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On Weakly 1-Absorbing Primary Ideals of Commutative Rings

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Abstract. Let R be a commutative ring with $1 \neq 0$. We introduce the concept of weakly 1-absorbing primary ideal, which is a generalization of 1-absorbing primary ideal. A proper ideal I of R is said to be *weakly 1-absorbing primary* if whenever nonunit elements $a, b, c \in R$ and $0 \neq abc \in I$, we have $ab \in I$ or $c \in \sqrt{I}$. A number of results concerning weakly 1-absorbing primary ideals are given, as well as examples of weakly 1-absorbing primary ideals. Furthermore, we give a corrected version of a result on 1-absorbing primary ideals of commutative rings.

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1 Introduction

Throughout this paper, all rings are commutative with nonzero identity. Let R be a commutative ring. By a proper ideal I of R, we mean an ideal I of R with $I \neq R$. Let I be a proper ideal of R. By $\sqrt{I}$ we denote the radical of I in R, that is, $\{a \in R \mid a^n \in I$ for some positive integer $n\}$. In particular, $\sqrt{\{0\}}$ denotes the set of all nilpotent elements of R. Set $Z_I(R) = \{r \in R \mid rs \in I$ for some $s \in R \backslash I\}$. A ring R is called a *reduced* ring if it has no nonzero nilpotent elements, i.e., $\sqrt{\{0\}} = \{0\}$. For two ideals I and J of R, the *residual division* of I and J is defined to be the ideal $(I : J) = \{a \in R \mid aJ \subseteq I\}$. Let R be a commutative ring with identity and

M a unitary R-module. Then

$$R(+)M = R \times M$$

with coordinate-wise addition and multiplication $(a, m)(b, n) = (ab, an + bm)$ is a commutative ring with identity $(1, 0)$, called the *idealization* of M. A ring R is called a *quasilocal* ring if R has exactly one maximal ideal. As usual, $\mathbb{Z}$ and $\mathbb{Z}_n$ will denote the ring of integers and the ring of integers modulo n, respectively.

Since prime and primary ideals have key roles in commutative ring theory, many authors have studied generalizations of prime and primary ideals (see [1]–[3] and [5]–[11]). Anderson and Smith introduced in [2] the notion of weakly prime ideals. A proper ideal I of R is said to be weakly prime if for $a, b \in R$ with $0 \neq ab \in I$, either $a \in I$ or $b \in I$. After that, Atani and Farzalipour introduced in [5] the concept of weakly primary ideals. A proper ideal I of R is said to be weakly primary if whenever $a, b \in R$ and $0 \neq ab \in I$, we have $a \in I$ or $b \in \sqrt{I}$. For a different generalization of prime ideals and weakly prime ideals, the concepts of 2-absorbing and weakly 2-absorbing ideals were defined. According to [6] and [7], a proper ideal I of a commutative ring R is called a *2-absorbing* (resp., *weakly 2-absorbing*) ideal if whenever $a, b, c \in R$ and $abc \in I$ (resp., $0 \neq abc \in I$), we have $ab \in I$ or $bc \in I$ or $ac \in I$. As a generalization of 2-absorbing and weakly 2-absorbing ideals, 2-absorbing primary and weakly 2-absorbing primary ideals were defined in [9] and [10], respectively. A proper ideal I of a commutative ring R is said to be *2-absorbing primary* (resp., *weakly 2-absorbing primary*) if whenever $a, b, c \in R$ and $abc \in I$ (resp., $0 \neq abc \in I$), we have $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. In a recent study [11], we call a proper ideal I of a commutative ring R a *1-absorbing primary* ideal if whenever nonunit elements $a, b, c \in R$ and $abc \in I$, we have $ab \in I$ or $c \in \sqrt{I}$.

In this paper, we introduce the concept of weakly 1-absorbing primary ideals of commutative rings. A proper ideal I of a commutative ring R is called a *weakly 1-absorbing primary* ideal of the ring R if whenever nonunit elements $a, b, c \in R$ and $0 \neq abc \in I$, we have $ab \in I$ or $c \in \sqrt{I}$. It is clear that a 1-absorbing primary ideal of a commutative ring R is a weakly 1-absorbing primary ideal of R. However, since $\{0\}$ is always weakly 1-absorbing primary, a weakly 1-absorbing primary ideal of a commutative ring R need not be a 1-absorbing primary ideal of R (see Example 2.2).

We prove (Theorem 2.4) that if a proper ideal I of a commutative ring R is weakly 1-absorbing primary such that $\sqrt{I}$ is a maximal ideal of R, then I is a primary ideal of R, and hence I is 1-absorbing primary. We show (Theorem 2.5) that if R is a commutative reduced ring and I is a weakly 1-absorbing primary ideal of R, then $\sqrt{I}$ is a prime ideal of R. If I is a proper nonzero ideal of a commutative von Neumann regular ring R, then we show (Theorem 2.6) that I is a weakly 1-absorbing primary ideal of R if and only if I is a 1-absorbing primary ideal of R, if and only if I is a primary ideal of R. Moreover, we show (Theorem 2.7) that if R is a commutative nonquasilocal ring and I is a proper ideal of R such that $\mathrm{ann}(i) = \{r \in R \mid ri = 0\}$ is not a maximal ideal of R for every element $i \in I$, then I is a weakly 1-absorbing primary ideal of R if and only if I is a weakly primary ideal of R. If I is a proper ideal of a commutative reduced divided ring R, then we show (Theorem 2.10) that I is a weakly 1-absorbing primary ideal of R if and only if I is a weakly primary ideal of R. If I is a weakly 1-absorbing primary ideal of a commutative ring R but not a 1-absorbing primary ideal of R, then we give (Theorem 2.14) sufficient conditions so that $I^3 = \{0\}$ (i.e., $I \subseteq \sqrt{\{0\}}$). In Theorem 2.12, we obtain some equivalent conditions for a proper ideal of a u-ring to be weakly 1-absorbing primary. We give (Theorem 2.19) a characterization of weakly 1-absorbing primary ideals in $R = R_1 \times R_2$, where R_1 and R_2 are commutative rings with identity that are not fields. If $R_1, R_2, \ldots, R_n$ are commutative rings with identity for some $2 \leq n < \infty$ and $R = R_1 \times \cdots \times R_n$, then it is shown (Theorem 2.20) that every proper ideal of R is weakly 1-absorbing primary if and only if $n = 2$ and R_1, R_2 are fields. For a weakly 1-absorbing primary ideal of a ring R, we show (Theorem 2.26) that $S^{-1}I$ is a weakly 1-absorbing primary ideal of $S^{-1}R$ for every multiplicatively closed subset S of R that is disjoint from I, and we show that the converse holds if $S \cap Z(R) = S \cap Z_I(R) = \emptyset$, where $Z(R)$ denotes the center of R. In addition, we give (Remark 2.25) a corrected version of [11, Theorem 17(1) and Corollaries 3 and 4].

2 Properties of Weakly 1-Absorbing Primary Ideals

Definition 2.1. Let R be a commutative ring and I be a proper ideal of R. We call I a *weakly 1-absorbing primary* ideal of R if whenever nonunit elements $a, b, c \in R$ and $0 \neq abc \in I$, we have $ab \in I$ or $c \in \sqrt{I}$.

It is clear that every 1-absorbing primary ideal of a commutative ring R is a weakly 1-absorbing primary ideal of R, and $I = \{0\}$ is a weakly 1-absorbing primary ideal of R. In the following example, we construct a weakly 1-absorbing primary ideal of a commutative ring R that is neither 1-absorbing primary nor weakly primary.

Example 2.2. (1) The ideal $I = \{0\}$ is a weakly 1-absorbing primary ideal of $R = \mathbb{Z}_6$, which is not a 1-absorbing primary ideal of R. Indeed, $2 \cdot 2 \cdot 3 \in I$, but neither $2 \cdot 2 \in I$ nor $3 \in \sqrt{I}$. Note that I is a weakly primary ideal of R.

(2) Let $A = \mathbb{Z}_2[[X, Y]]$, $I = (XY^2, YX^2)A$, $R = A/I$, and $J = (XY)A/I$. We show that J is a weakly 1-absorbing primary ideal of R, which is neither 1-absorbing primary nor weakly primary.

Assume that $abc \in J$ for some nonunit elements $a, b, c \in R$. Then $abc = XYZ + I$ for some nonunit element $Z \in A$. Hence, $abc = I \in J$ by the construction of J. Thus, J is a weakly 1-absorbing primary ideal of R. Since

$$(X + I)(X + I)(Y + I) = I \in J$$

and neither $X^2 + I \in J$ nor $Y + I \in \sqrt{J}$, we conclude that J is not a 1-absorbing primary ideal of R. Since $I \neq (X + I)(Y + I) = XY + I \in J$ and neither $X + I \in J$ nor $Y + I \in \sqrt{J}$, we conclude that J is not a weakly primary ideal of R.

We begin with the following trivial result without proof.

Theorem 2.3. *Let I be a proper ideal of a commutative ring R.*

(1) *If I is a weakly prime ideal, then I is a weakly 1-absorbing primary ideal.*
(2) *If I is a weakly primary ideal, then I is a weakly 1-absorbing primary ideal.*
(3) *If I is a 1-absorbing primary ideal, then I is a weakly 1-absorbing primary ideal.*
(4) *If I is a weakly 1-absorbing primary ideal, then I is a weakly 2-absorbing primary ideal.*
(5) *If R is an integral domain, then I is a weakly 1-absorbing primary ideal if and only if I is a 1-absorbing primary ideal of R.*
(6) *Let R be a quasilocal ring with maximal ideal $\sqrt{\{0\}}$. Then every proper ideal of R is a weakly 1-absorbing primary ideal of R.*

We recall that a proper ideal I of a commutative ring R is called a *semiprimary* ideal of R if $\sqrt{I}$ is a prime ideal of R. For an interesting article on semiprimary ideals of commutative rings, see [12]. For a recent related article on semiprimary ideals, we recommend [8]. We have the following result.

Theorem 2.4. *Let I be a weakly 1-absorbing primary ideal of a commutative ring R. If $\sqrt{I}$ is a maximal ideal of R, then I is a primary ideal of R, hence a 1-absorbing primary ideal of R. In particular, if I is a weakly 1-absorbing primary ideal of R but not a 1-absorbing primary ideal of R, then $\sqrt{I}$ is not a maximal ideal of R.*

Proof. Suppose that $\sqrt{I}$ is a maximal ideal of R. Then I is a semiprimary ideal of R. Since I is a semiprimary ideal of R and $\sqrt{I}$ is a maximal ideal of R, we conclude that I is a primary ideal of R by [14, p. 153]. Thus, I is a 1-absorbing primary ideal of R. □

Theorem 2.5. *Let R be a commutative reduced ring. If I is a nonzero weakly 1-absorbing primary ideal of R, then $\sqrt{I}$ is a prime ideal of R. In particular, if $\sqrt{I}$ is a maximal ideal of R, then I is a primary ideal of R, hence a 1-absorbing primary ideal of R.*

Proof. Suppose that $0 \neq ab \in \sqrt{I}$ for some $a, b \in R$. We may assume that a, b are nonunits. Then there exists an even positive integer $n = 2m$ ($m \geq 1$) such that $(ab)^n \in I$. Since $\sqrt{\{0\}} = \{0\}$, we have $(ab)^n \neq 0$. Hence, $0 \neq a^m a^m b^n \in I$. Thus $a^m a^m = a^n \in I$ or $b^n \in \sqrt{I}$, and so $\sqrt{I}$ is a weakly prime ideal of R. Since R is reduced and $I \neq \{0\}$, we conclude that $\sqrt{I}$ is a prime ideal of R by [2, Corollary 2]. The proof of the "in particular" statement is now clear by Theorem 2.4. □

Recall that a commutative ring R is called a *von Neumann regular* ring if and only if for every $x \in R$, there is a $y \in R$ such that $x^2 y = x$. It is known that a commutative ring R is a von Neumann regular ring if and only if for each $x \in R$, there exist an idempotent $e \in R$ and a unit $u \in R$ such that $x = eu$. For a recent article on von Neumann regular rings, see [4]. We have the following result.

Theorem 2.6. *Let R be a commutative von Neumann regular ring and I be a nonzero ideal of R. Then the following statements are equivalent:*

(1) *I is a weakly 1-absorbing primary ideal of R.*

(2) *I is a primary ideal of R.*
(3) *I is a 1-absorbing primary ideal of R.*

Proof. (1)⇒(2) Since R is a commutative von Neumann regular ring, we know that R is reduced. Hence, $\sqrt{I}$ is a prime ideal of R by Theorem 2.5. Since every prime ideal of a von Neumann regular ring is maximal, we conclude that $\sqrt{I}$ is a maximal ideal of R. Thus, I is a primary ideal of R by Theorem 2.4.

(2)⇒(3)⇒(1) It is clear. □

Let A, I, R, and J be as in Example 2.2(2). Then R is a quasilocal ring with maximal ideal $M = (X, Y)A/I$, and

$$\text{ann}(XY + I) = \{a \in R \mid a(XY + I) = 0\} = M.$$

We have the following result.

Theorem 2.7. *Let R be a commutative nonquasilocal ring and I be a proper ideal of R such that* $\text{ann}(i) = \{r \in R \mid ri = 0\}$ *is not a maximal ideal of R for any element $i \in I$. Then I is a weakly 1-absorbing primary ideal of R if and only if I is a weakly primary ideal of R.*

Proof. If I is a weakly primary ideal of R, then I is a weakly 1-absorbing primary ideal of R by Theorem 2.3(2). Hence, let I be a weakly 1-absorbing primary ideal of R and $0 \neq ab \in I$ for some elements $a, b \in R$. We show that $a \in I$ or $b \in \sqrt{I}$.

Assume that a, b are nonunit elements of R. Let

$$\text{ann}(ab) = \{c \in R \mid cab = 0\}.$$

Since $ab \neq 0$, $\text{ann}(ab)$ is a proper ideal of R. Let L be a maximal ideal of R such that $\text{ann}(ab) \subsetneq L$. Since R is a nonquasilocal ring, there is a maximal ideal M of R such that $M \neq L$. Let $m \in M \backslash L$. Hence, $m \notin \text{ann}(ab)$ and $0 \neq mab \in I$. Since I is a weakly 1-absorbing primary ideal of R, we have $ma \in I$ or $b \in \sqrt{I}$.

If $b \in \sqrt{I}$, then we are done. Assume that $b \notin \sqrt{I}$. Thus, $ma \in I$. Since $m \notin L$ and L is a maximal ideal of R, we conclude that $m \notin J(R)$. Hence, there exists an $r \in R$ such that $1 + rm$ is a nonunit element of R. Suppose that $1 + rm \notin \text{ann}(ab)$. Then $0 \neq (1 + rm)ab \in I$. Since I is a weakly 1-absorbing primary ideal of R and $b \notin \sqrt{I}$, we conclude that $(1 + rm)a = a + rma \in I$. Since $rma \in I$, we have $a \in I$ and we are done. Suppose that $1 + rm \in \text{ann}(ab)$. Since $\text{ann}(ab)$ is not a maximal ideal of R and $\text{ann}(ab) \subsetneq L$, there is a $w \in L\backslash\text{ann}(ab)$. Hence, $0 \neq wab \in I$. Since I is a weakly 1-absorbing primary ideal of R and $b \notin \sqrt{I}$, we conclude that $wa \in I$. Since $1+rm \in \text{ann}(ab) \subsetneq L$ and $w \in L\backslash\text{ann}(ab)$, we see that $1+rm+w$ is a nonzero nonunit element of L. So $0 \neq (1 + rm + w)ab \in I$. Since I is a weakly 1-absorbing primary ideal of R and $b \notin \sqrt{I}$, we conclude that $(1+rm+w)a = a+rma+wa \in I$. Since $rma, wa \in I$, we obtain $a \in I$. □

Question. Is the preceding theorem still valid without the assumption that $\text{ann}(i) = \{r \in R \mid ri = 0\}$ is not a maximal ideal of R for any element $i \in I$? We are unable to give a proof of Theorem 2.7 without this assumption.

In light of the proof of Theorem 2.7, we have the following result.

Theorem 2.8. *Let I be a weakly 1-absorbing primary ideal of a commutative ring R such that for every nonzero element $i \in I$, there exists a nonunit $w \in R$ such that $wi \neq 0$ and $w + u$ is a nonunit element of R for some unit $u \in R$. Then I is a weakly primary ideal of R.*

Proof. Let $0 \neq ab \in I$ and $b \notin \sqrt{I}$ for some $a, b \in R$. We may assume that a, b are nonunit elements of R. Then there is a nonunit $w \in R$ such that $wab \neq 0$ and $w + u$ is a nonunit element of R for some unit $u \in R$. Since $0 \neq wab \in I$, $b \notin \sqrt{I}$, and I is a weakly 1-absorbing primary ideal of R, we conclude that $wa \in I$. Since $0 \neq (w + u)ab \in I$, I is a weakly 1-absorbing primary ideal of R, and $b \notin \sqrt{I}$, we conclude that $wa + ua = (w + u)a \in I$. Since $wa \in I$ and $wa + ua \in I$, we conclude that $ua \in I$. Since u is a unit, we have $a \in I$. □

Corollary 2.9. *Let R be a commutative ring and $A = R[X]$. Suppose that I is a weakly 1-absorbing primary ideal of A. Then I is a weakly primary ideal of A.*

Proof. Since $Xi \neq 0$ for every nonzero $i \in I$ and $X + 1$ is a nonunit element of A, we are done by Theorem 2.8. □

Recall that a commutative ring R is called *divided* if for every prime ideal P of R and for every $x \in R \backslash P$, we have $x|p$ for every $p \in P$.

Theorem 2.10. *Let R be a commutative reduced divided ring and I be a proper ideal of R. Then the following statements are equivalent:*

(1) *I is a weakly 1-absorbing primary ideal of R.*

(2) *I is a weakly primary ideal of R.*

Proof. (1)⇒(2) Suppose that $0 \neq ab \in I$ for some $a, b \in R$ and $b \notin \sqrt{I}$. We may assume that a, b are nonunit elements of R. Since $\sqrt{I}$ is a prime ideal of R by Theorem 2.5, we conclude that $a \in \sqrt{I}$. Since R is divided, $b|a$. Thus, $a = bc$ for some $c \in R$. Observe that c is a nonunit element of R as $b \notin \sqrt{I}$ and $a \in \sqrt{I}$. Since $0 \neq ab = bcb \in I$, I is weakly 1-absorbing primary, and $b \notin \sqrt{I}$, we conclude that $a = bc \in I$. Thus, I is a weakly primary ideal of R.

(2)⇒(1) It is clear by Theorem 2.3(2). □

Recall that a commutative ring R is called a *chained* ring if for all $x, y \in R$, we have $x|y$ or $y|x$. Every chained ring is divided. So if R is a reduced chained ring, then a proper ideal I of R is a weakly 1-absorbing primary ideal if and only if it is a weakly primary ideal of R.

Theorem 2.11. *Let R be a Dedekind domain and I be a nonzero proper ideal of R. Then I is a weakly 1-absorbing primary ideal of R if and only if $\sqrt{I}$ is a prime ideal of R.*

Proof. Suppose that I is a weakly 1-absorbing primary ideal of R. Then $\sqrt{I}$ is a prime ideal of R by Theorem 2.5. The converse follows from [11, Theorem 14]. □

Let R be a commutative ring with $1 \neq 0$. If an ideal of R contained in a finite union of ideals must be contained in one of those ideals, then R is called a *u-ring*

[13]. In the next theorem, we give some characterizations of weakly 1-absorbing primary ideals in u-rings.

Theorem 2.12. *Let R be a commutative u-ring and I a proper ideal of R. Then the following statements are equivalent:*

(1) *I is a weakly 1-absorbing primary ideal of R.*

(2) *For all nonunit elements $a, b \in R$ with $ab \notin I$, $(I : ab) = (0 : ab)$ or $(I : ab) \subseteq \sqrt{I}$.*

(3) *For any nonunit element $a \in R$ and any ideal I_1 of R with $I_1 \nsubseteq \sqrt{I}$, if $(I : aI_1)$ is a proper ideal of R, then $(I : aI_1) = (\{0\} : aI_1)$ or $(I : aI_1) \subseteq (I : a)$.*

(4) *For all ideals I_1, I_2 of R with $I_1 \nsubseteq \sqrt{I}$, if $(I : I_1I_2)$ is a proper ideal of R, then $(I : I_1I_2) = (\{0\} : I_1I_2)$ or $(I : I_1I_2) \subseteq (I : I_2)$.*

(5) *For all ideals I_1, I_2, I_3 of R with $0 \neq I_1I_2I_3 \subseteq I$, we have $I_1I_2 \subseteq I$ or $I_3 \subseteq \sqrt{I}$.*

Proof. (1)⇒(2) Suppose that I is a weakly 1-absorbing primary ideal of R, and let $ab \notin I$ for some nonunit elements $a, b \in R$ and $c \in (I : ab)$. Then $abc \in I$. Since $ab \notin I$, c is a nonunit. If $abc = 0$, then $c \in (0 : ab)$. Assume that $0 \neq abc \in I$. Since I is weakly 1-absorbing primary, we have $c \in \sqrt{I}$. Hence, we conclude that $(I : ab) \subseteq (0 : ab) \cup \sqrt{I}$. Since R is a u-ring, we obtain $(I : ab) = (0 : ab)$ or $(I : ab) \subseteq \sqrt{I}$.

(2)⇒(3) If $aI_1 \subseteq I$, then we are done. Suppose that $aI_1 \nsubseteq I$ for some nonunit element $a \in R$ and $c \in (I : aI_1)$. It is clear that c is a nonunit. Then $acI_1 \subseteq I$. Now $I_1 \subseteq (I : ac)$. If $ac \in I$, then $c \in (I : a)$. Suppose that $ac \notin I$. Then $(I : ac) = (0 : ac)$ or $(I : ac) \subseteq \sqrt{I}$ by (2). So $I_1 \subseteq (0 : ac)$ or $I_1 \subseteq \sqrt{I}$. Since $I_1 \nsubseteq \sqrt{I}$ by hypothesis, we have $I_1 \subseteq (0 : ac)$; i.e., $c \in (\{0\} : aI_1)$. Hence, $(I : aI_1) \subseteq (\{0\} : aI_1) \cup (I : a)$. Since R is a u-ring, we have $(I : aI_1) = (\{0\} : aI_1)$ or $(I : aI_1) \subseteq (I : a)$.

(3)⇒(4) If $I_1 \subseteq \sqrt{I}$, then we are done. Let $I_1 \nsubseteq \sqrt{I}$ and $c \in (I : I_1I_2)$. Then $I_2 \subseteq (I : cI_1)$. Since $(I : I_1I_2)$ is proper, c is a nonunit. Hence, $I_2 \subseteq (\{0\} : cI_1)$ or $I_2 \subseteq (I : c)$ by (3). If $I_2 \subseteq (\{0\} : cI_1)$, then $c \in (\{0\} : I_1I_2)$. If $I_2 \subseteq (I : c)$, then $c \in (I : I_2)$. Therefore, we get $(I : I_1I_2) \subseteq (\{0\} : I_1I_2) \cup (I : I_2)$, which implies $(I : I_1I_2) = (\{0\} : I_1I_2)$ or $(I : I_1I_2) \subseteq (I : I_2)$, as needed.

(4)⇒(5) It is clear.

(5)⇒(1) Let $a, b, c \in R$ be nonunit elements and $0 \neq abc \in I$. Put $I_1 = aR$, $I_2 = bR$, and $I_3 = cR$. Then (1) is now clear by (5). □

Definition 2.13. Let I be a weakly 1-absorbing primary ideal of a commutative ring R and a, b, c be nonunit elements of R. We call (a, b, c) a 1-*triple-zero* of I if $abc = 0$, $ab \notin I$, and $c \notin \sqrt{I}$.

Observe that if I is a weakly 1-absorbing primary ideal of a commutative ring R but not 1-absorbing primary, then there exists a 1-triple-zero (a, b, c) of I for some nonunit elements $a, b, c \in R$.

Theorem 2.14. *Let I be a weakly 1-absorbing primary ideal of a commutative ring R and (a, b, c) be a 1-triple-zero of I. Then the following statements hold:*

(1) *$abI = \{0\}$.*

(2) *If $a, b \notin (I : c)$, then $bcI = acI = aI^2 = bI^2 = cI^2 = \{0\}$.*

(3) *If* $a, b \notin (I : c)$*, then* $I^3 = \{0\}$.

Proof. (1) Suppose that $abI \neq \{0\}$. Then $abx \neq 0$ for some nonunit $x \in I$. Hence, $0 \neq ab(c + x) \in I$. Since $ab \notin I$, $(c + x)$ is a nonunit element of R. Since I is a weakly 1-absorbing primary ideal of R and $ab \notin I$, we conclude that $(c + x) \in \sqrt{I}$. Since $x \in I$, we have $c \in \sqrt{I}$, a contradiction. Thus, $abI = \{0\}$.

(2) Suppose that $bcI \neq 0$. Then $bcy \neq 0$ for some nonunit element $y \in I$. Hence, $0 \neq bcy = b(a + y)c \in I$. Since $b \notin (I : c)$, we see that $a + y$ is a nonunit element of R. Since I is a weakly 1-absorbing primary ideal of R, $ab \notin I$, and $by \in I$, we conclude that $b(a + y) \notin I$, and hence $c \in \sqrt{I}$, a contradiction. Thus, $bcI = \{0\}$. We show that $acI = \{0\}$. Suppose that $acI \neq \{0\}$. Then $acy \neq 0$ for some nonunit element $y \in I$. Hence, $0 \neq acy = a(b + y)c \in I$. Since $a \notin (I : c)$, we conclude that $b + y$ is a nonunit element of R. Since I is a weakly 1-absorbing primary ideal of R, $ab \notin I$, and $ay \in I$, we have $a(b + y) \notin I$, and hence $c \in \sqrt{I}$, a contradiction. Thus, $acI = \{0\}$.

Now we prove $aI^2 = \{0\}$. Suppose that $axy \neq 0$ for some $x, y \in I$. Since $abI = \{0\}$ by (1) and $acI = \{0\}$ by (2), $0 \neq axy = a(b + x)(c + y) \in I$. Since $ab \notin I$, we see that $c + y$ is a nonunit element of R. Since $a \notin (I : c)$, we see that $b + x$ is a nonunit element of R. Since I is a weakly 1-absorbing primary ideal of R, we have $a(b + x) \in I$ or $(c + y) \in \sqrt{I}$. Since $x, y \in I$, we get $ab \in I$ or $c \in \sqrt{I}$, a contradiction. Thus, $aI^2 = \{0\}$. Next we show $bI^2 = \{0\}$. Let $bxy \neq 0$ for some $x, y \in I$. Since $abI = \{0\}$ by (1) and $bcI = \{0\}$ by (2), we obtain $0 \neq bxy = b(a + x)(c + y) \in I$. Since $ab \notin I$, we conclude that $c + y$ is a nonunit element of R. Since $b \notin (I : c)$, we see that $a+x$ is a nonunit element of R. Since I is a weakly 1-absorbing primary ideal of R, we have $b(a+x) \in I$ or $(c+y) \in \sqrt{I}$. Since $x, y \in I$, we conclude that $ab \in I$ or $c \in \sqrt{I}$, a contradiction. Hence, $bI^2 = \{0\}$. We show $cI^2 = \{0\}$. Let $cxy \neq 0$ for some $x, y \in I$. Since $acI = bcI = \{0\}$ by (2), $0 \neq cxy = (a + x)(b + y)c \in I$. Since $a, b \notin (I : c)$, we conclude that $a + x$ and $b + y$ are nonunit elements of R. Since I is a weakly 1-absorbing primary ideal of R, we have $(a + x)(b + y) \in I$ or $c \in \sqrt{I}$. Since $x, y \in I$, we get $ab \in I$ or $c \in \sqrt{I}$, a contradiction. Thus, $cI^2 = \{0\}$.

(3) Let $xyz \neq 0$ for some $x, y, z \in I$. Then $0 \neq xyz = (a + x)(b + y)(c + z) \in I$ by (1) and (2). Since $ab \notin I$, we conclude that $c + z$ is a nonunit element of R. Since $a, b \notin (I : c)$, $a + x$ and $b + y$ are nonunit elements of R. Since I is a weakly 1-absorbing primary ideal of R, we have $(a + x)(b + y) \in I$ or $c + z \in \sqrt{I}$. Since $x, y, z \in I$, we see that $ab \in I$ or $c \in \sqrt{I}$, a contradiction. Thus, $I^3 = \{0\}$. □

Theorem 2.15. (1) *Let I be a weakly* 1*-absorbing primary ideal of a commutative reduced ring R. Suppose that I is not a* 1*-absorbing primary ideal of R and (a, b, c) is a 1-triple-zero of I such that $a, b \notin (I : c)$. Then $I = \{0\}$.*

(2) *Let I be a nonzero weakly* 1*-absorbing primary ideal of a reduced ring R. Suppose that I is not a* 1*-absorbing primary ideal of R and (a, b, c) is a* 1*-triple-zero of I. Then $ac \in I$ or $bc \in I$.*

Proof. (1) Since $a, b \in (I : c)$, we have $I^3 = \{0\}$ by Theorem 2.14(3). Since R is reduced, we conclude that $I = \{0\}$.

(2) Suppose that neither $ac \in I$ nor $bc \in I$. Then $I = \{0\}$ by (1), a contradiction

since I is a nonzero ideal of R by hypothesis. Hence, if (a, b, c) is a 1-triple-zero of I, then $ac \in I$ or $bc \in I$. □

Theorem 2.16. *Let I be a weakly 1-absorbing primary ideal of a commutative ring R. If I is not a weakly primary ideal of R, then there exist an irreducible element $x \in R$ and a nonunit element $y \in R$ such that $xy \in I$, but neither $x \in I$ nor $y \in \sqrt{I}$. Furthermore, if $ab \in I$ for some nonunit elements $a, b \in R$ such that neither $a \in I$ nor $b \in \sqrt{I}$, then a is an irreducible element of R.*

Proof. Suppose that I is not a weakly primary ideal of R. Then there exist nonunit elements $x, y \in R$ such that $0 \neq xy \in I$ with $x \notin I$ and $y \notin \sqrt{I}$. Suppose that x is not an irreducible element of R. Then $x = cd$ for some nonunit elements $c, d \in R$. Since $0 \neq xy = cdy \in I$, I is weakly 1-absorbing primary, and $y \notin \sqrt{I}$, we conclude that $x = cd \in I$, a contradiction. Hence, x is an irreducible element of R. □

In general, the intersection of a family of weakly 1-absorbing primary ideals need not be a weakly 1-absorbing primary ideal. Indeed, consider the ring $R = \mathbb{Z}_{12}$. Then $I = (2)$ and $J = (3)$ are clearly weakly 1-absorbing primary ideals of R, but $I \cap J = \{0, 6\}$ is not a weakly 1-absorbing primary ideal of R (since $0 \neq 3 \cdot 3 \cdot 2 \in I \cap J$, but neither $3 \cdot 3 \in I \cap J$ nor $2 \in \sqrt{I \cap J}$). However, we have the following result.

Proposition 2.17. *Let $\{I_i \mid i \in \Lambda\}$ be a finite collection of weakly 1-absorbing primary ideals of a commutative ring R such that $Q = \sqrt{I_i} = \sqrt{I_j}$ for any distinct $i, j \in \Lambda$. Then $I = \bigcap_{i \in \Lambda} I_i$ is a weakly 1-absorbing primary ideal of R.*

Proof. Suppose that $0 \neq abc \in I = \bigcap_{i \in \Lambda} I_i$ for nonunit elements a, b, c of R and $ab \notin I$. Then $0 \neq abc \in I_k$ and $ab \notin I_k$ for some $k \in \Lambda$. This implies $c \in \sqrt{I_k} = Q = \sqrt{I}$. □

Proposition 2.18. *Let I be a weakly 1-absorbing primary ideal of a commutative ring R and c a nonunit element of $R \backslash I$. Then $(I : c)$ is a weakly primary ideal of R.*

Proof. Suppose that $0 \neq ab \in (I : c)$ for some nonunit $c \in R \backslash I$ and assume that $a \notin (I : c)$. Hence, b is a nonunit element of R. If a is a unit of R, then $b \in (I : c) \subseteq \sqrt{(I : c)}$ and we are done. So assume that a is a nonunit element of R. Since $0 \neq abc = acb \in I$, $ac \notin I$, and I is a weakly 1-absorbing primary ideal of R, we see that $b \in \sqrt{I} \subseteq \sqrt{(I : c)}$. Thus, $(I : c)$ is a weakly primary ideal of R. □

The next theorem gives a characterization for weakly 1-absorbing primary ideals of $R = R_1 \times R_2$, where R_1 and R_2 are commutative rings with identity that are not fields.

Theorem 2.19. *Let R_1 and R_2 be commutative rings with identity but not fields, $R = R_1 \times R_2$, and I be a nonzero proper ideal of R. Then the following statements are equivalent:*

(1) *I is a weakly 1-absorbing primary ideal of R.*

(2) *$I = I_1 \times R_2$ for some primary ideal I_1 of R_1 or $I = R_1 \times I_2$ for some primary ideal I_2 of R_2.*

(3) *I is a 1-absorbing primary ideal of R.*

(4) *I is a primary ideal of R.*

Proof. (1)⇒(2) Suppose that I is a weakly 1-absorbing primary ideal of R. Then I is of the form $I_1 \times I_2$ for some ideals I_1 and I_2 of R_1 and R_2, respectively. Assume that both I_1 and I_2 are proper. Since I is a nonzero ideal of R, we conclude that $I_1 \neq \{0\}$ or $I_2 \neq \{0\}$.

We may assume that $I_1 \neq \{0\}$. Let $0 \neq c \in I_1$. Then

$$0 \neq (1,0)(1,0)(c,1) = (c,0) \in I_1 \times I_2.$$

This implies $(1,0)(1,0) \in I_1 \times I_2$ or $(c,1) \in \sqrt{I_1 \times I_2} = \sqrt{I_1} \times \sqrt{I_2}$, that is, $I_1 = R_1$ or $I_2 = R_2$, a contradiction. Thus, either I_1 or I_2 is a proper ideal. Without loss of generality, assume that $I = I_1 \times R_2$ for some proper ideal I_1 of R_1. We show that I_1 is a primary ideal of R_1. Let $ab \in I_1$ for some $a, b \in R_1$. We can assume that a and b are nonunit elements of R_1. Since R_2 is not a field, there exists a nonunit nonzero element $x \in R_2$. Then $0 \neq (a,1)(1,x)(b,1) \in I_1 \times R_2$, which implies either $(a,1)(1,x) \in I_1 \times R_2$ or $(b,1) \in \sqrt{I_1 \times R_2} = \sqrt{I_1} \times R_2$; i.e., $a \in I_1$ or $b \in \sqrt{I_1}$.

(2)⇒(3) Since I is a primary ideal of R, I is a 1-absorbing primary ideal of R by [11, Theorem 1(1)].

(3)⇒(4) Since I a 1-absorbing primary ideal of R and R is not a quasilocal ring, we conclude that I is a primary ideal of R by [11, Theorem 3].

(4)⇒(1) It is clear. □

Theorem 2.20. *Assume that $R_1, \dots, R_n$ are commutative rings with $1 \neq 0$ for some $2 \leq n < \infty$ and let $R = R_1 \times \cdots \times R_n$. Then the following statements are equivalent:*

(1) *Every proper ideal of R is a weakly 1-absorbing primary ideal of R.*

(2) *$n = 2$ and R_1, R_2 are fields.*

Proof. (1)⇒(2) Suppose that every proper ideal of R is a weakly 1-absorbing primary ideal. Without loss of generality, we may assume that $n = 3$. Then $I = R_1 \times \{0\} \times \{0\}$ is a weakly 1-absorbing primary ideal of R. However, for a nonzero $a \in R_1$, we have $(0,0,0) \neq (1,0,1)(1,0,1)(a,1,0) = (a,0,0) \in I$, but neither $(1,0,1)(1,0,1) \in I$ nor $(a,1,0) \in \sqrt{I}$, a contradiction. Thus, $n = 2$. Assume that R_1 is not a field. Then there exists a nonzero proper ideal A of R_1. Hence, $I = A \times \{0\}$ is a weakly 1-absorbing primary ideal of R. However, for a nonzero $a \in A$, we have $(0,0) \neq (1,0)(1,0)(a,1) = (a,0) \in I$, but neither $(1,0)(1,0) \in I$ nor $(a,1) \in \sqrt{I}$, a contradiction. Similarly, one can easily show that R_2 is a field. Therefore, $n = 2$ and R_1, R_2 are fields.

(2)⇒(1) Let $n = 2$ and R_1, R_2 be fields. Then R has exactly three proper ideals, i.e., $\{(0,0)\}$, $\{0\} \times R_2$, and $R_1 \times \{0\}$ are the only proper ideals of R. Hence, it is clear that each proper ideal of R is a weakly 1-absorbing primary ideal of R. □

Since every ring that is a product of a finite number of fields is a von Neumann regular ring, in light of Theorems 2.6 and 2.20, we have the following result.

Corollary 2.21. *Assume that $R_1, \dots, R_n$ are commutative rings with $1 \neq 0$ for some $2 \leq n < \infty$ and let $R = R_1 \times \cdots \times R_n$. Then the following statements are equivalent:*

(1) *Every proper ideal of R is a weakly 1-absorbing primary ideal of R.*
(2) *Every proper ideal of R is a weakly primary ideal of R.*
(3) *$n = 2$ and R_1, R_2 are fields, and hence $R = R_1 \times R_2$ is a von Neumann regular ring.*

Theorem 2.22. *Let R_1 and R_2 be commutative rings and $f\colon R_1 \to R_2$ be a ring homomorphism with $f(1) = 1$.*
(1) *Suppose that f is injective and that $f(a)$ is a nonunit element of R_2 for every nonunit element $a \in R_1$. Let J be a weakly 1-absorbing primary ideal of R_2. Then $f^{-1}(J)$ is a weakly 1-absorbing primary ideal of R_1.*
(2) *If f is an epimorphism and I is a weakly 1-absorbing primary ideal of R_1 such that $\mathrm{Ker}(f) \subseteq I$, then $f(I)$ is a weakly 1-absorbing primary ideal of R_2.*

Proof. (1) Since $f(1) = 1$, $f^{-1}(J)$ is a proper ideal of R_1. Let $0 \neq abc \in f^{-1}(J)$ for some nonunit elements $a, b, c \in R$. Since $\mathrm{Ker}(f) = 0$, we have

$$0 \neq f(abc) = f(a)f(b)f(c) \in J,$$

where $f(a), f(b), f(c)$ are nonunit elements of R_2 by hypothesis. Therefore, we have $f(a)f(b) \in J$ or $f(c) \in \sqrt{J}$. So $ab \in f^{-1}(J)$ or $c \in \sqrt{f^{-1}(J)} = f^{-1}(\sqrt{J})$. Thus, $f^{-1}(J)$ is a weakly 1-absorbing primary ideal of R_1.

(2) Let $0 \neq xyz \in f(I)$ for some nonunit elements $x, y, z \in R$. Since f is onto, there exist nonunit elements $a, b, c \in I$ such that $x = f(a)$, $y = f(b)$, $z = f(c)$. Then $f(abc) = f(a)f(b)f(c) = xyz \in f(I)$. Since $\mathrm{Ker}(f) \subseteq I$, we have $0 \neq abc \in I$. Hence, $ab \in I$ or $c \in \sqrt{I}$. Thus, $xy \in f(I)$ or $z \in f(\sqrt{I})$. Since f is onto and $\mathrm{Ker}(f) \subseteq I$, we have $f(\sqrt{I}) = \sqrt{f(I)}$. We are done. □

The following example shows that the hypothesis in Theorem 2.22(1) is crucial.

Example 2.23. [11, Example 1] Let $A = K[x, y]$, where K is a field, $M = (x, y)A$, and $B = A_M$. Note that B is a quasilocal ring with maximal ideal M_M. Then $I = xM_M = (x^2, xy)B$ is a 1-absorbing primary ideal of B (see [11, Theorem 5]) and $\sqrt{I} = xB$. However, $xy \in I$, but neither $x \in I$ nor $y \in \sqrt{I}$. Thus, I is not a primary ideal of B. Let $f\colon B \times B \to B$ such that $f(x, y) = x$. Then f is a ring homomorphism from $B \times B$ onto B such that $f(1, 1) = 1$. However, $(1, 0)$ is a nonunit element of $B \times B$ and $f(1, 0) = 1$ is a unit of B. Thus, f does not satisfy the hypothesis of Theorem 2.22(1). Now $f^{-1}(I) = I \times B$ is not a weakly 1-absorbing primary ideal of $B \times B$ by Theorem 2.19.

Theorem 2.24. *Let I be a proper ideal of a commutative ring R.*
(1) *If J is a proper ideal of R with $J \subseteq I$ and I is a weakly 1-absorbing primary ideal of R, then I/J is a weakly 1-absorbing primary ideal of R/J.*
(2) *Let J be a proper ideal of R with $J \subseteq I$ such that $a + J$ is a nonunit element of R/J for every nonunit $a \in R$. If J is a 1-absorbing primary ideal of R and I/J is a weakly 1-absorbing primary ideal of R/J, then I is a 1-absorbing primary ideal of R.*
(3) *If $\{0\}$ is a 1-absorbing primary ideal of R and I is a weakly 1-absorbing primary ideal of R, then I is a 1-absorbing primary ideal of R.*

(4) *Let J be a proper ideal of R with $J \subseteq I$ such that $a + J$ is a nonunit element of R/J for every nonunit $a \in R$. If J is a weakly 1-absorbing primary ideal of R and I/J is a weakly 1-absorbing primary ideal of R/J, then I is a weakly 1-absorbing primary ideal of R.*

Proof. (1) Consider the natural epimorphism $\pi\colon R \to R/J$. Then $\pi(I) = I/J$. So we are done by Theorem 2.22(2).

(2) Suppose that $abc \in I$ for some nonunit elements $a, b, c \in R$. If $abc \in J$, then $ab \in J \subseteq I$ or $c \in \sqrt{J} \subseteq \sqrt{I}$ as J is a 1-absorbing primary ideal of R. Assume that $abc \notin J$. Then

$$J \neq (a + J)(b + J)(c + J) \in I/J,$$

where $a+J$, $b+J$, $c+J$ are nonunit elements of R/J by hypothesis. Thus, it follows that $(a + J)(b + J) \in I/J$ or $(c + J) \in \sqrt{I/J}$. Hence, $ab \in I$ or $c \in \sqrt{I}$.

(3) The proof follows from (2).

(4) Suppose that $0 \neq abc \in I$ for some nonunit elements $a, b, c \in R$. If $abc \in J$, then $ab \in J \subseteq I$ or $c \in \sqrt{J} \subseteq \sqrt{I}$ as J is a weakly 1-absorbing primary ideal of R. Assume that $abc \notin J$. Then $J \neq (a+J)(b+J)(c+J) \in I/J$, where $a+J$, $b+J$, $c+J$ are nonunit elements of R/J by hypothesis. Thus, we have $(a + J)(b + J) \in I/J$ or $(c + J) \in \sqrt{I/J}$. Hence, $ab \in I$ or $c \in \sqrt{I}$. □

In the following remark, we give the correct version of [11, Theorem 17(1) and Corollaries 3 and 4].

Remark 2.25. Mohammed Tamekkante pointed out to the first-named author that in [11] a fact is overlooked: if $f\colon R_1 \to R_2$ is a ring homomorphism such that $f(1) = 1$, then it is possible that $f(a) \in U(R_2)$ for some nonunit element $a \in R_1$. Overlooking this fact caused a problem in the proof of [11, Theorem 17(1) and Corollaries 3 and 4]. Here we state the correct version of [11, Theorem 17(1) and Corollaries 3 and 4].

(1) (cf. [11, Theorem 17(1)]) Let R_1, R_2 be commutative rings and $f\colon R_1 \to R_2$ be a ring homomorphism with $f(1) = 1$ such that if R_2 is a quasilocal ring, then $f(a)$ is a nonunit element of R_2 for every nonunit element $a \in R_1$. If J is a 1-absorbing primary ideal of R_2, then $f^{-1}(J)$ is a 1-absorbing primary ideal of R_1. (Note that if R_2 is not a quasilocal ring, then J is primary by [11, Theorem 3], and hence $f^{-1}(J)$ is a primary ideal of R_1. Since every primary ideal of a commutative ring A is a 1-absorbing primary ideal of A, we conclude that $f^{-1}(J)$ is a 1-absorbing primary ideal of R_1.)

(2) (cf. [11, Corollary 3]) Let I and J be proper ideals of a commutative ring R with $I \subseteq J$. If J is a 1-absorbing primary ideal of R, then J/I is a 1-absorbing primary ideal of R/I. Furthermore, assume that if R/I is a quasilocal ring, then $a + I$ is a nonunit element of R/I for every nonunit $a \in R$. If J/I is a 1-absorbing primary ideal of R/I, then J is a 1-absorbing primary ideal of R.

(3) (cf. [11, Corollary 4]) Let R be a commutative ring and $A = R[x]$. Then a proper ideal I of R is a 1-absorbing primary ideal of R if and only if $(I[x]+xA)/xA$ is a 1-absorbing primary ideal of A/xA. (The claim is clear since R is ring-isomorphic to A/xA.)

Note that Example 2.23 shows that the hypothesis in (1) is crucial.

Theorem 2.26. *Let S be a multiplicatively closed subset of a commutative ring R and I be proper ideal of R. Then the following assertions hold:*

(1) *If I is a weakly 1-absorbing primary ideal of R such that $I \cap S = \emptyset$, then $S^{-1}I$ is a weakly 1-absorbing primary ideal of $S^{-1}R$.*

(2) *If $S^{-1}I$ is a weakly 1-absorbing primary ideal of $S^{-1}R$ such that $S \cap Z(R) = \emptyset$ and $S \cap Z_I(R) = \emptyset$, then I is a weakly 1-absorbing primary ideal of R.*

Proof. (1) Suppose that $0 \neq \frac{a}{s_1}\frac{b}{s_2}\frac{c}{s_3} \in S^{-1}I$ for some nonunit elements $a, b, c \in R \backslash S$, $s_1, s_2, s_3 \in S$ and $\frac{a}{s_1}\frac{b}{s_2} \notin S^{-1}I$. Then $0 \neq uabc \in I$ for some $u \in S$. Since I is weakly 1-absorbing primary and $uab \notin I$, we have $c \in \sqrt{I}$. Thus,

$$\frac{c}{s_3} \in S^{-1}\sqrt{I} = \sqrt{S^{-1}I}.$$

So $S^{-1}I$ is a weakly 1-absorbing primary ideal of $S^{-1}R$.

(2) Suppose that $0 \neq abc \in I$ for some nonunit elements $a, b, c \in R$. Hence, $0 \neq \frac{abc}{1} = \frac{a}{1}\frac{b}{1}\frac{c}{1} \in S^{-1}I$ as $S \cap Z(R) = \emptyset$. Since $S^{-1}I$ is weakly 1-absorbing primary, either $\frac{a}{1}\frac{b}{1} \in S^{-1}I$ or $\frac{c}{1} \in \sqrt{S^{-1}I} = S^{-1}\sqrt{I}$. If $\frac{a}{1}\frac{b}{1} \in S^{-1}I$, then $uab \in I$ for some $u \in S$. Since $S \cap Z_I(R) = \emptyset$, we conclude that $ab \in I$. If $\frac{c}{1} \in S^{-1}\sqrt{I}$, then $(tc)^n \in I$ for some positive integer $n \geq 1$ and $t \in S$. Since $t^n \notin Z_I(R)$, we have $c^n \in I$, i.e., $c \in \sqrt{I}$. Thus, I is a weakly 1-absorbing primary ideal of R. □

Definition 2.27. Let I be a weakly 1-absorbing primary ideal of a commutative ring R and $I_1I_2I_3 \subseteq I$ for some proper ideals I_1, I_2, I_3 of R. If (a, b, c) is not a 1-triple-zero of I for any $a \in I_1$, $b \in I_2$, $c \in I_3$, then we call I a *free* 1-triple-zero with respect to $I_1I_2I_3$.

Theorem 2.28. *Let I be a weakly 1-absorbing primary ideal of a commutative ring R and J be a proper ideal of R with $abJ \subseteq I$ for some $a, b \in R$. If (a, b, j) is not a 1-triple-zero of I for any $j \in J$ and $ab \notin I$, then $J \subseteq \sqrt{I}$.*

Proof. Suppose that $J \nsubseteq \sqrt{I}$. Then there exists $c \in J \backslash \sqrt{I}$. Thus, $abc \in abJ \subseteq I$. If $abc \neq 0$, then it contradicts our assumption that $ab \notin I$ and $c \notin \sqrt{I}$. So $abc = 0$. Since (a, b, c) is not a 1-triple-zero of I and $ab \notin I$, we conclude that $c \in \sqrt{I}$, a contradiction. Hence, $J \subseteq \sqrt{I}$. □

Theorem 2.29. *Let I be a weakly 1-absorbing primary ideal of a commutative ring R and $\{0\} \neq I_1I_2I_3 \subseteq I$ for some proper ideals I_1, I_2, I_3 of R. If I is a free 1-triple-zero with respect to $I_1I_2I_3$, then $I_1I_2 \subseteq I$ or $I_3 \subseteq \sqrt{I}$.*

Proof. Let I be a free 1-triple-zero with respect to $I_1I_2I_3$, and $\{0\} \neq I_1I_2I_3 \subseteq I$. Assume that $I_1I_2 \nsubseteq I$. Then there exist $a \in I_1$ and $b \in I_2$ such that $ab \notin I$. Since I is a free 1-triple-zero with respect to $I_1I_2I_3$, we conclude that (a, b, c) is not a 1-triple-zero of I for any $c \in I_3$. Thus, $I_3 \subseteq \sqrt{I}$ by Theorem 2.28. □

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