Name————————————, ID ——————

Exam Two, MTH 213 , Fall 2021

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(Stop working at 13:00 pm/ submit your solution by 13:12 pm, DO NOT SUBMIT BY EMAIL) ——— 42

QUESTION 1. (**9 points**)(**SHOW THE WORK**)

(i) Let $n = 108$. Find $\phi(n)$. (**SHOW THE WORK**)

(ii) Find $(5)^{36003} \pmod{108}$ (**Show the work**)

(iii) Let $n = 11^3 \cdot 7^5 \cdot 5^{11}$ and k be the number of all positive integers $< n$ such that $gcd(integer,\ n) = 77$. Find the value of k. (**Show the work**)

QUESTION 2. (**6 points**)(**SHOW THE WORK**)

(i) Convert the number 124 (base 10) to base 5

(ii) Find $(235)_6 + (155)_6$ (Do not convert to base 10 then back to base 6, do it as I explained in the class, so you enjoy the beauty of math!)

QUESTION 3. (**SHOW THE WORK**)(**6 points**)

(i) Let D be a square 2×2 (i.e., each side is 2cm). What is the minimum number of points that you can locate on the sides of D (randomly), so that there are at least two points of them, say Q_1, Q_2, such that the distance between Q_1 and Q_2 is strictly less than $<$ 1/5

(ii) A group consists of 111 persons. The age of each person in the group is either 17 years or 18 years or 19 years or 20 years. There are at least m persons in the group that have the same age. Find the maximum value of m.

QUESTION 4. (**SHOW THE WORK**)(**9 points**)

(i) Let $F : (-\infty, 0] \to (a, 2]$ be a bijective function such that $F(x) = be^x - 7$. Find the values of a, b. You may draw $F(x)$

(ii) In two to three lines, convince me that $A = (Q \cap [3, 7]) \cup \{1, 7, 9, 11, 20\}$ is a countable set. Can we claim that $|A| = |Q|$? explain briefly note Q is the set of all rational numbers

(iii) Let F be a bijective function such that $F = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 7 & 6 & 2 & 8 & 1 & 5 & 9 \end{bmatrix}$ Find the smallest positive integer n such that $f^n = f\ o\ f\ o\ \cdots\ o\ f = I$ (the identity map)

QUESTION 5. (**SHOW THE WORK**)(**3 points**)
Use the TRUTH table and convince me that $(\overline{B} + A) \cdot A = \overline{B} \cdot A + A$

QUESTION 6. (**6 points**). Let $A = \{1, 2, \{2\}, 3, 4\}$, $B = \{\{2\}, 1, 7, 8, 9\}$, and $U = \{1, 2, \{2\}, 3, 4, 5, 6, 7, 8, 9, 20\}$ be the universal set.

(i) Find $A - B$

(ii) Find $\overline{B}$

(iii) How many elements does $P(A)$ have? (note $P(A)$ is the power set of A)

(iv) How many elements does $A \times B$ have? (note $A \times B$ is the Cartesian product of A with B)

(v) Let A, B as above. WRITE DOWN T or F (no need for justification).

a. $(2, \{2\}) \in B \times A$.
b. $\{3, 7\} \in P(A \times B)$
c. $\{7, 8, 1\} \subset P(B)$
d. $\{2\} \in P(A)$
e. $\{\{2\}, \{3\}\} \subset P(A)$
f. $\{\phi, \{3\}, \{2\}\} \subset P(A)$

QUESTION 7.)(**3 points**) Write down T or F (no need for justification)

(i) If $\exists$! x $\in Z$ such that $x^2 + 0.7x = 0$, then $\exists y \in Q$ such that $y^2 + 2 = 5$.

(ii) $\exists! y \in Q^*$ such that $xy = 1\ \forall x \in Q^*$

(iii) $\forall x \in Q^*, \exists!\ y \in R^*$ such that $x = y^2$

Q1) i) $n = 108 = 2\times2\times27 = 2\times2\times3\times3\times3 = 2^2\times3^3$

$\Phi(108) = (2-1)2^1\times(3-1)(3^2) = 36$

3

ii) $5^{36003} \pmod{108}$

$= 5^{36000} \pmod{108} \times 5^3 \pmod{108}$

$= \underbrace{5^{36\times1000} \pmod{108}}_{1} \times 125 \pmod{108} = 17$

by euler Fermat thm

3

iii) $n = 11^3\cdot7^5\cdot5^{11} = 77\cdot11^2\cdot7^4\cdot5^{11}$

$\Phi(\frac{n}{77}) = (11-1)(11)\cdot(7-1)(7^3)\cdot(5-1)(5^{10}) \approx 8.843\times10^{12}$

3

Q2) i) $(124)_{10} \rightarrow$ base 5

$$
\begin{array}{r} 24 \\ 5\overline{)124} \\ 120 \\ \hline 4 \end{array}
\quad
\begin{array}{r} 4 \\ 5\overline{)24} \\ 20 \\ \hline 4 \end{array}
\quad
\begin{array}{r} 0 \\ 5\overline{)4} \\ 0 \\ \hline 4 \end{array}
$$

$(444)_5$

3

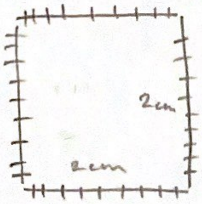
ii)
$$
\begin{array}{r} {\scriptstyle 1\,1} \\ (235)_6 \\ (155)_6 \\ \hline (434)_6 \end{array}
$$

$10 = \boxed{1}\times6 + \boxed{4}$

$9 = 1\times6+3$

3

Q3) i)



equally divided points on each side with distances between them adding up to 2 cm

$9 \times 4 = 36$

$\boxed{\text{at least 37 number of points}}$ 3

ii) 111 people, D

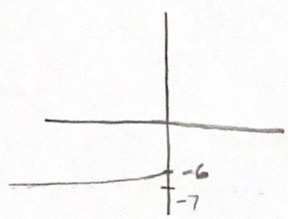
4 age groups, R

$m = \left\lceil \frac{111}{4} \right\rceil = 28$ 3

Q4) i) F is bijective $(-\infty, 0] \to (a, 2]$

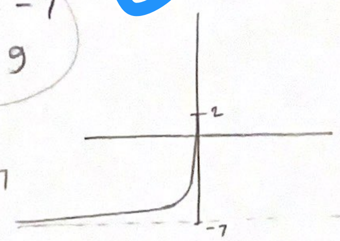
$F(x) = be^x - 7$

3



$a = -7$
$b = 9$

$2 = be^0 - 7$
$9 = b$



ii) $[3,7]$ is a subset of Q, so its countable

The $\cap$ of two countable sets is countable, $\{1,7,9,11,20\}$ is a finite set so it is also countable the $\cup$ of two countable sets is countable.

3

The cardinality of a countable infinite set is equal to the cardinality of another countable infinite set so $|A| = |Q|$ is correct.

iii) $f = \underbrace{(1\,3\,7)}_{3\text{ cycle}} \circ \underbrace{(2\,4\,6\,8\,5)}_{5\text{ cycle}}$

$n = \mathrm{LCM}(3,5) = \frac{3 \times 5}{\gcd(3,5)} = 15$

3

$f^{15} = I$

Q5) $(\overline{B}+A)\cdot A = \overline{B}\cdot A + A$

Two variable, $2^2 = 4$

B	A	$\overline{B}$	$\overline{B}+A$	$(\overline{B}+A)\cdot A$	$(\overline{B}\cdot A)$	$(\overline{B}\cdot A + A)$
1	1	0	1	1	0	1
1	0	0	0	0	0	0
0	1	1	1	1	1	1
0	0	1	1	0	0	0

the same

3

Q6) i) $A - B = \{2,3,4\}$ ✓

ii) $\overline{B} = U - B = \{2,3,4,5,6,20\}$ ✓

iii) # of elements in $P(A) = 2^5 = 32$

iv) $|A \times B| = |A| \times |B| = 5 \times 5 = 25$ ✓

3/3

v) a) F ✓

b) F ✓

c) F ✓

d) T ✓

e) T ✓

f) T ✓

3/3

Q7) i) T ✗ F

ii) F

iii) T ✗ F

1/3