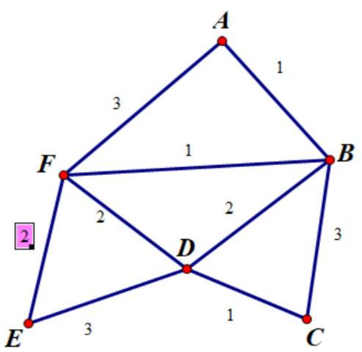
Final Exam, MTH 213 , Summer 2021

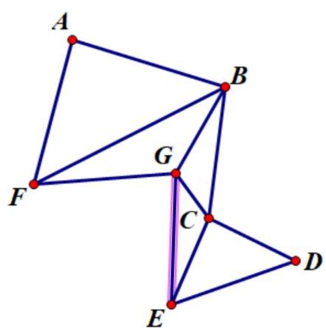
Ayman Badawi

(Stop working at 4:00 pm/ submit your solution by 4:15 pm, DO NOT SUBMIT BY EMAIL) ———— 50

QUESTION 1. (**6 points**)(**SHOW THE WORK**) Consider the following weighted graph of order 6 (i.e., number of vertices is 6). Use Dijkstra Algorithm and construct a minimum spanning tree.



QUESTION 2. (**9 points**)(**SHOW THE WORK**) Consider the below graph of order 7



(i) Is the graph Eulerian? explain. If yes, construct an Euler circuit.

(ii) Is the graph an Euler trail? explain. If yes, construct an Euler trail.

(iii) Is the graph Hamiltonian? explain. If yes, construct a Hamiltonian cycle (C_7)

QUESTION 3. (**SHOW THE WORK**)(**9 points**)

(i) Let C be a circle with circumference equals to 16 cm. What is the minimum number of points that you should locate on the circle so that there are at least two points, say Q_1, Q_2, where the arch-length between Q_1, Q_2 is strictly less than $\frac{1}{3}$.

(ii) Given 46 distinct integers. Then there are at least k integers out of the 46 numbers, say $n_1, ..., n_k$, such that $n_1 \ (mod\, 6) = n_2 \ (mod\, 6) = \cdots = n_k \ (mod\, 6)$. What is the maximum value of k.

(iii) Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 5 & 1 & 8 & 6 & 2 & 9 & 7 & 10 & 4 \end{pmatrix}$. Find the minimum positive integer n such that $f^n = I$ (the identity map)

QUESTION 4. (**SHOW THE WORK**)(**12 points**)

(i) Use math induction and prove that $2^{2n} - 1 + n^3 + 8n$ is divisible by 3 for every positive integer $n \geq 1$.

(ii) Use the 4th-method and prove that $\sqrt{29}$ is an irrational number (you may start by assuming that $\sqrt{29} = a/b$ where $gcd(a, b) = 1$ and a,b are odd integers.)

(iii) Given $a_n = 3a_{n-1} - 2a_{n-2} + 3^n + 5$. Find a general formula for a_n, where $a_0 = 6.5$, $a_1 = 21.5$.

QUESTION 5. (**SHOW THE WORK**)(**8 points**)

(i) Let x be the age of Ahmad. Given $14 \leq x \leq 60$, $x \ (mod\, 7) = 4$ and $x \ (mod\, 4) = 3$. Use the CRT and find x.

(ii) Let $A = \{1, 3, 8, 9, 11, 12, 13, 14\}$ Define " $=$ " on A such that for all $a, b \in A$, we have $a" = "b$ if and only if $(a - b) \ (mod\ 15) \in \{0, 5, 10\}$. Then " $=$ " is an equivalence relation (do not show that). Find all distinct equivalence classes of " $=$ ". If we view " $=$ " as a subset of $A \times A$, then how many elements does "=" have?.

QUESTION 6.)(**SHOW THE WORK**)(**6 points**) We have 5 men and 4 females. We need to form a committee of 3 persons.

(i) In how many ways can we form such committee where exactly two men are serving on the committee?

(ii) In how many ways can we form such committee where at least two females are serving on the committee?

(iii) Assume the names of the females are Mona, Ideal, Raneem, and Nada. In how many ways can we form such committee where Raneem, exactly one man, and one more female are serving on the committee?

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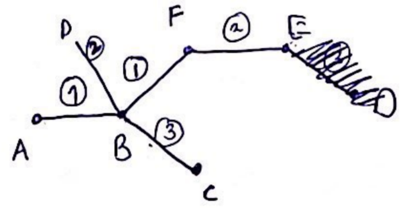
Daim Shaif

1

Q1)

V	A	B	C	D	E	F
A	$\boxed{0_A}$	1_A	∞	∞	∞	3_A
B		$\boxed{1_A}$	4_B	3_B	∞	2_B
F			4_B	3_B	4_F	$\boxed{2_B}$
D			4_B	$\boxed{3_B}$	4_F	
E			4_B 4_B	~~7_E~~	$\boxed{4_F}$	
C			$\boxed{4_B}$	~~7_F~~		



Q2) i) NO, because to be Eulerian every vertex in the graph should have an even degree. Here this is NOT the case for example degree (F) = 3

ii) Yes, A Euler trail requires exactly 2 ~~vertex~~ vertices with odd degree
degree of F & E are odd = 3.

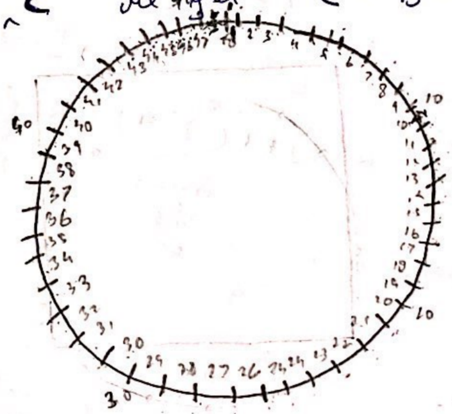
F – A – B – F – G – B – C – G – E – D – C – E

starts at odd degree vertex

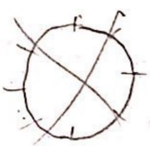
ends at odd degree vertex

iii) Yes if we start at vertex C we get C – B – A – F – G – E – D – C
and that is C_7

Q3i)



$\frac{16}{n} = \frac{1}{3}$



$\frac{16}{n} = \frac{1}{3} \Rightarrow n = 48$ we need 48 sections ~~to have $\frac{1}{3}$ each the~~ for the min. distance (arc length) between 2 points to be $\frac{1}{3}$.

And so at 48 points we have the min. distance between 2 points say Q_1, Q_2 to be $= \frac{1}{3}$

adding one more point to this will mean the min. distance (arc length) between any 2 points to be $< \frac{1}{3}$ $48 + 1 = 49$ points

Q3ii) $D = 46$ integers $\quad \left\lceil \frac{46}{6} \right\rceil = 8$

$C = 6$ possibilities $(0-5)$

there are atleast 8 integers s.t. $n_1 \pmod 6 = n_2 \pmod 6 = \dots = n_8 \pmod 6$

Q3iii) $(1\ 3) \circ (2\ 5\ 6) \circ (4\ 8\ 7\ 9\ 10)$

2 cycle $\quad$ 3 cycle $\quad$ 5 cycle

3

$LCM(2, 3, 5)$ $\quad$ or $\quad$ ~~$\frac{2 \times 3 \times 5}{5\ gcd(2 \times 3)}$~~

$$\begin{array}{c|c} 2 & 2, 3, 5 \\ 3 & 1, 3, 5 \\ 5 & 1, 1, 5 \\ & 1, 1, 1 \end{array} \quad = 2 \times 3 \times 5 = 30$$

$\underline{n = 30}$ $\quad f^{30} = I$

Q4i) 1. we prove for $n=1$

$2^2 - 1 + 1^3 + 8 = 12 \qquad 12|3$ ✓

2. Assume $2^{2k} - 1 + k^3 + 8k$ is divisible by 3 for some $n=k$

3. We show $2^{2(k+1)} - 1 + (k+1)^3 + 8(k+1)$ is divisible by 3 also

$2^{2k} \cdot 2^2 - 1 + k^3 + 3k^2 + 3k + 1 + 8k + 8$

$2^{2k} \cdot 2^2 - 1 + k^3 + 8k + [3k^2 + 3k + 9] - 1 \cdot 2^2 + 1 \cdot 2^2 - k^3 \cdot 2^2 + k^3 \cdot 2^2 - 8k \cdot 2^2 + 8k \cdot 2^2$

$[1 \cdot 2^2 - 1] + [k^3 - k^3 \cdot 2^2] + [8k - 8k \cdot 2^2]$

$[2^{2k} \cdot 2^2 - 1 \cdot 2^2 + k^3 \cdot 2^2 + 8k \cdot 2^2] + [3k^2 + 3k + 9] + 1(2^2 - 1) + k^3(1 - 2^2) + 8k(1 - 2^2)$

$2^2[2^{2k} - 1 + k^3 + 8k] + 3(k^2 + k + 3) + 1(2^2 - 1) + k^3(1 - 2^2) + 8k(1 - 2^2)$

$1 - 4 = -3$, $-3|3$ ✓

$1 - 4 = -3$, $-3|3$ ✓

$(4-1) = 3$, $3|3$ ✓

divisible by 3 ✓

divisible by 3 from part 2 ✓

$\therefore$ we proved $2^{2(k+1)} - 1 + (k+1)^3 + 8(k+1)$ is divisible by 3. And so through proving by induction we proved $2^{2n} - 1 + n^3 + 8n$ is divisible by 3 for $n \geq 1$

4

Daim Sharif

Q4ii) Reny - hence $\sqrt{29}$ is a rational #

$$\sqrt{29} = \frac{a}{b} \qquad \underbrace{29b^2}_{odd} = \underbrace{a^2}_{odd}$$

$$29 = \frac{a^2}{b^2}$$

~~a, b to be g~~
for gcd (a, b) to be 1, a, b should either be both odd or one even and other odd

let b, a be odd $\therefore$ $b = 2k_1 + 1$ and $a = 2k_2 + 1$ for some $k_1, k_2 \in \mathbb{Z}$

4

$$29(2k_1+1)^2 = (2k_2+1)^2$$

$$29(4k_1^2 + 4k_1 + 1) = 4k_2^2 + 4k_2 + 1 \qquad \div 4$$

$$29 \times 4k_1^2 + 29 \times 4k_1 + 28 = 4k_2^2 + 4k_2$$

$$\underbrace{29k_1^2 + 29k_1}_{even} + \underbrace{7}_{odd} = \underbrace{k_2^2 + k_2}_{even} \Leftarrow \text{contradiction} \therefore$$

even + odd = odd

$\therefore$ $\sqrt{29}$ is irrational.

Q4iii) for H:

$$a_n - 3a_{n-1} + 2a_{n-2} = 3^n + \underbrace{5}_{\text{degree } 0}$$

$$3^{n-1} = 3^n \cdot 3^{-1} = \frac{3^n}{3}$$

$$a_n - 3a_{n-1} + 2a_{n-2} = 0$$

$$\alpha^n - 3\alpha^{n-1} + 2\alpha^{n-2} = 0 \qquad \div \alpha^{n-2}$$

$$\alpha^2 - 3\alpha + 2 = 0$$

$$\alpha = 2, 1$$

$$H: \; c_1 2^n + \underbrace{c_2 1^n}_{constant}$$

for P: $f(n) = A3^n + Bn$

$$a_n - 3a_{n-1} + 2a_{n-2} = 3^n + 5$$

$$A3^n + Bn - 3(A3^{n-1} + B(n-1)) + 2(A3^{n-2} + B(n-2)) = 3^n + 5$$

$$\cancel{A3^n} + \cancel{B}n - \cancel{\frac{3A3^n}{3}} - 3\cancel{B}n + 3B + \frac{2A3^n}{9} + 2\cancel{B}n - 4B = 3^n + 5$$

$$\frac{2A3^n}{9} - B = 3^n + 5$$

$$\frac{2}{9}A\cancel{3^n} = \cancel{3^n} \qquad -B = 5$$

$$\frac{2}{9}A = 1 \qquad B = -5$$

$$A = \frac{9}{2}$$

$$P: f(n) = \frac{9}{2}3^n - 5n$$

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$a_n = c_1 2^n + c_2 + \frac{9}{2} \cdot 3^n - 5n$

$a_0 = 6.5 = c_1 + c_2 + \frac{9}{2}$

$2 = c_1 + c_2$

$a_1 = 21.5 = c_1 2^1 + c_2 + \frac{9}{2} \cdot 3^1 - 5$

$21.5 = 2c_1 + c_2 + \frac{17}{2}$

$13 = 2c_1 + c_2$

$2 = c_1 + c_2$
$13 = 2c_1 + c_2$ —

$11 = c_1$

$2 = 11 + c_2$

$2 - 11 = c_2$

$-9 = c_2$

$a_n = 11(2^n) - 9 + \frac{9}{2}(3^n) - 5n$

4

Q5i) $\Rightarrow$ $a_1 = 4$ $\quad a_2 = 3$

$m_1 = 7$ $\quad m_2 = 4$

gcd (between m_i's) = 1 $\therefore$ we can use CRT

$m = m_1 \times m_2 = 7 \times 4 = 28$

$n_1 = \frac{m}{m_1} = \frac{7 \times 4}{7} = 4$ $\quad n_2 = \frac{m}{m_2} = \frac{7 \times 4}{4} = 7$

$\Rightarrow n_1 = 4, \quad n_2 = 7$

$n_1^{-1} (\text{mod } m_1)$

$4^{-1} (\text{mod } 7) = 2$

$4 \times \boxed{2} (\text{mod } 7) = 1$

$\hookrightarrow 4^{-1}$

$n_2^{-1} (\text{mod } m_2)$

$7^{-1} (\text{mod } 4) = 3$

$7 \times \boxed{3} (\text{mod } 4) = 1$

$\hookrightarrow 7^{-1}$

$\Rightarrow n_1^{-1} = 2 \quad n_2^{-1} = 3$

$x = \sum_{i=1}^{2} a_i n_i n_i^{-1} = \overset{32}{(4)(4)(2)} + (3)(7)(3) = 95$

$x = 95 (\text{mod } 28) = 11$

answer is in the form $11 + mk \Rightarrow 11 + 28k$ where $k \in \mathbb{N}$

$11 + 28(1) = 39$

$x = 39$

4

Q5 ii) $[1] = \{1, 11\} \Rightarrow 2^2$

$[3] = \{3, 8, 13\} \Rightarrow 3^2$

$[9] = \{9, 14\} \Rightarrow 2^2$

$[12] = \{12\} \Rightarrow 1$

# of elements in "="

$= 2^2 + 3^2 + 2^2 + 1$

$= 18$

Q6 i) $M = 5 \quad F = 4$

$\binom{5}{2}\binom{4}{1} = 40$

ii) $\binom{4}{2}\binom{5}{1} + \binom{4}{3}\binom{5}{0} = 34$

iii) $\underline{1} \times \underline{5} \times \underline{3} = 15$

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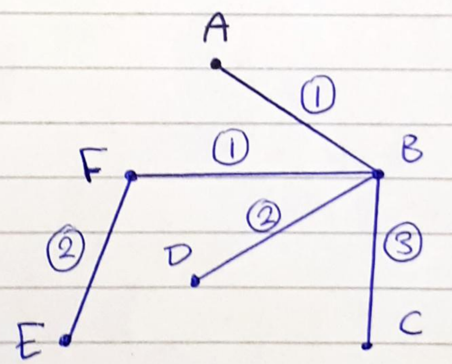
Q1]

V	A	B	C	D	E	F
A	$\boxed{0}$	1^A	∞	∞	∞	3^A
B		$\boxed{1}^A$	4^B	3^B	∞	2^B
F			4^B	3^B	4^F	$\boxed{2}^B$
D			4^B	$\boxed{3}^B$	4^F	
C			$\boxed{4}^B$		4^F	
E					$\boxed{4}^F$	

6



→ Minimum Spanning Tree

Q2] (i) No it is not Eulerian because not all of its vertices ~~are even~~. have even degrees

(ii) Yes it is an Euler trail because there are exactly 2 vertices with odd degrees.

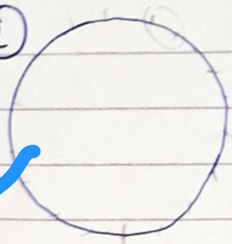
Euler trail example: F-A-B-F-G-B-C-G-E-D-C-E

(iii) Yes it is Hamiltonian because you can construct a cycle of the same order as the graph in which every vertex is visited exactly once.

(C_7): A-F-G-E-D-C-B-A

9

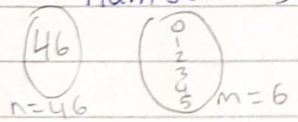
Q3] (i) 3 ✓



First divide the circumference into 48 equal parts. $\left(16x = \frac{1}{3},\ x = \frac{1}{48}\right)$
Then, to have at least 2 points strictly less than $\frac{1}{3}$, add 1 more point.
$48 + 1 = \boxed{49}$ Place at least 49 points

(ii) 3

$\left\lceil \frac{46}{6} \right\rceil = \lceil 7.6 \rceil = 8$, at least 8 values/integers out of the 46 numbers satisfy $n_1 \pmod 6 = n_2 \pmod 6 \ldots$



(iii) 3

$f^n = I$

$f: \underbrace{(1,3)}_{\text{2-cycles}} \circ \underbrace{(2,5,6)}_{\text{3-cycles}} \circ \underbrace{(4,8,7,9,10)}_{\text{5-cycles}}$

$LCM[2,3,5] = \frac{2 \times 3 \times 5}{\gcd(2,3,5)} = \boxed{30}$, $f^{30} = I$

Q4] $2^{2n} - 1 + n^3 + 8n$ is divisible by 3 $\forall\ n \geq 1$:

1) Prove for $n=1$: $2^2 - 1 + (1)^3 + 8(1) = 4 + 8 = 12$ $3 \mid 12$ ✓
2) Assume $2^{2k} - 1 + k^3 + 8k$ is div. by 3 for some $n = k$ ✓
3) Prove: $2^{2(k+1)} - 1 + (k+1)^3 + 8k$ is div. by 3 for $n = k+1$

(k+1)

$2^{2k+2} - 1 + (k+1)^3 + 8k$

$= 2^{2k} \cdot 2^2 - \cancel{1} + k^3 + 3k^2 + 3k + \cancel{1} + 8k$

$= 2^{2k} \cdot 2^2 + k^3 + 3k^2 + 11k$

Now add & subtract: 2^2, $2^2 k^3$, $2^2 \cdot 8k$

$= 2^{2k} \cdot 2^2 + 2^2 - 2^2 + 2^2 k^3 - 2^2 k^3 + 2^2 \cdot 8k - 2^2 \cdot 8k + k^3 + 3k^2 + 11k$

$= 2^2(2^{2k} - 1 + k^3 + 8k) + 2^2 - 2^2 k^3 - 2^2 8k + k^3 + 3k^2 + 11k$

$= 2^2(2^{2k} - 1 + k^3 + 8k) - 3k^3 + 3k^2 - 21k + 2^2$

$= 2^2\underbrace{(2^{2k} - 1 + k^3 + 8k)}_{\text{by step \#2 it is div. by 3}} + \underbrace{3(-k^3 + k^2 - 7k + \tfrac{4}{3})}_{\text{Factor of 3}}$

all div. by 3

So whole statement is div. by 3.

3/0

Q4] (ii) Deny: $\sqrt{29}$ is rational so:

$\sqrt{29} = \frac{a}{b}$ $\begin{cases} a,b \in \mathbb{Z} \\ \gcd(a,b)=1 \\ b \neq 0 \end{cases}$ Assume a,b are odd s.t:

$a = 2n+1$, $n \in \mathbb{Z}$

$b = 2m+1$, $m \in \mathbb{Z}$

$29b^2 = a^2$

$29(2m+1)^2 = (2n+1)^2$

$29[4m^2+4m+1] = [4n^2+4n+1]$

$29.4m^2 + 29.4m + 29 = 4n^2+4n+1$

$29.4m^2 + 29.4m + 28 = 4n^2+4n$

Now divide by 4:

$\underbrace{\underbrace{29m^2+29m}_{\text{even}}+7}_{\text{odd}} = \underbrace{n^2+n}_{\text{even}}$

odd $\neq$ even $\rightarrow$ contradiction $\therefore \sqrt{29}$ is irrational.

(iii) $a_n = 3a_{n-1} - 2a_{n-2} + 3^n + 5$ $\quad a_0 = 6.5$, $a_1 = 21.5$

$a_n - 3a_{n-1} + 2a_{n-2} = 3^n + 5$ so $a_n = H + P$:

H: $\frac{\alpha^n - 3\alpha^{n-1} + 2\alpha^{n-2}}{\alpha^{n-2}} = \frac{0}{\alpha^{n-2}}$

$\alpha^2 - 3\alpha + 2 = 0$

$\alpha = 2$ $\quad \alpha = 1$

H: $c_1(2)^n + c_2(1)^n$

$c_1(2)^n + c_2$

P: $f(n) = A3^n + Bn$

$A3^n + Bn - 3[A3^{n-1} + B(n-1)] + 2[A3^{n-2} + B(n-2)] = 3^n + 5$

$\cancel{A3^n} + \cancel{Bn} - \cancel{A3^n} - 3\cancel{B}n + 3B + \frac{2}{9}A3^n + \cancel{2B}n - 4B = 3^n + 5$

$\frac{2}{9}A3^n - B = 3^n + 5$

$\frac{2}{9}A\cancel{3^n} = \cancel{3^n}$ $\quad -B = 5$

$\boxed{A = \frac{9}{2}}$ $\quad \boxed{B = -5}$

Now: $a_n = c_1 2^n + c_2 + \frac{9}{2}3^n - 5n$

for a_0:

$6.5 = c_1 + c_2 + \frac{9}{2} \rightarrow c_1 + c_2 = 2$

for a_1: $21.5 = 2c_1 + c_2 + \frac{(9)(3)}{2} - 5$

$\rightarrow 2c_1 + c_2 = 13$

So $c_1 = 11$, $c_2 = -9$

$\therefore$ $\boxed{a_n = 11.2^n - 9 + \frac{9}{2}.3^n - 5n}$

4

4

Q5] (i) $14 \leq x \leq 60$

$x \equiv 4 \pmod 7$ $\quad$ $\gcd(7,4) = 1$ so C.R.T applies

$x \equiv 3 \pmod 4$

$m = 7 \times 4 = 28$

$n_1 = \frac{28}{7} = 4 \qquad n_2 = \frac{28}{4} = 7$

4

$n_1^{-1} = 4^{-1} \pmod 7 = 2 \qquad n_2^{-1} = 7^{-1} \pmod 4 = 3$

$n = (4 \times 4 \times 2) + (3 \times 7 \times 3) = 95 \pmod{28} = 11 \rightarrow$ smallest +ve int.

$x + mk \rightarrow 11 + 28k$, $k \in \mathbb{Z}$

let $k = 1$, $11 + 28 = \boxed{39}$ Ahmad is 39 years old

$14 \leq x \leq 60$ ✓

(ii) $A = \{1, 3, 8, 9, 11, 12, 13, 14\}$ $\quad$ $a \text{“}=\text{”} b$ iff $(a-b) \pmod{15} \in \{0, 5, 10\}$

<u>All classes:</u>

$\overline{1} = \{1, 11\}$

$\overline{3} = \{3, 8, 13\}$

$\overline{9} = \{9, 14\}$

$\overline{12} = \{12\}$

# of elements for “=” as a subset of $A \times A$:

$= 2^2 + 3^2 + 2^2 + 1^2$

$= \boxed{18}$ ✓ 4

Q6] 5 men, 4 female Committee of 3:

(i) exactly 2 men: $5C2 \times 4C1 = \boxed{40}$ ✓

(ii) at least 2 females: $(4C2 \times 5C1) + (4C3 \times 5C0) = \boxed{34}$

(iii) Raneem, exactly 1 man, 1 other female: ~~4C~~ $\boxed{1 \times 5C1 \times 3C1}$

$= \cancel{\boxed{60}} = \boxed{15}$ ✓

6 ✓