & \alpha-4 & -5 \\ 0 & \alpha & \alpha \end{vmatrix} = 0 \quad \text{solve for } \alpha.$$

↖

you need this to find E_α.

we chose 3rd row to find the determinant :

$$(-1)^{3+2}\,\alpha\begin{vmatrix}\alpha-2 & -3\\ -2 & -5\end{vmatrix} + (-1)\,\alpha^{3+3}\begin{vmatrix}\alpha-2 & -1\\ 2 & \alpha-4\end{vmatrix} = 0$$

$$-\alpha\left[-5\alpha+10+6\right] + \alpha\left[\alpha^2-6\alpha+8+2\right] = 0$$

$$5\alpha^2-16\alpha+\alpha^3-6\alpha^2+10\alpha=0$$

$$\alpha^3-\alpha^2-6\alpha=0$$

$$\alpha\left(\alpha^2-\alpha-6\right)=0$$

$$\alpha\left(\alpha-3\right)\left(\alpha+2\right)=0$$

$\alpha=0$ $\alpha=3$ $\alpha=-2$ 3 eigen values

$\alpha=0$ → E_0 : the solution of the homogenous system

$$\left[\begin{array}{c|c} 0I_3 - A & \begin{matrix}C\\0\end{matrix}\end{array}\right]$$

Augmented Matrix

$$\left[\begin{array}{ccc|c}-2 & -1 & -3 & 0\\ 2 & -4 & -5 & 0\\ 0 & 0 & 0 & 0\end{array}\right] \underset{\sim}{-\tfrac{1}{2}R_1} \left[\begin{array}{ccc|c}+1 & +1/2 & +3/2 & 0\\ 2 & -4 & -5 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

$-2R_1+R_2 \to R_2$

$$\left[\begin{array}{ccc|c}1 & 1/2 & 3/2 & 0\\ 0 & 1 & 8/5 & 0\\ 0 & 0 & 0 & 0\end{array}\right] \underset{\sim}{-1/5\,R_2} \left[\begin{array}{ccc|c}1 & 1/2 & 3/2 & 0\\ 0 & -5 & -8 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

another name for

$\dim(O) = IN(O)$

independent number

$-\frac{1}{2}R_2 + R_1 \rightarrow R_1$

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 7/10 & 0 \\ 0 & \boxed{1} & 8/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Read!

$$a_1 + 7/10\, a_3 = 0 \longrightarrow a_1 = \frac{-7}{10}a_3$$

$$a_2 + 8/5\, a_3 = 0 \longrightarrow a_2 = -8/5\, a_3$$

$$0 = 0 \longrightarrow 0 = 0 \qquad a_3 \in R$$

$$E_0 = \left\{ \left(\frac{-7}{10}a_3 , \frac{-8}{5}a_3 , a_3 \right) \middle| \; a_3 \in R \right\}$$

$$= \left\{ a_3 \left(\frac{-7}{10} , \frac{-8}{5} , a_3 \right) \middle| \; a_3 \in R \right\}$$

$$= \text{span} \left\{ \left(\frac{-7}{10} , \frac{-8}{5} , 1 \right) \right\}$$

E_α = set of <u>all</u> points in R^n, say $Q = (a_1, \ldots, a_n)$

where

$$A \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \alpha \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

E_3 = Augmentd matrix

$$\begin{bmatrix} \alpha - 2 & -1 & -3 \\ 2 & \alpha - 4 & -5 \\ 0 & \alpha & \alpha \end{bmatrix} \text{ becomes } \begin{bmatrix} 1 & -1 & -3 \\ 2 & -1 & -5 \\ 0 & 3 & 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} \textcircled{1} & -1 & -3 & 0 \\ 2 & -1 & -5 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right]$$

$-2R_1 + R_2 \rightarrow R_2$

$$\left[\begin{array}{ccc|c} \textcircled{1} & -1 & -3 & 0 \\ 0 & \textcircled{1} & 1 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right]$$

$R_2 + R_1 \rightarrow R_1$ $-3R_2 + R_3 \rightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

READ $a_1 = 2a_3$

$a_2 = -a_3$

$a_3 \in R$

$$E_3 = \left\{ (2a_3, -a_3, a_3) \mid a_3 \in R \right\}$$

$$= \text{span}\left\{ (2,-1,1) \right\}$$

Midterm 2 Material Starts here !

7 March 2022

Monday

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \\ -1 & 0 & -2 \end{bmatrix}$$

Def : **Null(A)** is the solution set to the homogeneous system

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \left[\begin{array}{c|c} A & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \end{array}\right]$$

Nullity (A) = dim (Null(A))

Def : (only for $n \times n$ matrix)

A, $n \times n$, we say A is non-singular (Invertible) if $\exists$ a matrix, denoted by A^{-1} such that $AA^{-1} = I_n$.

careful : $A^{-1} \neq \frac{1}{A}$

Know : A. $n \times n$, is invertible <u>iff</u> $|A| \neq 0$.

Q) $A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$

Find A^{-1} if possible.

A)

$$\begin{tikzcd}
\left[A \mid I_n \right] \arrow[r, "\text{row operation}"] \arrow[dr, "\text{row operation}"'] & \left[\text{If } I_n \mid A^{-1} \right] \\
& \left[\text{If not } I_n \mid A^{-1} \text{ does not exist} \right]
\end{tikzcd}$$

$\therefore$ A is non-invertible.

A is singular.

$$\underset{A}{\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]} \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 0 \end{array}\right] \xrightarrow{-2R_1 + R_2 \to R_2} \overset{B}{\left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]}$$

I_2 : [illegible]   : [illegible]

No way that we get I_2 on the left side :(

$\therefore$ A is non-invertible.

$BA = \begin{matrix} 1 & 2 \\ 0 & 0 \end{matrix}$

Believe it or not
but the original A
multiplied with B in this order
will give you that.

Result : $[A|B] \xrightarrow[\text{equivalent}]{} [D|E]$ then $EA = D$

Q. $A = \begin{matrix} 1 & 2 & -2 \\ -1 & -1 & 2 \\ 2 & 4 & -3 \end{matrix}$ Find A^{-1} if possible.

A.

$$\overset{A}{\left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 & 1 & 0 \\ 2 & 4 & -3 & 0 & 0 & 1 \end{array}\right]} \xrightarrow[-2R_1 + R_3 \to R_3]{R_1 + R_2 \to R_2} \overset{B}{\left[\begin{array}{ccc|ccc} \textcircled{1} & 2 & -2 & 1 & 0 & 0 \\ 0 & \textcircled{1} & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]}$$

$$\xrightarrow{-2R_2 + R_1 \to R_1} \overset{C}{\left[\begin{array}{ccc|ccc} \textcircled{1} & 0 & -2 & -1 & -2 & 0 \\ 0 & \textcircled{1} & 0 & 1 & 1 & 0 \\ 0 & 0 & \textcircled{1} & -2 & 0 & 1 \end{array}\right]} \xrightarrow{2R_3 + R_1 \to R_1} \overset{I_3}{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -2 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]}$$

I_3 A^{-1}

$$\boxed{AA^{-1} = A^{-1}A = I_n}$$

- $|A| = |B| = |C| = |I_3| = 1$ (for the prev question)

Know : $AA^{-1} = I_n$

$|AA^{-1}| = |I_n| = 1$

$|A||A^{-1}| = 1$, $|A| \neq 0$

$\therefore \; |A^{-1}| = \dfrac{1}{|A|}$

Know : $(A^{-1})^{-1} = A$

Q. $A^{-1} = \begin{matrix} 1 & 2 & 3 \\ 2 & 10 & 1 \\ -2 & 6 & 10 \end{matrix}$

Solve the system $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{matrix} 2 \\ 4 \\ 7 \end{matrix}$

Solution : $\xrightarrow{\text{augmented}} \left[\begin{array}{c|c} A & \begin{matrix} 2 \\ 4 \\ 7 \end{matrix} \end{array}\right]$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 10 & 1 \\ -2 & 6 & 10 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 10 \\ 6 \end{bmatrix} + 7 \begin{bmatrix} 3 \\ 1 \\ 10 \end{bmatrix} = \begin{bmatrix} 31 \\ 51 \\ 90 \end{bmatrix}$$

Solution set = $\{(31, 51, 90)\}$ but we expected a unique solution aslan because the $|A| \neq 0$.

Know : A, B are invertible n×n Then :

$(AB)^{-1} = B^{-1}A^{-1}$ is equal to the inverse of the 2nd × inv of 1st

Also : if C is n×m and D is m×n

then $(CD)^T = D^T C^T$

Special case : For 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad |A| \neq 0$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Q. $A = \begin{matrix} 3 & 7 \\ 2 & 1 \end{matrix}$

$|A| = (3)(1) - (7)(2) = 3 - 14 = -11$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{-11}\begin{bmatrix} 1 & -7 \\ -2 & 3 \end{bmatrix}$$

Know : If A n×m and B is n×m.

$(A \pm B)^T = A^T \pm B^T$

$(A^T)^T = A$

Q. A, 2x2

$$\left(A\begin{bmatrix}1&2\\-1&-1\end{bmatrix}+\begin{bmatrix}1&2\\4&5\end{bmatrix}\right)^T=\begin{bmatrix}1&1\\-1&1\end{bmatrix}$$

Find A.

① Transpose on both sides

$$A\begin{bmatrix}1&2\\-1&-1\end{bmatrix}+\begin{bmatrix}1&2\\4&5\end{bmatrix}=\begin{bmatrix}1&-1\\1&1\end{bmatrix}$$

$$A\begin{bmatrix}1&2\\-1&-1\end{bmatrix}=\begin{bmatrix}1&-1\\1&1\end{bmatrix}-\begin{bmatrix}1&2\\4&5\end{bmatrix}$$

$$A\underbrace{\begin{bmatrix}1&2\\-1&-1\end{bmatrix}}_{B}=\begin{bmatrix}0&-3\\-3&-4\end{bmatrix}$$

Find B^{-1}

$$B^{-1}=\frac{1}{|B|}\begin{bmatrix}-1&-2\\1&1\end{bmatrix}=\frac{1}{1}\begin{bmatrix}-1&-2\\1&1\end{bmatrix}=\begin{bmatrix}-1&-2\\1&1\end{bmatrix}$$

now multiply B^{-1} to both sides : (from the right)

$$A\underbrace{\underset{B}{\begin{bmatrix}1&2\\-1&-1\end{bmatrix}}\underset{B^{-1}}{\begin{bmatrix}-1&-2\\1&1\end{bmatrix}}}_{I_2}=\begin{bmatrix}0&-3\\-3&-4\end{bmatrix}\begin{bmatrix}-1&-2\\1&1\end{bmatrix}$$

$$A=\begin{bmatrix}0&-3\\-3&-4\end{bmatrix}\begin{bmatrix}-1&-2\\1&1\end{bmatrix}=\begin{bmatrix}-3&-3\\-1&2\end{bmatrix}$$

14 March 2022

$|A| \neq 0$ then A^{-1} exists.

$$|A^{-1}| = \frac{1}{|A|}$$

A, n×n, A^{-1} exists.

Know : $(A^T)^{-1} = (A^{-1})^T$

Know : A, n×n, assume A has atleast two identical rows or columns.

Then $|A| = 0$

$$A = \begin{bmatrix} \text{row } i \\ \text{row } k \\ \vdots \end{bmatrix} \xrightarrow{-R_i + R_k \to R_k} \begin{bmatrix} \text{row } i \\ 0 \quad 0 \ \dots \ 0 \\ \vdots \end{bmatrix} = B$$

since $|A| = |B|$

if we choose Row k to calculate the determinant

$|B| = 0$ which means $|A| = 0$.

* If i^{th} column and k^{th} column of A are identical, then i^{th} row and k^{th} row of A^T are identical.

Since $|A| = |A^T|$ and $|A^T| = 0$, we have $|A| = 0$.

System of linear equation, n×n,

$$= \left[A \,\middle|\, \text{constants} \right]$$ has unique solution <u>iff</u> $|A| \neq 0$

<u>iff</u> A^{-1} exists.

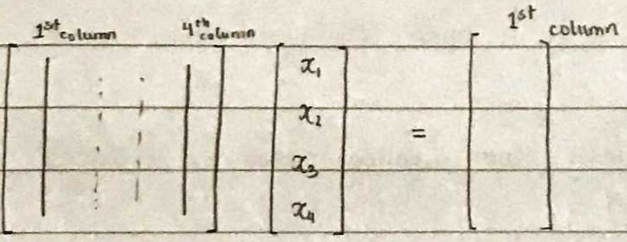
when A, $n \times n$, $\left[A \mid \text{constant} \right]$

$|A| = 0$ then $\longrightarrow$ consistent with infinitely many solutions

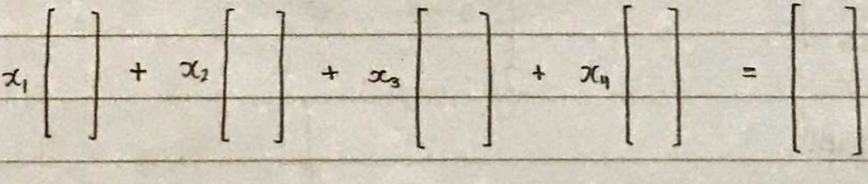
$\searrow$ has no solution, inconsistent

Midterm doubt :

A, 4×4



1st column and 4th column are identical.



Solutions :

① (1,0,0,0)

② (0,0,0,1)

③ (0.5, 0, 0, 0.5)

④ (0.2, 0, 0, 0.8)

} infinitely many as long as x_1 & x_4 will add up to 1.

Result : Assume $Q_1, \dots, Q_n$ are points in R^n

Then $Q_1, Q_2, \dots, Q_n$ are independent <u>iff</u>

$$\left| \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{bmatrix} \right| \neq 0$$

A, 4x4

$C_A(\alpha) = \left| \alpha I_4 - A \right|$ **Characteristic polynomial** (char)

It should be clear A, nxn, $\Rightarrow$ $\deg(C_A(\alpha)) = n$

$C_A(\alpha) = (\alpha - 3)^2 (\alpha + 5)^2$

eigen value of A:

3 → repeated twice

-5 → repeated twice

$|A|$ = multiplication of the eigenvalues (with repetition)

$\therefore |A| = 3 \times 3 \times -5 \times -5 = 225$

Know : If 0 is not an eigenvalue, then $|A|$ can never be 0. This means A is invertible. System has a unique solution.

α is an eigen value of A, nxn, $|A| \neq 0$

$\exists$ non-zero point $(a_1, \dots, a_n)$

such that
$$A \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \alpha \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

multiply A^{-1} from left :

$$A^{-1} A \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \alpha A^{-1} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$I_n \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \alpha A^{-1} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$\frac{1}{\alpha} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = A^{-1} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$\Rightarrow$ $\frac{1}{\alpha}$ is an eigenvalue of A^{-1}.

A, 3 x 3

$C_A(\alpha) = (\alpha-2)^2(\alpha-4)$

(i) Find $|A|$

$\alpha = 2$ repeated twice

$\alpha = 4$ repeated once

$|A| = 2 \times 2 \times 4 = 16$

(ii) Find $|A^{-1}|$

$$|A^{-1}| = \frac{1}{|A|} = \frac{1}{16}$$

(ii.5) Find eigen values of A^{-1}

- $1/2$ (repeated twice)
- $1/4$ (repeated once)

(iii) Given $E_2 = \text{span}\{(1,0,2)\}$ and $E_4 = \text{span}\{(0,2,3)\}$.

(a) Find $E_{1/2} = \text{span}\{(1,0,2)\}$

(b) Find $E_{1/4} = \text{span}\{(0,2,3)\}$

ⓘⓥ $A^{-1}\begin{bmatrix}0\\6\\9\end{bmatrix} = \begin{bmatrix}0\\6/4\\9/4\end{bmatrix} = \frac{1}{4}\begin{bmatrix}0\\6\\9\end{bmatrix}$

ⓥ $A\begin{bmatrix}0\\6\\9\end{bmatrix} = 4\begin{bmatrix}0\\6\\9\end{bmatrix} = \begin{bmatrix}0\\24\\36\end{bmatrix}$

Trace (A) , A must be nxn.

$$A = \begin{matrix}1 & 2 & 3\\ -4 & 1 & 6\\ 2 & 2 & 10\end{matrix}$$

Trace (A) = add the numbers on the main diagonal.

Trace (A) is the sum of numbers in the main diagonal.

Result : Trace (A) is the sum of eigen values with repetition.

Char (A) = $C_A(\alpha) = (\alpha+1)^2(\alpha-3)^3(\alpha+4)$, note A is 6x6.

Find

(i) All eigen values of A (by staring)

$\alpha = -1$ repeated twice

$\alpha = 3$ repeated three times

$\alpha = -4$ repeated once

(ii) Find Trace (A)

$-1 + -1 + 3 + 3 + 3 + -4 = 3$

$\therefore$ Trace (A) $= 3$

16 March 2022

(iii) $|A| = (-1)(-1)(3)(3)(3)(-4) = -108$

since $|A| \neq 0$, A is invertible

(iv) Find the eigen values of the inverse A^{-1}.

$\alpha = \frac{-1}{1} = -1$ repeated twice

$\alpha = \frac{1}{3}$ repeated 3 times

$\alpha = \frac{-1}{4}$ repeated once

(v) $\text{Trace}(A^{-1}) = -1 + -1 + 1/3 + 1/3 + 1/3 + -1/4 = -5/4$

(vi) $E_{1/3}$ (with respect to A^{-1}) $=$ E_3 (with respect to A)

$Q \in E_3 , \quad Q \neq (0, 0, \dots, 0)$

$AQ_T = 3Q^T$

$A^{-1}Q_T = 1/3\, Q^T$

(vii) Find the eigen values of A^2.

Assume α is an eigen value of A (A is nxn)

$\exists\, Q \in E_\alpha \quad (Q \neq (0,0,\cdots,0)$ and $Q = (a_1, a_2, \dots, a_n))$

$$A(AQ^T = \alpha Q^T)$$

$$A^2Q^T = \alpha \underline{AQ^T}$$

$$= \alpha\alpha Q^T$$

$$= \alpha^2 Q^T$$

we are using the same point

Know: If α is an eigen value of A, then α^K is an eigen value of A^K. $E_\alpha{}^K$ with respect to A^K.

Q: A, 3×3

$C_A(\alpha) = |\alpha I_3 - A| = (\alpha-4)^2(\alpha+4)$ degree = 3

now: $B = 2A^2 + 5A^{-1} - 4I_3$

B, 3×3

(i) Find $|B|$

(ii) Find Trace (B)

Know :

- α is an eigen value of A
- α^{-1} is an eigen value of A^{-1}
- α^k is an eigen value of A^k
- C is a constant, $C\alpha$ is an eigen value of CA.

(i) $|B| \neq |2A^2| + |5A^{-1}| - |4I_3|$

① What are the eigen values of A?

$\alpha = 4$ (twice) and $\alpha = -4$

$Q \in E_4$, $Q \neq (0,0,0)$

$AQ^T = 4Q^T$

$BQ^T = \left[2A^2 + 5A^{-1} - 4I_3\right]Q^T$

$= 2A^2Q^T + 5Q^TA^{-1} - 4Q^TI_3$

$= 2(4)^2Q^T + 5(1/4)Q^T - 4Q^T$

$= 32Q^T + 5/4\,Q^T - 4Q^T$

$= \underbrace{(32 + 5/4 - 4)}_{\text{eigen value of B}}Q^T$

For $\alpha = 4$ $2(4)^2 + 5(1/4) - 4 = 29.25$ this is an eigen value of B ! ← repeated twice

For $\alpha = -4$ $2(-4)^2 + 5(-1/4) - 4 = 26.75$ this is an eigen value of B! ← repeated once

$|B| = (29.25)(29.25)(26.75) = 22886.30$

(ii) Trace of B = 85.25

Def. A, n×n

We say A is **diagnolixable** if ∃ an invertible matrix Q and a diagonal matrix D such that :

$$Q^{-1}AQ = D \quad \text{which is also} \quad A = QDQ^{-1}$$

this means Q^{-1} exists & $|Q| \neq 0$ (when we say invertible)

$$A = \begin{matrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{matrix} \longrightarrow A^{-1} = \begin{matrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{matrix}$$

$$A^4 = \begin{matrix} 1^4 & 0 & 0 \\ 0 & 3^4 & 0 \\ 0 & 0 & 2^4 \end{matrix}$$

$$A^n = \begin{matrix} 1^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 2^n \end{matrix}$$

Calculate A^2

$$A^2 = (QDQ^{-1})(QDQ^{-1})$$

$$A^2 = QD^2Q^{-1}$$

so $A^3 = QD^3Q^{-1}$

∴ $A^n = QD^nQ^{-1}$

CRamer can be used only on/ when solving system of linear equations $n \times n$.

where the determinant of the coeffeient matrix $\neq 0$

$$|\text{co.eff matrix}| \neq 0$$

Example :

$$x_1 + 2x_2 - x_3 = 0$$

$$-x_1 + 5x_2 + 2x_3 = 2$$

$$2x_1 + 4x_2 + 10x_3 = 10$$

$$|\text{coeff matrix}| = |A| = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{vmatrix} 1 & 2 & -1 \\ -1 & 5 & 2 \\ 2 & 4 & 10 \end{vmatrix} \end{array}$$

solve for x_2

$$x_2 = \frac{|[x_1, \overset{\text{constant}}{c}, x_3]|}{|A|} = \frac{\begin{vmatrix} 1 & 0 & -1 \\ -1 & 2 & 2 \\ 2 & 10 & 10 \end{vmatrix}}{|A|}$$

Solve for x_3

$$x_3 = \frac{|[x_1, x_2, c]|}{|A|} = \frac{\begin{vmatrix} 1 & 2 & 0 \\ -1 & 5 & 2 \\ 2 & 4 & 10 \end{vmatrix}}{|A|}$$

21 March 2022

Adjoint method

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$$

$$\text{adjoint of } A = C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ \vdots & & & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}_{n\times n}$$

$$(i,k) \textbf{ entry} \text{ of } C = c_{ik} = \frac{(-1)^{i+k}\,|\text{After deleting } k^{th} \text{ row and } i^{th} \text{ column of } A|}{|A|}$$

Know :

$$A \times \text{adjoint}(A) = \overbrace{\det(A) \times I_n}^{\text{diagonal matrix}}$$

Assume $|A| \neq 0 \Rightarrow A^{-1}$ exists

$$A \underbrace{\left[\frac{1}{|A|}\,\text{adjoint}(A)\right]}_{A^{-1}} = I_n$$

Q:

$$A = \begin{matrix} 2 & 3 & 4 \\ -2 & 6 & 1 \\ -2 & -3 & 5 \end{matrix}$$

Find the (2,3) — entry of A^{-1}.

A:

$$(2,3) \text{ entry of } A^{-1} = \frac{(-1)^{2+3}\,|A \text{ after deleting the } 3^{rd} \text{ row } \& \text{ second column}|}{|A|}$$

Find $|A|$

$$\begin{array}{ccc} 2 & 3 & 4 \\ -2 & 6 & 1 \\ -2 & -3 & 5 \end{array} \quad \begin{array}{c} \\ R_1 + R_2 \rightarrow R_2 \\ R_1 + R_3 \rightarrow R_3 \end{array} \quad \begin{array}{ccc} 2 & 3 & 4 \\ 0 & 9 & 5 \\ 0 & 0 & 9 \end{array} \qquad |A| = 2 \times 9 \times 9 = 162$$

Find $|A$ after del 3rd R & 2nd C$|$

$$\begin{vmatrix} 2 & 4 \\ -2 & 1 \end{vmatrix} = (2)(1) - (4)(-2) = 10$$

(2,3) entry of $A^{-1} = \dfrac{(-1)(10)}{162} = \dfrac{-5}{81}$

Result : Assume $C_A(\alpha) = (\alpha - a_1)^{n_1} (\alpha - a_2)^{n_2} \dots (\alpha - a_k)^{n_k}$.

$0 < \dim(E_{a_i}) \leq n_i$

for every / for all

Know : A, $n \times n$ is diagnolizable <u>iff</u> $\forall$ eigenvalue a_i, $\dim(E_{a_i}) = n_i$

Q: A, 3×3, $C_A(\alpha) = (\alpha - 2)^2 (\alpha + 4)$

$E_{+2} = \text{span}\{(1,3,2)\} \longrightarrow \dim(E_2) = 1$

$E_{-4} = \text{span}\{(0,1,5)\} \longrightarrow \dim(E_{-4}) = 1$

Is A diagnolizable? no, because the dimension of E_2 is $\neq n_2 = 2$

Q: A, 5×5, $C_A(\alpha) = \overset{n_3}{(\alpha-3)^{\textcircled{2}}}\overset{n_2}{(\alpha+5)^{\textcircled{3}}}\overset{n_1}{(\alpha-6)^{\circ}}$

$E_3 = \text{Span}\{(1,1,1,1,1), (-1,1,1,1,1)\}$ $\dim(E_3) = 2 = n_3$

$E_{-5} = \text{Span}\{(-1,-1,1,1,1), (-1,-1,-1,1,1)\}$ $\dim(E_{-5}) = 2 = n_2$

$E_6 = \text{Span}\{(0,0,0,0,1)\}$ $\dim(E_6) = 1 = n_1$

$\therefore$ A is diagolizable.

Find a diagonal matrix $\underline{D}$ and an invertible matrix Q such that $Q^{-1}AQ = D$

A:

$$D = \begin{bmatrix} \text{eigenvalues of } A \text{ with repetition} \end{bmatrix} = \underset{5\times5}{\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}}$$

this can change. I can put the eigenvalues in any order on the diagonal.

$$Q = \underset{5\times5}{\begin{bmatrix} 1 & -1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}}$$

corresponds to D!

this can change, it depends on D.

wow!

$$\underset{5\times5}{\begin{bmatrix} & A & \end{bmatrix}}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 6\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

One big question from scratch:

$$A = \begin{matrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 1 & 3 \end{matrix}$$

If A is diagnolizable. Find a diagonal matrix D & an invertible matrix Q. such that $Q^{-1}AQ = D$

$$C_A(\alpha) = |\alpha I_3 - A| = \begin{vmatrix} \alpha-2 & 0 & 0 \\ 0 & \alpha-2 & 0 \\ -1 & 1 & \alpha-3 \end{vmatrix} = (\alpha-2)^2(\alpha-3)$$

lower triangular

set $(\alpha-2)^2(\alpha-3) = 0$

$\alpha = 2$ (twice)

$\alpha = 3$ (once)

E_2 = solution set of the homogenous system $= [2I_3 - A|0] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$

READ! $x_1 = x_2 + x_3$ $\quad x_2, x_3 \in R$

Solution set $= \{(x_2+x_3, x_2, x_3) \mid x_2, x_3 \in R\}$

$= \{x_2(1,1,0), x_3(1,0,1)\}$

$= \text{span}\{(1,1,0),(1,0,1)\}$

$= E_2$

$\dim(E_2) = 2$

$$E_3 = \boxed{3I_3 - A|0} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \xrightarrow{-R_1+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$R_2 + R_3 \to R_3$

$$\xrightarrow{\quad} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

READ! $x_1 = 0$

$x_2 = 0$

$x_3 \in R$

$$E_3 = \{(0,0,x_3) \mid x_3 \in R\}$$
$$= \{x_3 \ (0,0,1)\}$$
$$= \text{Span}\{(0,0,1)\} \qquad \dim(E_3) = 1$$

$\therefore$ A is diagnolizable

now

$$D = \begin{matrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{matrix}$$

$$Q = \begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{matrix}$$

Let $T: R^n \to R^m$ be a linear transformation.

T is invertible $\underline{\underline{\text{iff}}}$ $n = m$ and T is an isomorphism $\to$ one-to-one, onto

$$\dim(\text{domain}) = \underbrace{\dim(\text{Range})}_{m} + \dim(\text{zero})$$

$$n = m + 0$$

4 April 2022

After Break

Q. $T: R^3 \to R^3$

$T(a_1, a_2, a_3) = (a_1, a_2, a_3)$ } identity linear transformation

so if

$T(1,0,3) = (1,0,3)$

SMP : standard matrix presentation

SMF for T is

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

composition $f_1 \circ f_2$

if $f_1 = x^2$ and $f_2 = x+3$

$(f_1 \circ f_2)(x) = (x+3)^2$

$\overset{\downarrow}{f_1}(f_2(x))$

Note : $f_1 \circ f_2$ need not $= f_2 \circ f_1$

f is invertible

$(f \circ f^{-1})(x) = x$

Q) $T : R^2 \longrightarrow R^2$

$T(a_1, a_2) = (a_1 + 2a_2, \ -a_1 + a_2)$ another way to write T $T(a_1, a_2) = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

1) Is T invertible ?

Ans: ① Find the standard matrix representation of T.

$$M = \begin{array}{c} \begin{array}{cc} a_1 & a_2 \end{array} \\ \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \end{array}$$

② T is invertible iff M^{-1} exists.

$$|M| = (1)(1) - (2)(-1) = 3 \neq 0$$

M is invertible.

$\therefore$ T is invertible

2) If T is invertible, Find T^{-1}.

$T^{-1} : R^2 \rightarrow R^2$

$$T^{-1}(a_1, a_2) = M^{-1} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$M^{-1} = \frac{1}{3} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -2/3 \\ 1/3 & 1/3 \end{bmatrix}$$

$$\therefore T^{-1}(a_1, a_2) = \begin{bmatrix} 1/3 & -2/3 \\ 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1/3 a_1 - 2/3 a_2 \\ 1/3 a_1 + 1/3 a_2 \end{bmatrix}$$

$$\therefore T^{-1}(a_1, a_2) = (1/3 a_1 - 2/3 a_2, \ 1/3 a_1 + 1/3 a_2)$$

3) $(T \circ T^{-1})(a_1, a_2) = (a_1, a_2)$

$$T \circ T^{-1} = I$$

Fact : $T_1 : R^n \longrightarrow R^m$

$T_2 : R^k \longrightarrow R^n$

$M_1 \longrightarrow$ standard matrix for T_1

↳ dim(codomain) x dim(domain)

↳ $m \times n$

$M_2 \longrightarrow$ std matrix for T_2

↳ $n \times k$

Q. Find the standard n matrix presentation of $T_1 \circ T_2$.

A : $M = \underset{m \times n}{M_1} \underset{n \times k}{M_2}$

Fact: $T_1 : R_2 \longrightarrow R_2$

$T_1(a_1, a_2) = (a_1 + a_2, -a_1)$

$T_2 : R_2 \rightarrow R_2$

$T_2(a_1, a_2) = (3a_1 - a_2, a_1 + a_2)$

Find $T_1 \circ T_2 : R_2 \rightarrow R_2$

A: $T_1 \circ T_2 = M_1 M_2 \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$A = \begin{matrix} 1 & 2 \\ 0 & 1 \end{matrix}$

$f(x) = 3x^2 - 6x + 7$

Find f(A).

$$f(A) = \underset{\substack{2\times2\\ \text{Matrix}}}{3A^2} - \underset{\substack{2\times2\\ \text{Matrix}}}{6A} + 7 \quad = \text{undefined}$$

to fix this :

$$f(A) = 3A^2 - 6A + 7\mathbf{I_2}$$

Note : A, $n\times m$ where $n \neq m$

$$A^3 = AAA = \text{undefined}$$

$\downarrow \searrow$

$\underbrace{n\times m \quad n\times m}$

A^K is undefined

Q: A , $n\times n$,

A is invertible

$$A^{-5} = (A^{-1})^5$$

Note : $A^{-n} = (A^{-1})^n$

Note : $A^{1/2}$ = undefined

$A = \begin{matrix} 0 & 2 \\ 0 & 1 \end{matrix}$

$C_A(\alpha) = |\alpha I_2 - A| = \begin{vmatrix} \alpha & -2 \\ 0 & \alpha - 1 \end{vmatrix} = \alpha(\alpha-1)$

$C_A(A)$ = we will tak $\alpha(\alpha-1)$ and substitute α with A and I_2 with 1.

$= A(A - I_2)$

$= \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore \; A(A - I_2) = 0$

Caley's Theorem

A, $n \times n$

$$C_A(A) = \underset{n \times n}{\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}}$$

here $C_A(\alpha) = \alpha^n + \alpha_{n-1}\alpha^{n-1} + \cdots + \alpha_1\alpha + a_0$

Q)

$$A = \begin{array}{ccc} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{array}$$

$$C_A(\alpha) = \left|\alpha I_3 - A\right| = \begin{vmatrix} \alpha-1 & 0 & -2 \\ 0 & \alpha-2 & -3 \\ 0 & 0 & \alpha-4 \end{vmatrix} = (\alpha-1)(\alpha-2)(\alpha-4)$$

$$C_A(A) = (A - I_3)(A - 2I_3)(A - 4I_3)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Q)

$$A = \begin{array}{ccc} 1 & 2 & 4 \\ -1 & 2 & 5 \\ -1 & -2 & 6 \end{array}$$

Find (1,3) entry of A^{-1}. $= \dfrac{(-1)^{1+3} \begin{vmatrix} \text{delete } 3^{rd} \text{ row } \& \\ 1^{st} \text{ column} \end{vmatrix}}{|A|}$

this is to find

$$A^{-1} = \begin{array}{ccc} \times & \times & \overset{\text{(1,3) entry}}{\boxed{\times}} \\ \times & \times & \times \\ \times & \times & \times \end{array}$$

$$= \frac{(-1)^4 \begin{vmatrix} 2 & 4 \\ 2 & 5 \end{vmatrix}}{|A|}$$

matrix B

$$|A| = \text{prof did row operation} = \begin{matrix} 1 & 2 & 4 \\ 0 & 4 & 9 \\ 0 & 0 & 10 \end{matrix}$$

$R_1 + R_2 \to R_2$

$R_1 + R_3 \to R_3$

$$|A| = |B| = 40$$

now

$$= \frac{(-1)^4 \begin{vmatrix} 2 & 4 \\ 2 & 5 \end{vmatrix}}{40} = \frac{1}{20}$$

11 April 2022

$R^{n\times m}$: the set of all matrices

$$R^{2\times 3} = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \middle| \; a_1, a_2, a_3, \dots a_6 \in R \right\}$$

$$= \left\{ a_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \dots + a_6 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| \; a_1, a_2, \dots a_6 \in R \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \dots \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

$$R^{2\times 2} = \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

ex: Write $\begin{bmatrix} 5 & 7 \\ 1 & 2 \end{bmatrix}$ as a span of $R^{2\times 2}$

$$5 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 7 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Note :

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & c_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Note : $\dim(R^{n\times m}) = n \times m$

Q) $D = \left[\begin{bmatrix} a+b & -1 \\ 0 & a \end{bmatrix} \middle| \; a,b \in R \right]$

D "lives" inside $R^{2\times2}$

Convince me that D is not a subspace of $R^{2\times2}$.

A: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ does not belong in D ($\notin D$)

Hence D is not a subspace.

another way

$$D = \left\{ a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \right\}$$

$\therefore D \neq \text{span}\{\text{finite \# of matrices}\}$

P_n = set of all polynomials with degree $<n$

$$P_3 = \{ a_2x^2 + a_1x + a_0 \mid a_1, a_2, a_0 \in R \} = \text{span}\{x^2, x, 1\}$$

Is $5 \in P_3$? yes

Is $2x + \sqrt{3} \in P_3$? yes

Is $6x^2 - \sqrt{2}x + \sqrt{11} \in P_3$? yes

Is $2x^3 + 1 \in P_3$? no degree n cannot be 3, n has to be <3.

$1c_0 + c_1 x + c_2 x^2 = 0$

if $c_0, c_1, c_2 = 0$ then independent

because

$1c_0 + c_1 x + c_2 x^2 = 0 + 0x + 0x^2$

then $c_0 = 0$, $c_1 = 0$, $c_2 = 0$

$\rightarrow \quad P_4 = \text{Span}\{1, x, x^2, x^3\}$

Note : $\dim(P_n) = n$

Convince me that $D = \{a_0 + a_2 x + a_2 x^2 \mid a_0, a_1, a_2 \in R\}$ is a subspace.

A: $\{a_0(1) + a_2[x + x^2]\}$

$= \text{Span}\{1, x + x^2\}$ it is a subspace.

Note :

$R^{n \times m} \underset{\text{isomorphic}}{\approx} R^{nm}$ they are the same as subspaces / vectorspaces

ex: $R^{2 \times 2} \approx R^4$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \longleftrightarrow (1, 2, 3, 4)$$

ex: $R^{3 \times 4} \longleftrightarrow R^{12}$ they are the same.

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \longleftrightarrow (a_1, a_2, \ldots, a_{12})$$

Note: If the points are independent, then the matrices are independent. If the points are dependent, then the matrices are dependent.

Q) $D = \text{span}\left\{ \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \right\}$

Find dim(D). Write it as a span of basis.

A) Do the calculation in the co-space of $R^{2\times2}$

$$\underset{R^{2\times2}}{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}} \longleftrightarrow \underset{R^4}{(1,2,0,1)}$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \longleftrightarrow (-1,-1,1,1)$$

$$\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \longleftrightarrow (1,3,1,3)$$

now $\begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 3 & 1 & 3 \end{bmatrix} \xrightarrow[-R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{-R_2+R_3 \to R_3} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

now we know that the first two matrices are independent but the last is dependent.

$\dim(D) = 2$

basis for $D = \left\{ \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \right\}$ or any of the equivalent $\left\{ \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \right\}$

$D = \text{span}\left\{ \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \right\}$

span of independent matrices

13 April 2022

Q: Find a basis for $R^{2\times2}$, say B

such that $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \in B$.

* $\dim(R^{2\times2}) = 4$

which means you have to give me 4 independent matrices, each 2×2.

A: consider the co-space R^4

$$\underset{R^{2\times2}}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} \longrightarrow \underset{R^4}{(1,1,1,1)}$$

$$\underset{R^{2\times2}}{\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}} \longrightarrow \underset{R^4}{(-1,-1,1,1)}$$

then

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ \vdots & & & \end{bmatrix} \xrightarrow{R_1+R_2\to R_2} \begin{bmatrix} \textcircled{1} & 1 & 1 & 1 \\ 0 & 0 & \textcircled{2} & 2 \\ \vdots & & & \end{bmatrix}$$

add the other leaders

$$\begin{bmatrix} \textcircled{1} & 1 & 1 & 1 \\ 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & \textcircled{2} & 2 \\ 0 & 0 & 0 & \textcircled{1} \end{bmatrix} \begin{matrix} \\ \longleftarrow \text{we added this} \\ \\ \longleftarrow \text{we added this} \end{matrix}$$

now

$$B = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Fact : $P_n \approx R^n$ they are isomorphic as subspaces

$P_4 \longleftrightarrow R^4$

$a_3x^3 + a_2x^2 + ax + a_0 \longleftrightarrow (a_3, a_2, a_1, a_0)$

ex: $2x^3 - 10x + 15 \longleftrightarrow (2, 0, -10, 15)$

ex: $13x^2 - 10x + x^3 + 2 \longleftrightarrow (1, 13, -10, 2)$

Q: $D = \left\{ (a_2 + a_1)x^3 + a_2x^2 - a_1x + a_1 \;\middle|\; a_1, a_2 \in R \right\}$

D "lives" in P_4

(a) Convince me that D is a subspace of P_4.

factor D

$$D = \left\{ a_2(x^3 + x^2) + a_1(x^3 - x + 1) \;\middle|\; a_1, a_2 \in R \right\}$$

$$= \text{span}\left\{ (x^3 + x^2), (x^3 - x + 1) \right\}$$

(b) Find the basis for R_4

To check for independent, we use the co-space R_4.

$x^3 + x^2 \longleftrightarrow (1, 1, 0, 0)$

$x^3 - x + 1 \longleftrightarrow (1, 0, -1, 1)$

put R^4 in matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{-R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix}$$

By staring, $(1, 1, 0, 0)$ and $(1, 0, -1, 1)$ are independent.

$$B = \left\{ (x^3 + x^2), (x^3 - x + 1) \right\}$$

18 April 2022

Q) $T: P_3 \longrightarrow R^3$

$$T(a_2x^2 + a_1x + a_0) = (a_2 + a_1 + a_0, a_1, a_0)$$

Is T a linear transformation ? yes

* Find the co-matrix presentation of T.

A) $L: R^3 \rightarrow R^3$

$$L(a_2, a_1, a_0) = (a_2 + a_1 + a_0, a_1, a_0)$$

The co-matrix representation of T is the matrix presentation of L

$$\begin{array}{cc} & \begin{array}{ccc} a_2 & a_1 & a_0 \end{array} \\ M = & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

each point is a row!

* Is T invertible ?

<u>iff</u> L is invertible <u>iff</u> M is invertible.

How do we find the inverse of M?

$$\left[\begin{array}{ccc|ccc} \textcircled{1} & 1 & 1 & 1 & 0 & 0 \\ 0 & \textcircled{1} & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_2 + R_1 \rightarrow R_1} \left[\begin{array}{ccc|ccc} \textcircled{1} & 0 & 1 & 1 & -1 & 0 \\ 0 & \textcircled{1} & 0 & 0 & 1 & 0 \\ 0 & 0 & \textcircled{1} & 0 & 0 & 1 \end{array}\right]$$

$$-R_3 + R_1 \rightarrow R_3$$

Since L is invertible, T is invertible.

$$\left[\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3} \middle| \underbrace{\begin{matrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{M^{-1}} \right]$$

* Find T^{-1}.

when you inverse a function the domain becomes the codomain & the codomain becomes the domain.

$L^{-1} : R^3 \longrightarrow R^3$

$$L^{-1}(a_1, a_2, a_3) = M^{-1}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$= (a_1 - a_2 - a_3 , a_2 , a_3)$$

now $T^{-1} : R^3 \longrightarrow P_3$

$$T^{-1}(a_1, a_2, a_3) = (a_1 - a_2 - a_3)x^2 + a_2 x + a_3$$

* Find $T^{-1}(\overset{a_1}{1}, \overset{a_2}{1}, \overset{a_3}{0}) = 0x^2 + x + 0 = x$

* What are the $Z(T)$? 0

$\text{Dim}(\text{Range}) + \text{Dim}(Z(T)) = \dim(\text{domain})$

when invertible, it has to be onto & one-to-one.

Result (Know) :

A linear transformation T is one-to-one <u>iff</u> $Z(T) = \{0\text{-element}\}$

Result : Assume D is a <u>subspace</u> and the $\dim(D) < \infty$

The following are equivalent:

1) $\forall\, a, b \in D$, $a + b \in D$ closed under addition

2) $\forall\, c \in R$ and $a \in D$, $ca \in D$ closed under scalar multiplication

Q Convince me that $D = \left\{ \begin{bmatrix} a+b & a \\ a & a+1 \end{bmatrix} \middle| \; a,b \in R \right\}$ is not a subspace.

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin D$

$\therefore$ D is not a subspace.

Q $D = \left\{ A \in R^{3\times3} \mid |A| = 0 \right\}$

Convince me that D is not a subspace of $R^{3\times3}$.

now use the new definitions of the subspace to prove so.

lets prove $\forall a,b \in D$, $a+b \in D$ is not true

$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\qquad$ $b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$|a| = 0$ $\qquad$ $|b| = 0$

$\therefore a \in D$ $\qquad$ $\therefore b \in D$

now $a+b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$|a+b| = 1$

$\therefore a+b \notin D$

$\therefore$ D is not a subspace.

20 April 2022

Q) $D = \left\{ f(x) \in P_3 \mid f(0)=0 \text{ or } f(1)=0 \right\}$ 0 "lives" inside P_3

Show D is not a subspace.

$f_1(x) = x$ lives in D? yes $f_1(x) \in D$ because $f_1(0) = 0$

$f_2(x) = 1 - x$ lives in D? yes $f_2(x) \in D$ because $f_2(1) = 0$

now $f_1(x) + f_2(x) = x + 1 - x = 1$

$f_3(x) = 1 \notin D$.

because $f_3(0) \neq 0$ and $f_3(1) \neq 0$

∴ The axiom closed under addition failed.

∴ D is not a subspace.

Q) $D = \left\{ A \in R^{2\times2} \mid A^T = -A \right\}$

Show D is a subspace.

Solution: Show both 1. Closure under addition

2. closure under scalar multiplication

① Let $a, b \in D$

show $a + b \in D$

$$\left.\begin{aligned} a^T &= -a \\ b^T &= -b \end{aligned}\right\} \text{because } a,b \text{ live in } D$$

$(a+b)^T = a^T + b^T = -a + -b = -(a+b)$

② Let $a \in D$ and $c \in R$. Show $ca \in D$

cont

$$D = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 \\ -a_3 & -a_4 \end{bmatrix} \right\}$$

by starring

$$a_1 = -a_1 = 0$$

$$\boxed{\begin{array}{l} a_3 = -a_2 \\ a_2 = -a_3 \end{array}}$$

$$a_4 = -a_4 = 0$$

now

$$D = \left\{ \begin{bmatrix} 0 & a_2 \\ -a_2 & 0 \end{bmatrix} \middle| a_2 \in R \right\}$$

$$D = \left\{ a_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \middle| a_2 \in R \right\}$$

$$D = \text{span} \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

Find dim(D).

$\dim(D) = 1$

Q) $D = \{ f(x) \in P_3 \mid f(0) = 0 \text{ and } f(1) = 0 \}$

- Show D is a subspace.
- Find dim (D)

(a) $D = \{ \underbrace{a_2x^2 + a_1x + a_0}_{f(x)} \mid f(0) = a_0 = 0 \ \& \ f(1) = a_2 + a_1 + a_0 = 0 \}$

we conclude $a_0 = 0$ & $a_1 = -a_2$

$D = \{ a_2x^2 - a_2x \mid a_2 \in R \}$

$D = \{ a_2(x^2 - x) \mid a_2 \in R \}$

$D = \text{span}\{ x^3 - x \}$

(b) $\dim(D) = 1$

Q)

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}} \xrightarrow{2R_1} \begin{bmatrix} 2 & 4 \\ 3 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 3 & 0 \\ 2 & 4 \end{bmatrix} \xrightarrow{2R_1 + R_2 \to R_2} \underset{B}{\begin{bmatrix} 3 & 0 \\ 8 & 4 \end{bmatrix}}$$

Find 3 **elementary matrices** E_1, E_2, E_3 such that $E_1E_2E_3A = B$

$E_3 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ is the row operation $2R_1$

$E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is the row operation $R_1 \leftrightarrow R_2$

$E_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ is the row operation $2R_1 + R_2 \to R_2$

you take the identity matrix and perform the row operations on it.

Q)

$$A = \begin{matrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 3 \end{matrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{matrix} 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 3 \end{matrix} \xrightarrow{-2R_3} \underbrace{\begin{matrix} 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ -2 & -2 & 0 & -6 \end{matrix}}_{B}$$

Find Elementary matrices E_1, E_2 such that $E_1 E_2 A = B$

$$E_1 = \left.\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{matrix}\right\} -2R_3$$

note: they are 3×3 matrices even though B is 3×4, that is because it is in origin an identity matrix & multiplication is legal.

$$E_2 = \left.\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix}\right\} R_1 \leftrightarrow R_2$$

Dot Product over R^n

$(a_1, \dots, a_n) \cdot (b_1, \dots, b_n) = a_1 b_1 + a_2 b_2 + \dots + a_n b_n \in R$

Def : Let $Q_1, Q_2, \dots, Q_m \in R^n$.

we say $Q_1, Q_2, \dots, Q_m$ are **orthogonal** iff $Q_i \cdot Q_k = 0$ where $i \neq k$.

Q) Convince me $\{(1,2), (0,4)\}$ is a basis for R^2.

$\downarrow$ find matrix, prove they are independent

$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$ both independent

$R^2 = \text{span}\{(1,2), (0,4)\}$

25 April 2022

Q) $D = \text{span}\left\{ \overset{Q_1}{\overline{(1,2,1)}}, \overset{Q_2}{\overline{(-1,1,1)}} \right\}$

D "lives" in $\mathbb{R}^3$ (3 variables) but not equal to $\mathbb{R}^3$ ($\dim(D) \neq 3$)

- $\dim(D) = 2$

$\therefore$ they are independent matrices

(a) Find an orthogonal basis of D.

↳ should be 2 indpendent points

↳ should be 2 points where their dot product is 0.

How do we find these orthogonal basis ?

we use the **Gram - Schmidt Algorithm** .

$$O = \{ w_1, w_2 \}$$

$\mathbf{w_1} = Q_1 = 1, 2, 1$

$$\mathbf{w_2} = Q_2 - \frac{Q_2 \cdot Q_1}{|Q_1|^2} Q_1$$

> $|Q_1|^2$ is called the norm of Q_1...squared.

$|Q_1|^2 = a_1^2 + a_2^2 + \dots + a_n^2$

$$w_2 = (-1,1,1) - \frac{(1,2,1) \cdot (-1,1,1)}{1^2 + 2^2 + 1^2} (1,2,1)$$

$$= (-1,1,1) - \frac{2}{6} (1,2,1)$$

$$= (-1,1,1) - (1/3, 2/3, 1/3)$$

$$= (-4/3, 1/3, 2/3)$$

to check our answer ⇒ $w_1 \bullet w_2$

$= (1, 2, 1) \bullet (-4/3 , 1/3 , 2/3)$

$= -\frac{4}{3} + \frac{2}{3} + \frac{2}{3}$

$= 0$

∴ correct !

$$O = \left\{ w_1 , w_2 \right\} = \left\{ (1, 2, 1), (-4/3 , 1/3 , 2/3) \right\}$$

What do you do if you have more than two points ?!

$$D = \text{span}\left\{ \underbrace{Q_1 , \ldots , Q_K}_{\text{independent points}} \right\}$$

$\dim(D) = K$

Find an orthogonal basis of D.

Solution: $O = \left\{ w_1 , \ldots , w_K \right\}$ ⟵ the dot product of any & every two points is zero.

$w_1 = Q_1$

$$w_2 = Q_2 - \frac{Q_2 \cdot w_1}{|w_1|^2} w_1$$

$$w_3 = Q_3 - \frac{Q_3 \cdot w_1}{|w_1|^2} w_1 - \frac{Q_3 \cdot w_2}{|w_2|^2} w_2$$

and so on...

$$W_m = Q_m - \frac{Q_m \cdot w_1}{|w_1|^2} w_1 - \underline{\quad} w_2 - \ldots\ldots - W_{m-1}$$

Result : If $Q_1, Q_2, \dots, Q_k$ are non-zero points in R^n & are orthogonal then $Q_1, \dots, Q_k$ are independent.

Remember !! Independent does not mean Orthogonal

only orthogonal implies independence.

Ex : $Q_1 = (2,4)$

$Q_2 = (-2,4)$

$Q_1 \cdot Q_2 = (2)(-2) + (4)(4) = 12$

$\therefore$ Q_1 & Q_2 are not orthogonal

BUT Q_1 & Q_2 are independent.

$Q = \begin{bmatrix} 2 & 4 \\ -2 & 4 \end{bmatrix}$ $\quad |Q| = (2)(4) - (-2)(4) \neq 0$

$\therefore$ $|Q| \neq 0$

9 May 2022

Inner product on polynomials

$$\langle f_1, f_2 \rangle = \int_a^b f_1 f_2 \, dx$$

Q. $D = \text{span}\left\{1,\ x^2+1\right\} \subseteq P_3 \longrightarrow \dim(D) = 2$

Find orthogonal basis for D where $\langle f_1, f_2 \rangle = \int_0^1 f_1 f_2 \, dx$

To Find basis $O = \{w_1, w_2\}$

you do either $\rightarrow \langle w_1, w_2 \rangle = 0$

$\searrow \int_0^1 w_1 w_2 = 0$

Lets check if our f_1, f_2 are orthogonal:

$$\int_0^1 1 \cdot (x^2+1)\, dx = \frac{1}{3}x^3 + x \Big|_{x=0}^{x=1} = \frac{4}{3} \neq 0 \quad \therefore \text{not orthogonal}$$

So we have to find basis that are orthogonal:

If f is a polynomial $|f| = \sqrt{\int_a^b f^2 dx}$

so the norm is $|f|^2 = \int_a^b f^2 \, dx$

$w_1 = Q_1 = f_1 = 1$

$$w_2 = f_2 - \frac{\int_0^1 f_1 \cdot f_2 \, dx}{|f_1|^2} \times f_1$$

$$= (x^2+1) - \frac{\int_0^1 (x^2+1) \cdot 1}{\int_0^1 1 \, dx} \times 1$$

$$= x^2 + 1 - \frac{4}{3}$$

$$= x^2 - \frac{1}{3}$$

now:

$$O = \left\{1,\ x^2 - 1/3\right\}$$

$$D = \text{span}\left\{1,\ x^2 - \frac{1}{3}\right\} \longleftarrow \text{span of orthogonal basis.}$$

to check: integrate ↑ & you'll get 0.

Q: $D = \text{span}\left\{ \overset{\overset{f_1}{\downarrow}}{x},\ \overset{\overset{f_2}{\downarrow}}{x^2},\ \overset{\overset{f_3}{\downarrow}}{x^4} \right\}$ "lives" in P_5.

Inner product on D is defined.

$$\langle f_1 , f_2 \rangle = \int_0^1 f_1 f_2 \, dx$$

Find $O = \{ w_1 , w_2 , w_3 \}$

- $\boxed{w_1 = f_1 = x}$
- $w_2 = f_2 - \dfrac{\int_0^1 w_1 f_2 \, dx}{|w_1|^2} w_1$
- $w_3 = f_3 - \dfrac{\int_0^1 w_2 f_3}{|w_2|^2} w_2 - \dfrac{\int_0^1 w_1 f_3}{|w_1|^2} w_1$

for w_2 $\quad \displaystyle\int_0^1 w_1 f_2 \, dx = \int_0^1 x \, x^2 \, dx = \frac{1}{4}$

$$|w_1|^2 = \int_0^1 w_1^2 \, dx = \int_0^1 x^2 \, dx = \frac{1}{3}$$

so $\boxed{w_2 = x^2 - \dfrac{1/4}{1/3} x = x^2 - \dfrac{3}{4} x}$

for w_3

$$\left.\begin{aligned} &\int_0^1 w_2 f_3 \, dx = \\ &|w_2|^2 = \\ &\int_0^1 w_1 f_3 \, dx = \\ &|w_1|^2 = \end{aligned}\right\} \text{find these to get } w_3 .$$

" All subspaces in our MTH221 are called **vector spaces** " set, vector

$(V, +, \cdot)$ is called a vector space if :

① $\forall\, x, y \in V,\ x + y \in V$ closed under addition

② $\forall\, c \in R$ and $\forall\, x \in V$ so $cx \in V$ closed under scalar multiplication

③ $\exists$ zero element in V, call it 0

④ $\forall\, x \in V,\ \exists\, -x \in V$

⑤ $\forall\, c_1, c_2 \in R,\ \forall\, x \in V \quad (c_1 + c_2)x = c_1 x + c_2 x$

⑥ $\forall\, c_1, c_2 \in R,\ \forall\, x \in V \quad (c_1 c_2)x = c_1(c_2 x)$

⑦ $\forall\, c \in R,\ x, y \in V \quad c(x + y) = cx + cy$

if you want to prove it is not a vector space prove one of these wrong!

$f(x) = \frac{1}{x} \in D$

$D = C[1,2]$ set of all continuous function on $[1,2]$.

this is a subspace BUT cannot written as span

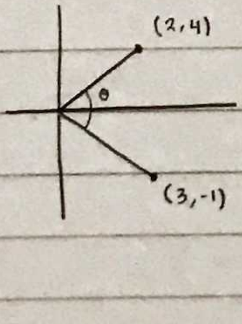
$\dim(D) = \infty$

solution: Let $f_1, f_2 \in D$ (f_1, f_2 are cont. on $[1,2]$)

from calclus 1, $f_1 + f_2$ is cont. on $[1,2]$

proved axiom 1

Application #1



$$\cos\theta = \frac{(2,4)\cdot(3,-1)}{|(2,4)||(3,-1)|}$$

$$= \frac{6-4}{\sqrt{20}\,\sqrt{10}}$$

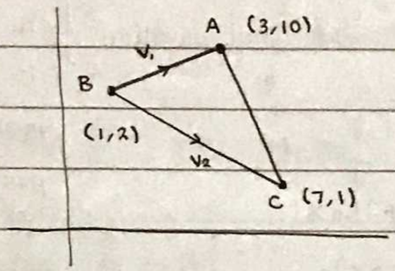
$$= \frac{2}{\sqrt{200}}$$

$$\therefore\ \theta = \cos^{-1}\left(\frac{2}{\sqrt{200}}\right)$$

$$= 81.8^\circ$$

Application #2

Find the area of ABC.



→ it is crucial for v_1 & v_2 to have the same initial point.

$$V_1 = (\Delta x, \Delta y) = (2,8)$$

$$V_2 = (6,-1)$$

$$\therefore \text{Area} = \left| \frac{\begin{vmatrix} 2 & 8 \\ 6 & -1 \end{vmatrix}}{2} \right|$$

$$= \left| \frac{-2-48}{2} \right|$$

$$= 25$$