Name————————————, **ID** ——————

MTH 205 Differential Equations Fall 2020, 1–261

MTH205-Course Portfolio-Fall 2020

Ayman Badawi

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1 Section : Course Syllabus

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COURSE SYLLABUS

Warning: During this difficult time, "trust" relationship between students and instructor will definitely facilitate our work, to ensure that this "trust" is not violated, suspicious Respondus reports (after exams) will be sent to the Associate Dean.

A Course Title & Number

Differential Equations – MTH205

B Pre/Co-requisite(s)

Pre-requisite: MTH104 (Calculus II)

C Number of credits

3

D Faculty Name

Ayman Badawi

E Term/ Year

Fall 2020

F Sections

Course No.	Sec. No.	Room	Days	Start	End
11103 – MTH205	07	Online	UTR	13:00	13:50

G Instructor Information

Instructor	Office	Telephone	Email
Ayman Badawi	**NAB 262**		**abadawi@aus.edu**

Office Hours: **UTR: 15:00 – 16** **or** **by appointment (send me an EMAIL)**

H Course Description from Catalog

Covers mathematical formulation of ordinary differential equations, methods of solution and applications of first order and second order differential equations, power series solutions, solutions by Laplace transforms and solutions of first order linear systems.

I Course Learning Outcomes

Upon completion of the course, students will be able to:

- Explain basic definitions, concepts, vocabulary, and mathematical notation of differential equations. Exam one and Final Exam
- Demonstrate the necessary manipulative skills (usually Algebra Skills) required to solve equations of first order and higher-order constant-coefficient linear differential equations. First Exam and Final Exam
- Demonstrate the necessary manipulative skills (usually Algebra Skills) required to find particular solutions of second order differential equations. Exam Two and Final Exam
- Apply Laplace transform to solve IVPs and systems of linear differential equations. First Exam and Final Exam
- Understand the fundamental properties of power series, and how to use them to solve linear differential equations with variable coefficients. Final Exam
- Formulate and give reasonable approximation solutions to applied physical problems arising in science and engineering. Exam Two and Final Exam

J Textbook, Instructional Material, and Resources

- **MAIN : CLASS NOTES, My personal webpage (old exams, quizzes)** http://ayman-badawi.com/MTH%20205.html
- **Problems with solutions for each section will be posted on I-Learn**
- **(Optional) Zill D.G.,** ***A First Course in Differential Equations with Modeling and Applications, International Metric Edition***, 11th ed,, 2018, CENGAGE Learning Custom Publishing.

- **(Optional) WebAssign:** To purchase the access code and get the discount, you need the following details:
  Cengage Brain URL : https://login.cengagebrain.co.uk/cb/
  Product ISBN : 9781337786911 (← click here)
  Discount Code : **MEBACKTOUNIVERISTY25**

K Teaching and Learning Methodologies

This is a traditional lecture based course. Students are tested and given feedback throughout the semester via regular homework, quizzes, and exams

L Grading Scale, Grading Distribution, and Due Dates

Grading Scale

[92 , 100]	**4.0**	**A**	**[72 , 77)**	**2.3**	**C**
[89 , 92)	**3.7**	**A-**	**[66, 72)**	**2.0**	**C**
[85 , 89)	**3.3**	**B+**	**[62 , 66)**	**1.7**	**C-**
[81 , 85)	**3.0**	**B**	**[50 , 62)**	**1.0**	**D**
[77 , 81)	**2.7**	**B-**	**[0 , 50)**	**0**	**F**

Grading Distribution

Assessmet	Weight	Date
Quizzes	15%	TBA
Exam 1	25%	**Sunday , Oct 11, 6:00pm - 7:15pm**
Exam 2	25%	**Sunday, Nov 29, 6:00pm - 7:15pm**
Final Exam	35%	**TBA**
Total	100%	

M Explanation of Assessments

There will be quizzes, two midterm tests, and a comprehensive final exam.

- Most quizzes will be pre-announced at least one lecture in advance. No make-up quizzes will be given. However the lowest quiz will not be counted toward your final grade.
- With a valid written excuse and making immediate arrangements with the instructor, a missed exam might be replaced with the grade of the final exam and/or the average grade of all tests (including final) and/or quizzes.
- The final exam is common and comprehensive. The date and time of the final exam will be scheduled by the registrar's office.

N Student Academic Integrity Code Statement

Student must adhere to the Academic Integrity code stated in the 2019-2020 undergraduate catalog

SCHEDULE

Note: ***Tests and other graded assignments due dates are set.*** *No addendum, make-up exams, or extra assignments to improve grades will be given.*

#	WEEK	CHAPTER/SECTIONS	NOTES
1	Week one	**4.1 Notations and Fundamental Theorem of IVP** **7.1 Definition of the Laplace Transform** **7.2 Inverse Transforms and Transforms of Derivative**	
2	Week two	**Continue with 7.2 and solving linear diff. equations using Laplace Transformation**	
3	Week three	**7.4 Derivatives of Transform, Transforms of integrals and Periodic Functions and solving linear diff. equations**	
4	Week four	**7.5 The Dirac Delta Function and solving linear diff. equations**	
5	Week five	**7.6 Solving Systems of Linear Diff Equations**	
6	Week six	**4.3 Homogeneous Linear Equations with Constant Coefficients** **4.7 Cauchy-Euler Equation**	
7	Week seven	**4.4 Undetermined Coefficients – Superposition Approach** **4.6 Variation of Parameters**	
8	Week eight	**4.2 Reduction of Order**	
9	Week nine	**2.3 Linear Equations and Bernoulli Equation** **2.4 Exact Equations**	

10	Week ten	**2.1 Solution Curves Without the Solution** **2.2 Separable Equations**	
11	Week eleven	**2.5 Solutions by Substitution**	
12	Week twelve	**3.1 Applications of First order linear ODE** • **Formulate and give reasonable approximation solutions to applied problems arising in science and engineering.**	
13	Week thirteen	• **Applications of second order diff equation** **Formulate and give reasonable approximation solutions to applied problems arising in science and engineering.**	
14	Week fourteen	**6.1 Review of Power Series** **6.2 Solutions of basic linear diff. equation using the concept of power series**	
15	Week fifteen	**More on first and second linear diff. equations, Bernoulli, Exact, separable, pictures for diff. equations without finding the exact diff. equation**	
	One day or two days (depends!)	**Reviews/Final Exam (Comprehensive)**	

Math 205 Suggested Problems (if you choose to use the textbook)

TEXT: *A First Course in Differential Equations with Modeling Application, by D.G. Zill, 11th Edition.*

Section	Page	Exercises
1.1	10	1-8, 12, 15, 19, 27, 32
1.2	17	4, 8, 14, 17, 18, 23, 24, 25, 27
1.3	28	1, 5, 13, 14, 17
2.1	43	1, 9, 13, 21, 22, 25, 27, 29
2.2	51	3, 6, 7, 8, 13, 14, 17, 25, 27, 30, 36(a)
2.3	61	5, 9, 12, 13, 17, 23, 24, 25, 28, 29, 31
2.4	69	2, 3, 6, 8, 12, 16, 24, 32, 35, 37
2.5	74	3, 5, 8, 11, 15, 18, 22, 23, 25, 28
3.1	90	1, 3, 6, 7, 14, 15, 23, 26, 27
4.1	127	1, 3, 5, 6, 9, 13, 15, 17, 19, 23, 26, 31, 36, 38, 40
4.2	131	2, 3, 9, 11, 17
4.3	137	3, 5, 11, 15, 16, 22, 23, 24, 31, 33, 43-48, 56, 57, 59
4.4	147	1, 5, 8, 11, 13, 15, 19, 20, 24, 26, 29, 32, 45
4.6	161	1, 3, 9, 15, 19, 25
4.7	168	1, 3, 4, 5, 6, 11, 14, 15, 17, 19, 29, 45
5.1	205	1, 2, 4, 5, 9, 11, 17-20, 21, 23, 29, 31, 45, 47
6.1	237	23, 24, 25, 27, 29, 31,33
6.2	246	1, 2, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21
7.1	280	4, 13, 15, 18, 21, 25, 29, 31,33, 37
7.2	288	2, 3, 7, 9, 11, 15, 19, 24, 33, 34, 36, 39
7.3	297	1, 3, 6, 7, 15, 22, 23, 26, 29, 37, 39, 43, 45, 47, 49, 51, 54, 55, 58 63, 65
7.4	309	1, 5, 7, 8, 11, 23, 25, 27, 29, 37, 39, 41, 45, 49, 51
7.5	315	1, 3, 6, 10
7.6	319	1, 3, 6, 7, 9, 12

2 Academic Integrity Measures

Academic Integrity Measures in Online Exams

List the measures taken to ensure the academic integrity of the exam.

Quizzes 1-6, all students were in the lecture room (blackboard Ultra room). All students had 20-25 minutes. All questions are essay. Students submitted their solution in a folder that I created on I-learn.

Students used lockdown browser for exams one, two and final exam. All questions are essay. Students submitted their solution in a folder that I created on I-learn. The outcome (scores) was not significantly different from a normal in-class exams (see the scores of the students in the excel-sheet)

I am completely satisfied with the outcome of MTH205.

3 Section : Instructor Teaching Material-Handouts

3.1 Questions with Solutions on Chapter 7.1 (Find Laplace)

19. $\mathscr{L}\{2t^4\} = 2\frac{4!}{s^5}$

20. $\mathscr{L}\{t^5\} = \frac{5!}{s^6}$

21. $\mathscr{L}\{4t - 10\} = \frac{4}{s^2} - \frac{10}{s}$

22. $\mathscr{L}\{7t + 3\} = \frac{7}{s^2} + \frac{3}{s}$

23. $\mathscr{L}\{t^2 + 6t - 3\} = \frac{2}{s^3} + \frac{6}{s^2} - \frac{3}{s}$

24. $\mathscr{L}\{-4t^2 + 16t + 9\} = -4\frac{2}{s^3} + \frac{16}{s^2} + \frac{9}{s}$

25. $\mathscr{L}\{t^3 + 3t^2 + 3t + 1\} = \frac{3!}{s^4} + 3\frac{2}{s^3} + \frac{3}{s^2} + \frac{1}{s}$

26. $\mathscr{L}\{8t^3 - 12t^2 + 6t - 1\} = 8\frac{3!}{s^4} - 12\frac{2}{s^3} + \frac{6}{s^2} -$

27. $\mathscr{L}\{1 + e^{4t}\} = \frac{1}{s} + \frac{1}{s-4}$

28. $\mathscr{L}\{t^2 - e^{-9t} + 5\} = \frac{2}{s^3} - \frac{1}{s+9} + \frac{5}{s}$

29. $\mathscr{L}\{1 + 2e^{2t} + e^{4t}\} = \frac{1}{s} + \frac{2}{s-2} + \frac{1}{s-4}$

30. $\mathscr{L}\{e^{2t} - 2 + e^{-2t}\} = \frac{1}{s-2} - \frac{2}{s} + \frac{1}{s+2}$

31. $\mathscr{L}\{4t^2 - 5\sin 3t\} = 4\frac{2}{s^3} - 5\frac{3}{s^2+9}$

32. $\mathscr{L}\{\cos 5t + \sin 2t\} = \frac{s}{s^2+25} + \frac{2}{s^2+4}$

33. $\mathscr{L}\{\sinh kt\} = \frac{1}{2}\mathscr{L}\{e^{kt} - e^{-kt}\} = \frac{1}{2}\left[\frac{1}{s-k} - \frac{1}{s+k}\right] = \frac{k}{s^2-k^2}$

34. $\mathscr{L}\{\cosh kt\} = \frac{1}{2}\mathscr{L}\{e^{kt} + e^{kt}\} = \frac{s}{s^2-k^2}$

35. $\mathscr{L}\{e^t \sinh t\} = \mathscr{L}\left\{e^t\,\frac{e^t - e^{-t}}{2}\right\} = \mathscr{L}\left\{\frac{1}{2}e^{2t} - \frac{1}{2}\right\} = \frac{1}{2(s-2)} - \frac{1}{2s}$

36. $\mathscr{L}\{e^{-t}\cosh t\} = \mathscr{L}\left\{e^{-t}\,\frac{e^t + e^{-t}}{2}\right\} = \mathscr{L}\left\{\frac{1}{2} + \frac{1}{2}e^{-2t}\right\} = \frac{1}{2s} + \frac{1}{2(s+2)}$

3.2 Questions with Solutions on Chapter 7.2 (Find Laplace Inverse)

Exercises 7.2

Inverse Transforms and Transforms of Derivatives

1. $\mathscr{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{2}{s^3}\right\} = \frac{1}{2}t^2$

2. $\mathscr{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{1}{6}\mathscr{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \frac{1}{6}t^3$

3. $\mathscr{L}^{-1}\left\{\frac{1}{s^2} - \frac{48}{s^5}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^2} - \frac{48}{24}\cdot\frac{4!}{s^5}\right\} = t - 2t^4$

4. $\mathscr{L}^{-1}\left\{\left(\frac{2}{s} - \frac{1}{s^3}\right)^2\right\} = \mathscr{L}^{-1}\left\{4\cdot\frac{1}{s^2} - \frac{4}{6}\cdot\frac{3!}{s^4} + \frac{1}{120}\cdot\frac{5!}{s^6}\right\} = 4t - \frac{2}{3}t^3 + \frac{1}{120}t^5$

5. $\mathscr{L}^{-1}\left\{\frac{(s+1)^3}{s^4}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s} + 3\cdot\frac{1}{s^2} + \frac{3}{2}\cdot\frac{2}{s^3} + \frac{1}{6}\cdot\frac{3!}{s^4}\right\} = 1 + 3t + \frac{3}{2}t^2 + \frac{1}{6}t^3$

6. $\mathscr{L}^{-1}\left\{\frac{(s+2)^2}{s^3}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s} + 4\cdot\frac{1}{s^2} + 2\cdot\frac{2}{s^3}\right\} = 1 + 4t + 2t^2$

7. $\mathscr{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s-2}\right\} = t - 1 + e^{2t}$

8. $\mathscr{L}^{-1}\left\{\frac{4}{s} + \frac{6}{s^5} - \frac{1}{s+8}\right\} = \mathscr{L}^{-1}\left\{4\cdot\frac{1}{s} + \frac{1}{4}\cdot\frac{4!}{s^5} - \frac{1}{s+8}\right\} = 4 + \frac{1}{4}t^4 - e^{-8t}$

9. $\mathscr{L}^{-1}\left\{\frac{1}{4s+1}\right\} = \frac{1}{4}\mathscr{L}^{-1}\left\{\frac{1}{s+1/4}\right\} = \frac{1}{4}e^{-t/4}$

10. $\mathscr{L}^{-1}\left\{\frac{1}{5s-2}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{5}\cdot\frac{1}{s-2/5}\right\} = \frac{1}{5}e^{2t/5}$

11. $\mathscr{L}^{-1}\left\{\frac{5}{s^2+49}\right\} = \mathscr{L}^{-1}\left\{\frac{5}{7}\cdot\frac{7}{s^2+49}\right\} = \frac{5}{7}\sin 7t$

12. $\mathscr{L}^{-1}\left\{\frac{10s}{s^2+16}\right\} = 10\cos 4t$

13. $\mathscr{L}^{-1}\left\{\frac{4s}{4s^2+1}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+1/4}\right\} = \cos\frac{1}{2}t$

14. $\mathscr{L}^{-1}\left\{\frac{1}{4s^2+1}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{2}\cdot\frac{1/2}{s^2+1/4}\right\} = \frac{1}{2}\sin\frac{1}{2}t$

Exercises 7.2 Inverse Transforms and Transforms of Derivatives

15. $\mathscr{L}^{-1}\left\{\frac{2s-6}{s^2+9}\right\} = \mathscr{L}^{-1}\left\{2\cdot\frac{s}{s^2+9} - 2\cdot\frac{3}{s^2+9}\right\} = 2\cos 3t - 2\sin 3t$

16. $\mathscr{L}^{-1}\left\{\frac{s+1}{s^2+2}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+2} + \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{s^2+2}\right\} = \cos\sqrt{2}t + \frac{\sqrt{2}}{2}\sin\sqrt{2}\,t$

17. $\mathscr{L}^{-1}\left\{\frac{1}{s^2+3s}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{3}\cdot\frac{1}{s} - \frac{1}{3}\cdot\frac{1}{s+3}\right\} = \frac{1}{3} - \frac{1}{3}e^{-3t}$

use partial fraction, /(s^2 + 3) = a/s +b/(s+3)
find a, b by cover method

18. $\mathscr{L}^{-1}\left\{\frac{s+1}{s^2-4s}\right\} = \mathscr{L}^{-1}\left\{-\frac{1}{4}\cdot\frac{1}{s} + \frac{5}{4}\cdot\frac{1}{s-4}\right\} = -\frac{1}{4} + \frac{5}{4}e^{4t}$

use partial fraction

19. $\mathscr{L}^{-1}\left\{\frac{s}{s^2+2s-3}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{4}\cdot\frac{1}{s-1} + \frac{3}{4}\cdot\frac{1}{s+3}\right\} = \frac{1}{4}e^{t} + \frac{3}{4}e^{-3t}$

use partial fraction

20. $\mathscr{L}^{-1}\left\{\frac{1}{s^2+s-20}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{9}\cdot\frac{1}{s-4} - \frac{1}{9}\cdot\frac{1}{s+5}\right\} = \frac{1}{9}e^{4t} - \frac{1}{9}e^{-5t}$

use partial fraction from 20 to 24
and cover method.

21. $\mathscr{L}^{-1}\left\{\frac{0.9s}{(s-0.1)(s+0.2)}\right\} = \mathscr{L}^{-1}\left\{(0.3)\cdot\frac{1}{s-0.1} + (0.6)\cdot\frac{1}{s+0.2}\right\} = 0.3e^{0.1t} + 0.6e^{-0.2t}$

22. $\mathscr{L}^{-1}\left\{\frac{s-3}{(s-\sqrt{3})(s+\sqrt{3})}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2-3} - \sqrt{3}\cdot\frac{\sqrt{3}}{s^2-3}\right\} = \cosh\sqrt{3}\,t - \sqrt{3}\sinh\sqrt{3}\,t$

23. $\mathscr{L}^{-1}\left\{\frac{s}{(s-2)(s-3)(s-6)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{2}\cdot\frac{1}{s-2} - \frac{1}{s-3} + \frac{1}{2}\cdot\frac{1}{s-6}\right\} = \frac{1}{2}e^{2t} - e^{3t} + \frac{1}{2}e^{6t}$

24. $\mathscr{L}^{-1}\left\{\frac{s^2+1}{s(s-1)(s+1)(s-2)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{s-1} - \frac{1}{3}\cdot\frac{1}{s+1} + \frac{5}{6}\cdot\frac{1}{s-2}\right\}$

$$= \frac{1}{2} - e^{t} - \frac{1}{3}e^{-t} + \frac{5}{6}e^{2t}$$

3.3 Questions with Solutions on Chapter 7.2 (Solving IVP using Laplace)

Solve y' - y = 1 , y (0) = 1

NOTE: Instead of writing Y(s), the author kept it as L{y(t)}
We know L{y(t)} = Y(s).

$$s\,\mathscr{L}\{y\} - y(0) - \mathscr{L}\{y\} = \frac{1}{s}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = -\frac{1}{s} + \frac{1}{s-1}.$$

Thus

$$y = -1 + e^t.$$

2y' + y = 0, y(0) = 3

$$2s\,\mathscr{L}\{y\} - 2y(0) + \mathscr{L}\{y\} = 0.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{6}{2s+1} = \frac{3}{s+1/2}.$$

Thus

$$y = 3e^{-t/2}.$$

y' + 6y = e^{4t}, y(0) = 2

$$s\mathscr{L}\{y\} - y(0) + 6\mathscr{L}\{y\} = \frac{1}{s-4}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1}{(s-4)(s+6)} + \frac{2}{s+6} = \frac{1}{10}\cdot\frac{1}{s-4} + \frac{19}{10}\cdot\frac{1}{s+6}.$$

Thus

$$y = \frac{1}{10}e^{4t} + \frac{19}{10}e^{-6t}.$$

y'' + 5y' + 4y = 0, y(0) = 1, y'(0) = 0

$\overline{25}$.

35.

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) + 5\left[s\,\mathscr{L}\{y\} - y(0)\right] + 4\,\mathscr{L}\{y\} = 0.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{s+5}{s^2+5s+4} = \frac{4}{3}\,\frac{1}{s+1} - \frac{1}{3}\,\frac{1}{s+4}.$$

Thus

$$y = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}.$$

36. y'' - 4y' = 6e^{3t} - 3e^{-t}, y(0) = 1, y'(0) = -1

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) - 4\left[s\,\mathscr{L}\{y\} - y(0)\right] = \frac{6}{s-3} - \frac{3}{s+1}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\begin{aligned}\mathscr{L}\{y\} &= \frac{6}{(s-3)(s^2-4s)} - \frac{3}{(s+1)(s^2-4s)} + \frac{s-5}{s^2-4s}\\ &= \frac{5}{2}\cdot\frac{1}{s} - \frac{2}{s-3} - \frac{3}{5}\cdot\frac{1}{s+1} + \frac{11}{10}\cdot\frac{1}{s-4}.\end{aligned}$$

Thus

$$y = \frac{5}{2} - 2e^{3t} - \frac{3}{5}e^{-t} + \frac{11}{10}e^{4t}.$$

3.4 Questions with Solutions More-Questions-Solutions-Laplace-IVP

Name [redacted], ID [redacted]

Quiz 2, MTH 205, Fall 2019

Ayman Badawi

$\frac{20}{20}$

QUESTION 1. (i) $\ell^{-1}\{\frac{e^{-s}}{(s-4)^2}\}$

$\ell^{-1}\left\{\frac{1}{(S-4)^2}\right\} = e^{4t} \cdot t$

$\boxed{\ell^{-1}\left\{\frac{e^{-s}}{(S-4)^2}\right\} = \left(e^{4(t-1)}(t-1)\right) U(t-1)}$ $\;3/3$

$\downarrow U_1$

$\ell^{-1}\left\{\frac{1}{(S-4)^2}\right\} = te^{4t}$

(ii) $\ell^{-1}\{\frac{e^{-s}+2}{s^2+4}\}$ $\;\ell^{-1}\left\{\frac{e^{-S}}{s^2+4}\right\} + \ell^{-1}\left\{\frac{2}{s^2+4}\right\}$

$= \boxed{\frac{1}{2}\sin(2(t-1))\,U(t-1) + \sin 2t}$ $\;3/3$

QUESTION 2. Solve for $y(t)$, $y'' + 6y' + 13y = 0$, where $y(0) = 0$, $y'(0) = 2$.

$\Rightarrow s^2Y(s) - s\cancel{y(0)}^{0} - y'(0) + 6sY(s) - \cancel{6y(0)}^{0} + 13Y(s) = 0$

$\Rightarrow Y(s)(s^2+6s+13) - 2 = 0$ $\quad 3/3$

$\Rightarrow Y(s) = \frac{2}{s^2+6s+13}$

$\Rightarrow \ell^{-1}\left\{\frac{2}{s^2+6s+13}\right\} = y(t)$ $\quad 4/3$

$\Rightarrow \ell^{-1}\left\{\frac{2}{s^2+6s+9-9+13}\right\} = y(t)$ $\quad \boxed{K=2}$

$\Rightarrow \ell^{-1}\left\{\frac{2}{(S+3)^2+4}\right\} = y(t)$

$\Rightarrow \boxed{y(t) = e^{-3t}\sin 2t}$

1. $\mathscr{L}\left\{te^{10t}\right\} = \dfrac{1}{(s-10)^2}$

2. $\mathscr{L}\left\{te^{-6t}\right\} = \dfrac{1}{(s+6)^2}$

3. $\mathscr{L}\left\{t^3e^{-2t}\right\} = \dfrac{3!}{(s+2)^4}$

4. $\mathscr{L}\left\{t^{10}e^{-7t}\right\} = \dfrac{10!}{(s+7)^{11}}$

5. $\mathscr{L}\left\{t\left(e^t+e^{2t}\right)^2\right\} = \mathscr{L}\left\{te^{2t}+2te^{3t}+te^{4t}\right\} = \dfrac{1}{(s-2)^2} + \dfrac{2}{(s-3)^2} + \dfrac{1}{(s-4)^2}$

6. $\mathscr{L}\left\{e^{2t}(t-1)^2\right\} = \mathscr{L}\left\{t^2e^{2t}-2te^{2t}+e^{2t}\right\} = \dfrac{2}{(s-2)^3} - \dfrac{2}{(s-2)^2} + \dfrac{1}{s-2}$

7. $\mathscr{L}\left\{e^t\sin 3t\right\} = \dfrac{3}{(s-1)^2+9}$

8. $\mathscr{L}\left\{e^{-2t}\cos 4t\right\} = \dfrac{s+2}{(s+2)^2+16}$

9. $\mathscr{L}\{(1-e^t+3e^{-4t})\cos 5t\} = \mathscr{L}\{\cos 5t - e^t\cos 5t + 3e^{-4t}\cos 5t\}$

$$= \frac{s}{s^2+25} - \frac{s-1}{(s-1)^2+25} + \frac{3(s+4)}{(s+4)^2+25}$$

10. $\mathscr{L}\left\{e^{3t}\left(9-4t+10\sin\frac{t}{2}\right)\right\} = \mathscr{L}\left\{9e^{3t} - 4te^{3t} + 10e^{3t}\sin\frac{t}{2}\right\} = \frac{9}{s-3} - \frac{4}{(s-3)^2} + \frac{5}{(s-3)^2+1/}$

11. $\mathscr{L}^{-1}\left\{\frac{1}{(s+2)^3}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{2}\frac{2}{(s+2)^3}\right\} = \frac{1}{2}t^2e^{-2t}$

12. $\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^4}\right\} = \frac{1}{6}\mathscr{L}^{-1}\left\{\frac{3!}{(s-1)^4}\right\} = \frac{1}{6}t^3e^t$

13. $\mathscr{L}^{-1}\left\{\frac{1}{s^2-6s+10}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{(s-3)^2+1^2}\right\} = e^{3t}\sin t$

14. $\mathscr{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{2}\frac{2}{(s+1)^2+2^2}\right\} = \frac{1}{2}e^{-t}\sin 2t$

15. $\mathscr{L}^{-1}\left\{\frac{s}{s^2+4s+5}\right\} = \mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1^2} - 2\frac{1}{(s+2)^2+1^2}\right\} = e^{-2t}\cos t - 2e^{-2t}\sin t$

16. $\mathscr{L}^{-1}\left\{\frac{2s+5}{s^2+6s+34}\right\} = \mathscr{L}^{-1}\left\{2\frac{(s+3)}{(s+3)^2+5^2} - \frac{1}{5}\frac{5}{(s+3)^2+5^2}\right\} = 2e^{-3t}\cos 5t - \frac{1}{5}e^{-3t}\sin 5t$

17. $\mathscr{L}^{-1}\left\{\frac{s}{(s+1)^2}\right\} = \mathscr{L}^{-1}\left\{\frac{s+1-1}{(s+1)^2}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{(s+1)^2}\right\} = e^{-t} - te^{-t}$

18. $\mathscr{L}^{-1}\left\{\frac{5s}{(s-2)^2}\right\} = \mathscr{L}^{-1}\left\{\frac{5(s-2)+10}{(s-2)^2}\right\} = \mathscr{L}^{-1}\left\{\frac{5}{s-2} + \frac{10}{(s-2)^2}\right\} = 5e^{2t} + 10te^{2t}$

find y(t), where y^\ + 4y = e^{-4t}, y(0) = 2.

21. The Laplace transform of the differential equation is

$$s\mathscr{L}\{y\} - y(0) + 4\mathscr{L}\{y\} = \frac{1}{s+4}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1}{(s+4)^2} + \frac{2}{s+4}.$$

Thus

$$y = te^{-4t} + 2e^{-4t}.$$

37. $\mathscr{L}\{(t-1)\mathscr{U}(t-1)\} = \dfrac{e^{-s}}{s^2}$

38. $\mathscr{L}\{e^{2-t}\mathscr{U}(t-2)\} = \mathscr{L}\left\{e^{-(t-2)}\mathscr{U}(t-2)\right\} = \dfrac{e^{-2s}}{s+1}$

39. $\mathscr{L}\{t\mathscr{U}(t-2)\} = \mathscr{L}\{(t-2)\mathscr{U}(t-2) + 2\mathscr{U}(t-2)\} = \dfrac{e^{-2s}}{s^2} + \dfrac{2e^{-2s}}{s}$
Alternatively, (16) of this section in the text could be used:

$$\mathscr{L}\{t\,\mathscr{U}(t-2)\} = e^{-2s}\,\mathscr{L}\{t+2\} = e^{-2s}\left(\frac{1}{s^2}+\frac{2}{s}\right).$$

40. $\mathscr{L}\{(3t+1)\mathscr{U}(t-1)\} = 3\,\mathscr{L}\{(t-1)\mathscr{U}(t-1)\} + 4\,\mathscr{L}\{\mathscr{U}(t-1)\} = \dfrac{3e^{-s}}{s^2} + \dfrac{4e^{-s}}{s}$
Alternatively, (16) of this section in the text could be used:

$$\mathscr{L}\{(3t+1)\,\mathscr{U}(t-1)\} = e^{-s}\,\mathscr{L}\{3t+4\} = e^{-s}\left(\frac{3}{s^2}+\frac{4}{s}\right).$$

41. $\mathscr{L}\{\cos 2t\,\mathscr{U}(t-\pi)\} = \mathscr{L}\{\cos 2(t-\pi)\,\mathscr{U}(t-\pi)\} = \dfrac{se^{-\pi s}}{s^2+4}$
Alternatively, (16) of this section in the text could be used:

$$\mathscr{L}\{\cos 2t\,\mathscr{U}(t-\pi)\} = e^{-\pi s}\,\mathscr{L}\{\cos 2(t+\pi)\} = e^{-\pi s}\,\mathscr{L}\{\cos 2t\} = e^{-\pi s}\,\frac{s}{s^2+4}.$$

42. $\mathscr{L}\left\{\sin t\,\mathscr{U}\left(t-\dfrac{\pi}{2}\right)\right\} = \mathscr{L}\left\{\cos\left(t-\dfrac{\pi}{2}\right)\mathscr{U}\left(t-\dfrac{\pi}{2}\right)\right\} = \dfrac{se^{-\pi s/2}}{s^2+1}$
Alternatively, (16) of this section in the text could be used:

$$\mathscr{L}\left\{\sin t\,\mathscr{U}\left(t-\frac{\pi}{2}\right)\right\} = e^{-\pi s/2}\,\mathscr{L}\left\{\sin\left(t+\frac{\pi}{2}\right)\right\} = e^{-\pi s/2}\,\mathscr{L}\{\cos t\} = e^{-\pi s/2}\,\frac{s}{s^2+1}.$$

43. $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s^3}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{2}\cdot\dfrac{2}{s^3}e^{-2s}\right\} = \dfrac{1}{2}(t-2)^2\,\mathscr{U}(t-2)$

44. $\mathscr{L}^{-1}\left\{\dfrac{(1+e^{-2s})^2}{s+2}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{s+2}+\dfrac{2e^{-2s}}{s+2}+\dfrac{e^{-4s}}{s+2}\right\} = e^{-2t}+2e^{-2(t-2)}\,\mathscr{U}(t-2)+e^{-2(t-4)}$

45. $\mathscr{L}^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+1}\right\} = \sin(t-\pi)\,\mathscr{U}(t-\pi) = -\sin t\,\mathscr{U}(t-\pi)$

Name ________, ID ________

MTH 205 Differential Equations Fall 2019, 1–6

Nadeen Tarek

Exam I, MTH 205, Fall 2019

Ayman Badawi

Total = $\frac{80}{80}$

QUESTION 1. (12 points)

(i) $\ell^{-1}\{\frac{s}{(s+3)^4}\}$

Aerr $\ell\{\ \} = \frac{1}{2}e^{-3t}t^2 - \frac{1}{2}e^{-3t}t$

Note $\frac{s}{(s+3)^4} = \frac{s+3-3}{(s+3)^4} = \frac{1}{(s+3)^3} - \frac{3}{(s+3)^4}$

$\frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{(s+3)^3} + \frac{D}{(s+3)^4} = \frac{s}{(s+3)^4}$

$A(s+3)^3 + B(s+3)^2 + C(s+3) + D = s$

$s=-1$

$8A + 4B + 2C + D = -1$

$\boxed{8A + 4B + 2C = 2}$

$s=-3$

$\boxed{D=-3}$

$s=0$

$27A + 9B + 3C + D = 0$

$\boxed{27A + 9B + 3C = 3}$

$s=1$

$64A + 16B + 4C - 3 = 1$

$\boxed{64A + 16B + 4C = 4}$

$A=0 \quad C=1$

$B=0$

✓

$\frac{1}{(s+3)^3} - \frac{3}{(s+3)^4}$

$= \frac{1}{2}e^{-3t}t^2 - \frac{3}{6}e^{-3t}t^3$

(ii) $\ell^{-1}\{\frac{e^{-2s}}{s^2+4s+13}\}$

$\frac{4}{2} = (2)^2$

$= \frac{1/e^{-2s}}{(s+2)^2+9} = U_2 f(t-2)$

F(s)

$f(t) = \ell^{-1} \frac{1}{(s+2)^2+9} = \frac{1}{3}e^{-2t}\sin(3t)$

$= \boxed{U_2 \frac{1}{3} e^{-2(t-2)} \sin(3(t-2))}$

✓

3.5 Questions with Solutions More-Questions-Solutions-Laplace-IVP

Name [redacted], ID [redacted]

Quiz 2, MTH 205, Fall 2019

Ayman Badawi

20/20

QUESTION 1. (i) $\ell^{-1}\{\frac{e^{-s}}{(s-4)^2}\}$ $\ell^{-1}\left\{\frac{1}{(s-4)^2}\right\} = e^{4t}\cdot t$

$$\boxed{\ell^{-1}\left\{\frac{e^{-s}}{(s-4)^2}\right\} = \left(e^{4(t-1)}(t-1)\right)U(t-1)}$$ ✓ 3/3

U_1

$\ell^{-1}\left\{\frac{1}{(s-4)^2}\right\} = te^{4t}$

(ii) $\ell^{-1}\{\frac{e^{-s}+2}{s^2+4}\}$ $\ell^{-1}\left\{\frac{e^{-s}}{s^2+4}\right\} + \ell^{-1}\left\{\frac{2}{s^2+4}\right\}$

$$= \boxed{\frac{1}{2}\sin(2(t-1))U(t-1) + \sin 2t}$$ ✓ 3/3

QUESTION 2. Solve for $y(t)$, $y'' + 6y' + 13y = 0$, where $y(0) = 0$, $y'(0) = 2$.

$\Rightarrow s^2Y(s) - \cancelto{0}{sy(0)} - y'(0) + 6sY(s) - \cancelto{0}{6y(0)} + 13Y(s) = 0$

$\Rightarrow Y(s)(s^2+6s+13) - 2 = 0$

$\Rightarrow Y(s) = \frac{2}{s^2+6s+13}$

3/3

$\Rightarrow \ell^{-1}\left\{\frac{2}{s^2+6s+13}\right\} = y(t)$

$\Rightarrow \ell^{-1}\left\{\frac{2}{s^2+6s+9-9+13}\right\} = y(t)$

$\Rightarrow \ell^{-1}\left\{\frac{2}{(s+3)^2+4}\right\} = y(t)$

$\Rightarrow \boxed{y(t) = e^{-3t}\sin 2t}$ ✓

4/3

$\boxed{K=2}$

QUESTION 3. Solve for $y(t)$, $y'' - 4y' + 4y = tU_3(t)$, where $y(0) = y'(0) = 0$

$(t-3)U_3(t)$

$$\boxed{(s-2)^2 = s^2 - 4s + 4}$$

$$s^2Y(s) - \cancel{sy(0)}^{0} - \cancel{y'(0)}^{0} - 4sY(s) \cancel{-4y(0)}^{0} + 4Y(s) = \frac{e^{-3s}}{s^2}$$

$$\rightarrow (s^2 - 4s + 4)Y(s) = \frac{e^{-3s}}{s^2}$$

$$\Downarrow$$

$$\boxed{\ell\{(t-3)U(t-3)\} = \frac{1}{s^2}e^{-3s}}$$

3/3

$$\rightarrow Y(s) = \frac{e^{-3s}}{s^2(s-2)^2}$$

$$\rightarrow y(t) = \ell^{-1}\left\{\frac{e^{-3s}}{s^2(s-2)^2}\right\} \checkmark$$

Partial fraction

$$\frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s-2)} + \frac{D}{(s-2)^2}$$

$$\boxed{B = 1/4}, \boxed{D = 1/4}$$

$$As(s^2 - 4s + 4) + B(s^2 - 4s + 4) + Cs^2(s-2) + Ds^2$$

$$As^3 - 4As^2 + 4As + Bs^2 - 4Bs + 4B + Cs^3 - 2Cs^2 + Ds^2$$

$$\boxed{A + C = 0} \; ①$$

$$\boxed{-4A + B - 2C + D = 0} \; ②$$

$$\boxed{4A - 4B = 0} \; ③$$

$$\boxed{4B = 1} \; ④$$

$$\boxed{A = 1/4}, \boxed{C = -1/4}$$

3/3

$$\rightarrow y(t) = \ell^{-1}\left\{\frac{1}{s^2(s-2)^2}\right\}$$

$$\rightarrow = \ell^{-1}\left\{\frac{1/4}{s} + \frac{1/4}{s^2} - \frac{1/4}{(s-2)} + \frac{1/4}{(s-2)^2}\right\}$$

$$= 1/4 + 1/4(t-3) - 1/4e^{2(t-3)} + 1/4(t-3)e^{2(t-3)}$$

$$\boxed{y(t) = U(t-3)\left[1/4 + 1/4(t-3) - 1/4e^{2(t-3)} + 1/4e^{2(t-3)}(t-3)\right]}$$

QUESTION 4. Let $f(t) = \begin{cases} 3 & if\ 2 \le t < 5 \\ 0 & if\ t \ge 5 \end{cases}$. Find $\ell\{f(t)\}$

$$f(t) = (U_2(t) - U_5(t))3 + \cancel{[U_5(t)]0}$$

$$f(t) = 3U_2 - 3U_5$$

$$\ell\{f(t)\} = \frac{3}{s}e^{-2s} - \frac{3}{s}e^{-5s}$$

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1/1 ✓

$$\boxed{\ell\{f(t)\} = \frac{3}{s}(e^{-2s} - e^{-5s})}$$

QUESTION 2. Find $y(t)$, where $y^{(2)} - 4y(t) = 4U_4(t)sin(2t - 8)$, $y(0) = y'(0) = 0$ (note $U_4(t) = U(t-4)$))

$s^2Y(s) - \underset{0}{\cancel{sy(0)}} - \underset{0}{\cancel{y'(0)}} - 4Y(s) = \frac{8e^{-4s}}{s^2+4}$

$Y(s)\,(s^2-4) = \frac{8e^{-4s}}{s^2+4}$

$Y(s) = 8e^{-4s}\,\boxed{\frac{1}{(s^2-4)(s^2+4)}}\ F(s)$ $\quad 8\,f(t-4)\,U_4$

$F(s) = \frac{1}{(s^2-4)(s^2+4)}$

$F(s) = \mathcal{L}^{-1}\,\frac{1}{8}\left[\frac{1}{s^2-4} - \frac{1}{s^2+4}\right]$

$f(t)\,\cancel{F(s)} = \frac{1}{8*2}\sinh(2t) - \frac{1}{8*2}\sin(2t) = \frac{1}{16}\left[\sinh(2t) - \sin(2t)\right]$

$y(t) = \frac{8}{16}\left[\sinh(2(t-4)) - \sin(2(t-4))\right]U_4$

$= \frac{1}{2}\left[\sinh(2(t-4)) - \sin(2(t-4))\right]U_4$ ✓ 6/6

$\mathcal{L}\{4U_4\sin(2t-8)\}$

$4[e^{-4s}\mathcal{L}\{\sin(2(t+4)-8$

$4[e^{-4s}\mathcal{L}\{\sin(2t)]$

$4[e^{-4s}\frac{2}{s^2+4}]$

$\cancel{s^2}+4-\cancel{s^2}+4$

$\frac{1}{s^2-4} - \frac{1}{s^2+4} = \frac{8}{s^2-4}$

QUESTION 3. Find $y(t)$, where $y' - 2y(t) = 2^t$, $y(0) = 0$

$sY(s) - \underset{0}{\cancel{y(0)}} - 2Y(s) = \frac{1}{s-\ln 2}$

$\cancel{sY}\ Y(s)\,(s-2) = \frac{1}{s-\ln 2}$

$Y(s) = \frac{1}{(s-\ln 2)(s-2)}$

$\mathcal{L}^{-1}\,Y(s) = \mathcal{L}^{-1}\,\frac{1}{-2+\ln 2}\left[\frac{1}{s-\ln 2} - \frac{1}{s-2}\right]$

$= \frac{1}{-2+\ln 2}\left[e^{\ln 2\,t} - e^{2t}\right]$

$e^{t\ln 2}$ $\quad \mathcal{L}\{e^{\ln 2\,t}$

$= \frac{1}{s-\ln 2}$

$\cancel{s}-2\ \cancel{-s}+\ln 2$

$\frac{1}{s-\ln 2} - \frac{1}{s-2} = \frac{-2+\ln 2}{}$

X/X

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In Problems 21–30 use the Laplace transform to solve the given initial-value problem.

21. $y' + 4y = e^{-4t}, \quad y(0) = 2$

22. $y' - y = 1 + te^{t}, \quad y(0) = 0$

23. $y'' + 2y' + y = 0, \quad y(0) = 1, y'(0) = 1$

24. $y'' - 4y' + 4y = t^3e^{2t}, \quad y(0) = 0, y'(0) = 0$

25. $y'' - 6y' + 9y = t, \quad y(0) = 0, y'(0) = 1$

26. $y'' - 4y' + 4y = t^3, \quad y(0) = 1, y'(0) = 0$

27. $y'' - 6y' + 13y = 0, \quad y(0) = 0, y'(0) = -3$

28. $2y'' + 20y' + 51y = 0, \quad y(0) = 2, y'(0) = 0$

29. $y'' - y' = e^{t}\cos t, \quad y(0) = 0, y'(0) = 0$

30. $y'' - 2y' + 5y = 1 + t, \quad y(0) = 0, y'(0) = 4$

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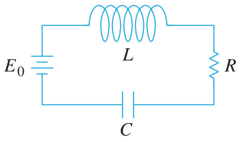


FIGURE 7.3.9 Series circuit in Problem 35

36. Use the Laplace transform to find the charge $q(t)$ in an RC series circuit when $q(0) = 0$ and $E(t) = E_0e^{-kt}$, $k > 0$. Consider two cases: $k \neq 1/RC$ and $k = 1/RC$.

7.3.2 TRANSLATION ON THE *t*-AXIS

In Problems 37–48 find either $F(s)$ or $f(t)$, as indicated.

37. $\mathscr{L}\{(t-1)\,\mathscr{U}(t-1)\}$

38. $\mathscr{L}\{e^{2-t}\,\mathscr{U}(t-2)\}$

39. $\mathscr{L}\{t\,\mathscr{U}(t-2)\}$

40. $\mathscr{L}\{(3t+1)\,\mathscr{U}(t-1)\}$

41. $\mathscr{L}\{\cos 2t\,\mathscr{U}(t-\pi)\}$

42. $\mathscr{L}\left\{\sin t\,\mathscr{U}\left(t-\frac{\pi}{2}\right)\right\}$

43. $\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\}$

44. $\mathscr{L}^{-1}\left\{\frac{(1+e^{-2s})^2}{s+2}\right\}$

45. $\mathscr{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$

46. $\mathscr{L}^{-1}\left\{\frac{se^{-\pi s/2}}{s^2+4}\right\}$

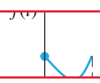
47. $\mathscr{L}^{-1}\left\{\frac{e^{-s}}{s(s+1)}\right\}$

48. $\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s^2(s-1)}\right\}$

[redacted]

[redacted]

53.



[redacted]

it step
ction.

55. $f(t) = \begin{cases} 2, & 0 \le t < 3 \\ -2, & t \ge 3 \end{cases}$

56. $f(t) = \begin{cases} 1, & 0 \le t < 4 \\ 0, & 4 \le t < 5 \\ 1, & t \ge 5 \end{cases}$

57. $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t^2, & t \ge 1 \end{cases}$

In Problems 63–70 use the Laplace transform to solve the given initial-value problem.

63. $y' + y = f(t), \quad y(0) = 0,$ where $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ 5, & t \ge 1 \end{cases}$

64. $y' + y = f(t), \quad y(0) = 0,$ where

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ -1, & t \ge 1 \end{cases}$$

65. $y' + 2y = f(t), \quad y(0) = 0,$ where

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$$

66. $y'' + 4y = f(t), \quad y(0) = 0, y'(0) = -1,$ where

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$$

67. $y'' + 4y = \sin t\, \mathscr{U}(t - 2\pi), \quad y(0) = 1, y'(0) = 0$

68. $y'' - 5y' + 6y = \mathscr{U}(t - 1), \quad y(0) = 0, y'(0) = 1$

69. $y'' + y = f(t), \quad y(0) = 0, y'(0) = 1,$ where

$$f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 1, & \pi \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$$

70. $y'' + 4y' + 3y = 1 - \mathscr{U}(t - 2) - \mathscr{U}(t - 4) + \mathscr{U}(t - 6),$
$y(0) = 0, y'(0) = 0$

22. The Laplace transform of the differential equation is

$$s\,\mathscr{L}\{y\} - \mathscr{L}\{y\} = \frac{1}{s} + \frac{1}{(s-1)^2}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1}{s(s-1)} + \frac{1}{(s-1)^3} = -\frac{1}{s} + \frac{1}{s-1} + \frac{1}{(s-1)^3}\,.$$

Thus

$$y = -1 + e^t + \frac{1}{2}t^2e^t.$$

23. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) + 2[s\,\mathscr{L}\{y\} - y(0)] + \mathscr{L}\{y\} = 0.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{s+3}{(s+1)^2} = \frac{1}{s+1} + \frac{2}{(s+1)^2}\,.$$

Thus

$$y = e^{-t} + 2te^{-t}.$$

24. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) - 4\,[s\,\mathscr{L}\{y\} - y(0)] + 4\,\mathscr{L}\{y\} = \frac{6}{(s-2)^4}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain $\mathscr{L}\{y\} = \dfrac{1}{20}\dfrac{5!}{(s-2)^6}$. Thus, $y = \dfrac{1}{20}t^5e^{2t}$.

25. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) - 6\,[s\,\mathscr{L}\{y\} - y(0)] + 9\,\mathscr{L}\{y\} = \frac{1}{s^2}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1+s^2}{s^2(s-3)^2} = \frac{2}{27}\frac{1}{s} + \frac{1}{9}\frac{1}{s^2} - \frac{2}{27}\frac{1}{s-3} + \frac{10}{9}\frac{1}{(s-3)^2}\,.$$

Thus

$$y = \frac{2}{27} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{10}{9}te^{3t}.$$

26. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) - 4\,[s\,\mathscr{L}\{y\} - y(0)] + 4\,\mathscr{L}\{y\} = \frac{6}{s^4}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{s^5 - 4s^4 + 6}{s^4(s-2)^2} = \frac{3}{4}\frac{1}{s} + \frac{9}{8}\frac{1}{s^2} + \frac{3}{4}\frac{2}{s^3} + \frac{1}{4}\frac{3!}{s^4} + \frac{1}{4}\frac{1}{s-2} - \frac{13}{8}\frac{1}{(s-2)^2}\,.$$

Thus

$$y = \frac{3}{4} + \frac{9}{8}t + \frac{3}{4}t^2 + \frac{1}{4}t^3 + \frac{1}{4}e^{2t} - \frac{13}{8}te^{2t}.$$

27. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) - 6\left[s\,\mathscr{L}\{y\} - y(0)\right] + 13\,\mathscr{L}\{y\} = 0.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = -\frac{3}{s^2 - 6s + 13} = -\frac{3}{2}\,\frac{2}{(s-3)^2 + 2^2}\,.$$

Thus

$$y = -\frac{3}{2}e^{3t}\sin 2t.$$

28. The Laplace transform of the differential equation is

$$2[s^2\mathscr{L}\{y\} - sy(0)] + 20[s\,\mathscr{L}\{y\} - y(0)] + 51\,\mathscr{L}\{y\} = 0.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{4s+40}{2s^2 + 20s + 51} = \frac{2s+20}{(s+5)^2 + 1/2} = \frac{2(s+5)}{(s+5)^2 + 1/2} + \frac{10}{(s+5)^2 + 1/2}\,.$$

Thus

$$y = 2e^{-5t}\cos(t/\sqrt{2}\,) + 10\sqrt{2}\,e^{-5t}\sin(t/\sqrt{2}\,).$$

29. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) - \left[s\,\mathscr{L}\{y\} - y(0)\right] = \frac{s-1}{(s-1)^2 + 1}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1}{s(s^2 - 2s + 2)} = \frac{1}{2}\,\frac{1}{s} - \frac{1}{2}\,\frac{s-1}{(s-1)^2 + 1} + \frac{1}{2}\,\frac{1}{(s-1)^2 + 1}\,.$$

Thus

$$y = \frac{1}{2} - \frac{1}{2}e^t\cos t + \frac{1}{2}e^t\sin t.$$

30. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) - 2\left[s\,\mathscr{L}\{y\} - y(0)\right] + 5\,\mathscr{L}\{y\} = \frac{1}{s} + \frac{1}{s^2}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{4s^2 + s + 1}{s^2(s^2 - 2s + 5)} = \frac{7}{25}\,\frac{1}{s} + \frac{1}{5}\,\frac{1}{s^2} + \frac{-7s/25 + 109/25}{s^2 - 2s + 5}$$

$$= \frac{7}{25}\,\frac{1}{s} + \frac{1}{5}\,\frac{1}{s^2} - \frac{7}{25}\,\frac{s-1}{(s-1)^2 + 2^2} + \frac{51}{25}\,\frac{2}{(s-1)^2 + 2^2}\,.$$

37. $\mathscr{L}\{(t-1)\,\mathscr{U}(t-1)\} = \dfrac{e^{-s}}{s^2}$

38. $\mathscr{L}\{e^{2-t}\,\mathscr{U}(t-2)\} = \mathscr{L}\left\{e^{-(t-2)}\,\mathscr{U}(t-2)\right\} = \dfrac{e^{-2s}}{s+1}$

39. $\mathscr{L}\{t\,\mathscr{U}(t-2)\} = \mathscr{L}\{(t-2)\,\mathscr{U}(t-2) + 2\,\mathscr{U}(t-2)\} = \dfrac{e^{-2s}}{s^2} + \dfrac{2e^{-2s}}{s}$
Alternatively, (16) of this section in the text could be used:

$$\mathscr{L}\{t\,\mathscr{U}(t-2)\} = e^{-2s}\,\mathscr{L}\{t+2\} = e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right).$$

40. $\mathscr{L}\{(3t+1)\,\mathscr{U}(t-1)\} = 3\,\mathscr{L}\{(t-1)\,\mathscr{U}(t-1)\} + 4\,\mathscr{L}\{\mathscr{U}(t-1)\} = \dfrac{3e^{-s}}{s^2} + \dfrac{4e^{-s}}{s}$
Alternatively, (16) of this section in the text could be used:

$$\mathscr{L}\{(3t+1)\,\mathscr{U}(t-1)\} = e^{-s}\,\mathscr{L}\{3t+4\} = e^{-s}\left(\frac{3}{s^2} + \frac{4}{s}\right).$$

41. $\mathscr{L}\{\cos 2t\,\mathscr{U}(t-\pi)\} = \mathscr{L}\{\cos 2(t-\pi)\,\mathscr{U}(t-\pi)\} = \dfrac{se^{-\pi s}}{s^2+4}$
Alternatively, (16) of this section in the text could be used:

$$\mathscr{L}\{\cos 2t\,\mathscr{U}(t-\pi)\} = e^{-\pi s}\,\mathscr{L}\{\cos 2(t+\pi)\} = e^{-\pi s}\,\mathscr{L}\{\cos 2t\} = e^{-\pi s}\,\frac{s}{s^2+4}.$$

42. $\mathscr{L}\left\{\sin t\,\mathscr{U}\left(t-\dfrac{\pi}{2}\right)\right\} = \mathscr{L}\left\{\cos\left(t-\dfrac{\pi}{2}\right)\mathscr{U}\left(t-\dfrac{\pi}{2}\right)\right\} = \dfrac{se^{-\pi s/2}}{s^2+1}$
Alternatively, (16) of this section in the text could be used:

$$\mathscr{L}\left\{\sin t\,\mathscr{U}\left(t-\frac{\pi}{2}\right)\right\} = e^{-\pi s/2}\,\mathscr{L}\left\{\sin\left(t+\frac{\pi}{2}\right)\right\} = e^{-\pi s/2}\,\mathscr{L}\{\cos t\} = e^{-\pi s/2}\,\frac{s}{s^2+1}.$$

43. $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s^3}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{2}\cdot\dfrac{2}{s^3}e^{-2s}\right\} = \dfrac{1}{2}(t-2)^2\,\mathscr{U}(t-2)$

44. $\mathscr{L}^{-1}\left\{\dfrac{(1+e^{-2s})^2}{s+2}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{s+2} + \dfrac{2e^{-2s}}{s+2} + \dfrac{e^{-4s}}{s+2}\right\} = e^{-2t} + 2e^{-2(t-2)}\,\mathscr{U}(t-2) + e^{-2(t-4)}$

45. $\mathscr{L}^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+1}\right\} = \sin(t-\pi)\,\mathscr{U}(t-\pi) = -\sin t\,\mathscr{U}(t-\pi)$

46. $\mathscr{L}^{-1}\left\{\frac{se^{-\pi s/2}}{s^2+4}\right\} = \cos 2\left(t-\frac{\pi}{2}\right)\mathscr{U}\left(t-\frac{\pi}{2}\right) = -\cos 2t\,\mathscr{U}\left(t-\frac{\pi}{2}\right)$

47. $\mathscr{L}^{-1}\left\{\frac{e^{-s}}{s(s+1)}\right\} = \mathscr{L}^{-1}\left\{\frac{e^{-s}}{s} - \frac{e^{-s}}{s+1}\right\} = \mathscr{U}(t-1) - e^{-(t-1)}\mathscr{U}(t-1)$

48. $\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s^2(s-1)}\right\} = \mathscr{L}^{-1}\left\{-\frac{e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{e^{-2s}}{s-1}\right\} = -\mathscr{U}(t-2) - (t-2)\mathscr{U}(t-2) + e^{t-2}\mathscr{U}(t-2)$

49. (c) 50. (e) 51. (f) 52. (b) 53. (a) 54. (d)

55. $\mathscr{L}\{2 - 4\mathscr{U}(t-3)\} = \frac{2}{s} - \frac{4}{s}e^{-3s}$

56. $\mathscr{L}\{1 - \mathscr{U}(t-4) + \mathscr{U}(t-5)\} = \frac{1}{s} - \frac{e^{-4s}}{s} + \frac{e^{-5s}}{s}$

57. $\mathscr{L}\left\{t^2\mathscr{U}(t-1)\right\} = \mathscr{L}\left\{\left[(t-1)^2 + 2t - 1\right]\mathscr{U}(t-1)\right\} = \mathscr{L}\left\{\left[(t-1)^2 + 2(t-1) + 1\right]\mathscr{U}(t-1)\right\}$

$$= \left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)e^{-s}$$

Alternatively, by (16) of this section in the text,

$$\mathscr{L}\{t^2\,\mathscr{U}(t-1)\} = e^{-s}\mathscr{L}\{t^2 + 2t + 1\} = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right).$$

58. $\mathscr{L}\left\{\sin t\,\mathscr{U}\left(t-\frac{3\pi}{2}\right)\right\} = \mathscr{L}\left\{-\cos\left(t-\frac{3\pi}{2}\right)\mathscr{U}\left(t-\frac{3\pi}{2}\right)\right\} = -\frac{se^{-3\pi s/2}}{s^2+1}$

59. $\mathscr{L}\{t - t\,\mathscr{U}(t-2)\} = \mathscr{L}\{t - (t-2)\mathscr{U}(t-2) - 2\mathscr{U}(t-2)\} = \frac{1}{s^2} - \frac{e^{-2s}}{s^2} - \frac{2e^{-2s}}{s}$

60. $\mathscr{L}\{\sin t - \sin t\,\mathscr{U}(t-2\pi)\} = \mathscr{L}\{\sin t - \sin(t-2\pi)\,\mathscr{U}(t-2\pi)\} = \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1}$

61. $\mathscr{L}\{f(t)\} = \mathscr{L}\{\mathscr{U}(t-a) - \mathscr{U}(t-b)\} = \frac{e^{-as}}{s} - \frac{e^{-bs}}{s}$

62. $\mathscr{L}\{f(t)\} = \mathscr{L}\{\mathscr{U}(t-1) + \mathscr{U}(t-2) + \mathscr{U}(t-3) + \cdots\} = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} + \frac{e^{-3s}}{s} + \cdots = \frac{1}{s}\frac{e^{-s}}{1-e^{-s}}$

63. The Laplace transform of the differential equation is

$$s\mathscr{L}\{y\} - y(0) + \mathscr{L}\{y\} = \frac{5}{s}e^{-s}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{5e^{-s}}{s(s+1)} = 5e^{-s}\left[\frac{1}{s} - \frac{1}{s+1}\right].$$

Thus

$$y = 5\,\mathscr{U}(t-1) - 5e^{-(t-1)}\,\mathscr{U}(t-1).$$

64. The Laplace transform of the differential equation is

$$s\,\mathscr{L}\{y\} - y(0) + \mathscr{L}\{y\} = \frac{1}{s} - \frac{2}{s}e^{-s}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1}{s(s+1)} - \frac{2e^{-s}}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1} - 2e^{-s}\left[\frac{1}{s} - \frac{1}{s+1}\right].$$

Thus

$$y = 1 - e^{-t} - 2\left[1 - e^{-(t-1)}\right]\mathscr{U}(t-1).$$

65. The Laplace transform of the differential equation is

$$s\,\mathscr{L}\{y\} - y(0) + 2\,\mathscr{L}\{y\} = \frac{1}{s^2} - e^{-s}\frac{s+1}{s^2}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1}{s^2(s+2)} - e^{-s}\frac{s+1}{s^2(s+2)} = -\frac{1}{4}\frac{1}{s} + \frac{1}{2}\frac{1}{s^2} + \frac{1}{4}\frac{1}{s+2} - e^{-s}\left[\frac{1}{4}\frac{1}{s} + \frac{1}{2}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s+2}\right]$$

Thus

$$y = -\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}e^{-2t} - \left[\frac{1}{4} + \frac{1}{2}(t-1) - \frac{1}{4}e^{-2(t-1)}\right]\mathscr{U}(t-1).$$

66. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) + 4\,\mathscr{L}\{y\} = \frac{1}{s} - \frac{e^{-s}}{s}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1-s}{s(s^2+4)} - e^{-s}\frac{1}{s(s^2+4)} = \frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4} - e^{-s}\left[\frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4}\right].$$

Thus

$$y = \frac{1}{4} - \frac{1}{4}\cos 2t - \frac{1}{2}\sin 2t - \left[\frac{1}{4} - \frac{1}{4}\cos 2(t-1)\right]\mathscr{U}(t-1).$$

67. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) + 4\,\mathscr{L}\{y\} = e^{-2\pi s}\frac{1}{s^2+1}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{s}{s^2+4} + e^{-2\pi s}\left[\frac{1}{3}\frac{1}{s^2+1} - \frac{1}{6}\frac{2}{s^2+4}\right].$$

Thus

$$y = \cos 2t + \left[\frac{1}{3}\sin(t-2\pi) - \frac{1}{6}\sin 2(t-2\pi)\right]\mathscr{U}(t-2\pi).$$

3.6 Questions with Solutions on Chapter 7.4, Questions-Solutions-Laplace-When-multiply-with-t^n

EXERCISES 7.4

Answers to selected odd-numbered problems begin on page ANS-11.

7.4.1 DERIVATIVES OF A TRANSFORM

In Problems 1–8 use Theorem 7.4.1 to evaluate the given Laplace transform.

1. $\mathscr{L}\{te^{-10t}\}$

2. $\mathscr{L}\{t^3e^t\}$

3. $\mathscr{L}\{t\cos 2t\}$

4. $\mathscr{L}\{t\sinh 3t\}$

5. $\mathscr{L}\{t^2\sinh t\}$

6. $\mathscr{L}\{t^2\cos t\}$

7. $\mathscr{L}\{te^{2t}\sin 6t\}$

8. $\mathscr{L}\{te^{-3t}\cos 3t\}$

In Problems 9–14 use the Laplace transform to solve the given initial-value problem. Use the table of Laplace transforms in Appendix III as needed.

9. $y' + y = t\sin t, \quad y(0) = 0$

10. $y' - y = te^t\sin t, \quad y(0) = 0$

11. $y'' + 9y = \cos 3t, \quad y(0) = 2, \quad y'(0) = 5$

12. $y'' + y = \sin t, \quad y(0) = 1, \quad y'(0) = -1$

13. $y'' + 16y = f(t), \quad y(0) = 0, \quad y'(0) = 1$, where

$$f(t) = \begin{cases} \cos 4t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$$

14. $y'' + y = f(t), \quad y(0) = 1, \quad y'(0) = 0$, where

$$f(t) = \begin{cases} 1, & 0 \le t < \pi/2 \\ \sin t, & t \ge \pi/2 \end{cases}$$

In Problems 15 and 16 use a graphing utility to graph the indicated solution.

15. $y(t)$ of Problem 13 for $0 \le t < 2\pi$

16. $y(t)$ of Problem 14 for $0 \le t < 3\pi$

1. $\mathscr{L}\{te^{-10t}\} = -\dfrac{d}{ds}\left(\dfrac{1}{s+10}\right) = \dfrac{1}{(s+10)^2}$

2. $\mathscr{L}\{t^3e^t\} = (-1)^3\dfrac{d^3}{ds^3}\left(\dfrac{1}{s-1}\right) = \dfrac{6}{(s-1)^4}$

3. $\mathscr{L}\{t\cos 2t\} = -\dfrac{d}{ds}\left(\dfrac{s}{s^2+4}\right) = \dfrac{s^2-4}{\left(s^2+4\right)^2}$

4. $\mathscr{L}\{t\sinh 3t\} = -\dfrac{d}{ds}\left(\dfrac{3}{s^2-9}\right) = \dfrac{6s}{\left(s^2-9\right)^2}$

5. $\mathscr{L}\{t^2\sinh t\} = \dfrac{d^2}{ds^2}\left(\dfrac{1}{s^2-1}\right) = \dfrac{6s^2+2}{\left(s^2-1\right)^3}$

6. $\mathscr{L}\{t^2\cos t\} = \dfrac{d^2}{ds^2}\left(\dfrac{s}{s^2+1}\right) = \dfrac{d}{ds}\left(\dfrac{1-s^2}{(s^2+1)^2}\right) = \dfrac{2s\left(s^2-3\right)}{\left(s^2+1\right)^3}$

7. $\mathscr{L}\left\{te^{2t}\sin 6t\right\} = -\dfrac{d}{ds}\left(\dfrac{6}{(s-2)^2+36}\right) = \dfrac{12(s-2)}{\left[(s-2)^2+36\right]^2}$

8. $\mathscr{L}\left\{te^{-3t}\cos 3t\right\} = -\dfrac{d}{ds}\left(\dfrac{s+3}{(s+3)^2+9}\right) = \dfrac{(s+3)^2-9}{\left[(s+3)^2+9\right]^2}$

9. The Laplace transform of the differential equation is

$$s\,\mathscr{L}\{y\} + \mathscr{L}\{y\} = \frac{2s}{(s^2+1)^2}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{2s}{(s+1)(s^2+1)^2} = -\frac{1}{2}\,\frac{1}{s+1} - \frac{1}{2}\,\frac{1}{s^2+1} + \frac{1}{2}\,\frac{s}{s^2+1} + \frac{1}{(s^2+1)^2} + \frac{s}{(s^2+1)^2}\,.$$

Thus

$$y(t) = -\frac{1}{2}e^{-t} - \frac{1}{2}\sin t + \frac{1}{2}\cos t + \frac{1}{2}(\sin t - t\cos t) + \frac{1}{2}t\sin t$$

$$= -\frac{1}{2}e^{-t} + \frac{1}{2}\cos t - \frac{1}{2}t\cos t + \frac{1}{2}t\sin t.$$

10. The Laplace transform of the differential equation is

$$s\,\mathscr{L}\{y\} - \mathscr{L}\{y\} = \frac{2(s-1)}{((s-1)^2+1)^2}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{2}{((s-1)^2+1)^2}.$$

Thus

$$y = e^t\sin t - te^t\cos t.$$

11. The Laplace transform of the differential equation is

$$s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) + 9\,\mathscr{L}\{y\} = \frac{s}{s^2+9}.$$

Letting $y(0) = 2$ and $y'(0) = 5$ and solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{2s^3+5s^2+19s+45}{(s^2+9)^2} = \frac{2s}{s^2+9} + \frac{5}{s^2+9} + \frac{s}{(s^2+9)^2}.$$

Thus

$$y = 2\cos 3t + \frac{5}{3}\sin 3t + \frac{1}{6}t\sin 3t.$$

12. The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) + \mathscr{L}\{y\} = \frac{1}{s^2+1}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{s^3-s^2+s}{(s^2+1)^2} = \frac{s}{s^2+1} - \frac{1}{s^2+1} + \frac{1}{(s^2+1)^2}.$$

Thus

$$y = \cos t - \sin t + \left(\frac{1}{2}\sin t - \frac{1}{2}t\cos t\right) = \cos t - \frac{1}{2}\sin t - \frac{1}{2}t\cos t.$$

13. The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) + 16\mathscr{L}\{y\} = \mathscr{L}\left\{\cos 4t - \cos 4t\,\mathscr{U}(t-\pi)\right\}$$

or by (16) of Section 7.3,

$$(s^2+16)\,\mathscr{L}\{y\} = 1 + \frac{s}{s^2+16} - e^{-\pi s}\,\mathscr{L}\{\cos 4(t+\pi)\}$$

$$= 1 + \frac{s}{s^2+16} - e^{-\pi s}\,\mathscr{L}\{\cos 4t\}$$

$$= 1 + \frac{s}{s^2+16} - \frac{s}{s^2+16}e^{-\pi s}.$$

3.7 Questions with Solutions on Chapter 7.4, Questions-Solutions-Related-to-Convolution

[redacted]

In Problems 19–30 use Theorem 7.4.2 to evaluate the given Laplace transform. Do not evaluate the integral before transforming.

19. $\mathscr{L}\{1 * t^3\}$ **20.** $\mathscr{L}\{t^2 * te^t\}$

21. $\mathscr{L}\{e^{-t} * e^t \cos t\}$ **22.** $\mathscr{L}\{e^{2t} * \sin t\}$

23. $\mathscr{L}\left\{\int_0^t e^\tau \, d\tau\right\}$ **24.** $\mathscr{L}\left\{\int_0^t \cos \tau \, d\tau\right\}$

25. $\mathscr{L}\left\{\int_0^t e^{-\tau} \cos \tau \, d\tau\right\}$ **26.** $\mathscr{L}\left\{\int_0^t \tau \sin \tau \, d\tau\right\}$

27. $\mathscr{L}\left\{\int_0^t \tau e^{t-\tau} \, d\tau\right\}$ **28.** $\mathscr{L}\left\{\int_0^t \sin \tau \cos (t - \tau) \, d\tau\right\}$

29. $\mathscr{L}\left\{t \int_0^t \sin \tau \, d\tau\right\}$ **30.** $\mathscr{L}\left\{t \int_0^t \tau e^{-\tau} \, d\tau\right\}$

In Problems 31–34 use (8) to evaluate the given inverse transform.

31. $\mathscr{L}^{-1}\left\{\frac{1}{s(s - 1)}\right\}$ **32.** $\mathscr{L}^{-1}\left\{\frac{1}{s^2(s - 1)}\right\}$

33. $\mathscr{L}^{-1}\left\{\frac{1}{s^3(s - 1)}\right\}$ **34.** $\mathscr{L}^{-1}\left\{\frac{1}{s(s - a)^2}\right\}$

In Problems 37–46 use the Laplace transform to solve the given integral equation or integrodifferential equation.

37. $f(t) + \int_0^t (t - \tau) f(\tau) \, d\tau = t$

38. $f(t) = 2t - 4 \int_0^t \sin \tau f(t - \tau) \, d\tau$

39. $f(t) = te^t + \int_0^t \tau f(t - \tau) \, d\tau$

40. $f(t) + 2 \int_0^t f(\tau) \cos (t - \tau) \, d\tau = 4e^{-t} + \sin t$

41. $f(t) + \int_0^t f(\tau) \, d\tau = 1$

42. $f(t) = \cos t + \int_0^t e^{-\tau} f(t - \tau) \, d\tau$

43. $f(t) = 1 + t - \frac{8}{3} \int_0^t (\tau - t)^3 f(\tau) \, d\tau$

44. $t - 2f(t) = \int_0^t (e^\tau - e^{-\tau}) f(t - \tau) \, d\tau$

45. $y'(t) = 1 - \sin t - \int_0^t y(\tau) \, d\tau, \quad y(0) = 0$

46. $\frac{dy}{dt} + 6y(t) + 9 \int_0^t y(\tau) \, d\tau = 1, \quad y(0) = 0$

In Problems 47 and 48 solve equation (10) subject to $i(0) = 0$

[redacted]

[redacted]

22. $\mathscr{L}\left\{e^{2t} * \sin t\right\} = \dfrac{1}{(s-2)(s^2+1)}$

23. $\mathscr{L}\left\{\displaystyle\int_0^t e^{\tau}\,d\tau\right\} = \dfrac{1}{s}\mathscr{L}\{e^t\} = \dfrac{1}{s(s-1)}$

24. $\mathscr{L}\left\{\displaystyle\int_0^t \cos\tau\,d\tau\right\} = \dfrac{1}{s}\mathscr{L}\{\cos t\} = \dfrac{s}{s(s^2+1)} = \dfrac{1}{s^2+1}$

25. $\mathscr{L}\left\{\displaystyle\int_0^t e^{-\tau}\cos\tau\,d\tau\right\} = \dfrac{1}{s}\,\mathscr{L}\left\{e^{-t}\cos t\right\} = \dfrac{1}{s}\,\dfrac{s+1}{(s+1)^2+1} = \dfrac{s+1}{s\,(s^2+2s+2)}$

26. $\mathscr{L}\left\{\displaystyle\int_0^t \tau\sin\tau\,d\tau\right\} = \dfrac{1}{s}\,\mathscr{L}\{t\sin t\} = \dfrac{1}{s}\left(-\dfrac{d}{ds}\,\dfrac{1}{s^2+1}\right) = -\dfrac{1}{s}\,\dfrac{-2s}{(s^2+1)^2} = \dfrac{2}{(s^2+1)^2}$

27. $\mathscr{L}\left\{\displaystyle\int_0^t \tau e^{t-\tau}\,d\tau\right\} = \mathscr{L}\{t\}\,\mathscr{L}\{e^t\} = \dfrac{1}{s^2(s-1)}$

28. $\mathscr{L}\left\{\displaystyle\int_0^t \sin\tau\cos(t-\tau)\,d\tau\right\} = \mathscr{L}\{\sin t\}\,\mathscr{L}\{\cos t\} = \dfrac{s}{(s^2+1)^2}$

29. $\mathscr{L}\left\{t\displaystyle\int_0^t \sin\tau\,d\tau\right\} = -\dfrac{d}{ds}\mathscr{L}\left\{\displaystyle\int_0^t \sin\tau\,d\tau\right\} = -\dfrac{d}{ds}\left(\dfrac{1}{s}\,\dfrac{1}{s^2+1}\right) = \dfrac{3s^2+1}{s^2\,(s^2+1)^2}$

30. $\mathscr{L}\left\{t\displaystyle\int_0^t \tau e^{-\tau}d\tau\right\} = -\dfrac{d}{ds}\mathscr{L}\left\{\displaystyle\int_0^t \tau e^{-\tau}d\tau\right\} = -\dfrac{d}{ds}\left(\dfrac{1}{s}\,\dfrac{1}{(s+1)^2}\right) = \dfrac{3s+1}{s^2(s+1)^3}$

31. $\mathscr{L}^{-1}\left\{\dfrac{1}{s(s-1)}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1/(s-1)}{s}\right\} = \displaystyle\int_0^t e^{\tau}d\tau = e^t - 1$

32. $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2(s-1)}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1/s(s-1)}{s}\right\} = \displaystyle\int_0^t (e^{\tau}-1)d\tau = e^t - t - 1$

33. $\mathscr{L}^{-1}\left\{\dfrac{1}{s^3(s-1)}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1/s^2(s-1)}{s}\right\} = \displaystyle\int_0^t (e^{\tau}-\tau-1)d\tau = e^t - \dfrac{1}{2}t^2 - t - 1$

34. Using $\mathscr{L}^{-1}\left\{\dfrac{1}{(s-a)^2}\right\} = te^{at}$, (8) in the text gives

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s-a)^2}\right\} = \int_0^t \tau e^{a\tau}\,d\tau = \frac{1}{a^2}(ate^{at} - e^{at} + 1).$$

35. **(a)** The result in (4) in the text is $\mathscr{L}^{-1}\{F(s)G(s)\} = f * g$, so identify

$$F(s) = \frac{2k^3}{(s^2+k^2)^2} \qquad \text{and} \qquad G(s) = \frac{4s}{s^2+k^2}\,.$$

Solving for $\mathscr{L}\{f\}$ we obtain

$$\mathscr{L}\{f\} = \frac{2s^2+2}{s^2(s^2+5)} = \frac{2}{5}\frac{1}{s^2} + \frac{8}{5\sqrt{5}}\frac{\sqrt{5}}{s^2+5}.$$

Thus

$$f(t) = \frac{2}{5}t + \frac{8}{5\sqrt{5}}\sin\sqrt{5}\,t.$$

39. The Laplace transform of the given equation is

$$\mathscr{L}\{f\} = \mathscr{L}\left\{te^t\right\} + \mathscr{L}\{t\}\mathscr{L}\{f\}.$$

Solving for $\mathscr{L}\{f\}$ we obtain

$$\mathscr{L}\{f\} = \frac{s^2}{(s-1)^3(s+1)} = \frac{1}{8}\frac{1}{s-1} + \frac{3}{4}\frac{1}{(s-1)^2} + \frac{1}{4}\frac{2}{(s-1)^3} - \frac{1}{8}\frac{1}{s+1}.$$

Thus

$$f(t) = \frac{1}{8}e^t + \frac{3}{4}te^t + \frac{1}{4}t^2e^t - \frac{1}{8}e^{-t}$$

40. The Laplace transform of the given equation is

$$\mathscr{L}\{f\} + 2\mathscr{L}\{\cos t\}\mathscr{L}\{f\} = 4\mathscr{L}\left\{e^{-t}\right\} + \mathscr{L}\{\sin t\}.$$

Solving for $\mathscr{L}\{f\}$ we obtain

$$\mathscr{L}\{f\} = \frac{4s^2+s+5}{(s+1)^3} = \frac{4}{s+1} - \frac{7}{(s+1)^2} + 4\frac{2}{(s+1)^3}.$$

Thus

$$f(t) = 4e^{-t} - 7te^{-t} + 4t^2e^{-t}.$$

41. The Laplace transform of the given equation is

$$\mathscr{L}\{f\} + \mathscr{L}\{1\}\mathscr{L}\{f\} = \mathscr{L}\{1\}.$$

Solving for $\mathscr{L}\{f\}$ we obtain $\mathscr{L}\{f\} = \dfrac{1}{s+1}$. Thus, $f(t) = e^{-t}$.

42. The Laplace transform of the given equation is

$$\mathscr{L}\{f\} = \mathscr{L}\{\cos t\} + \mathscr{L}\left\{e^{-t}\right\}\mathscr{L}\{f\}.$$

Solving for $\mathscr{L}\{f\}$ we obtain

$$\mathscr{L}\{f\} = \frac{s}{s^2+1} + \frac{1}{s^2+1}.$$

Thus

$$f(t) = \cos t + \sin t.$$

43. The Laplace transform of the given equation is

$$\mathscr{L}\{f\} = \mathscr{L}\{1\} + \mathscr{L}\{t\} - \mathscr{L}\left\{\frac{8}{3}\int_0^t (t-\tau)^3 f(\tau)\,d\tau\right\}$$

$$= \frac{1}{s} + \frac{1}{s^2} + \frac{8}{3}\mathscr{L}\{t^3\}\,\mathscr{L}\{f\} = \frac{1}{s} + \frac{1}{s^2} + \frac{16}{s^4}\mathscr{L}\{f\}.$$

Solving for $\mathscr{L}\{f\}$ we obtain

$$\mathscr{L}\{f\} = \frac{s^2(s+1)}{s^4-16} = \frac{1}{8}\frac{1}{s+2} + \frac{3}{8}\frac{1}{s-2} + \frac{1}{4}\frac{2}{s^2+4} + \frac{1}{2}\frac{s}{s^2+4}\,.$$

Thus

$$f(t) = \frac{1}{8}e^{-2t} + \frac{3}{8}e^{2t} + \frac{1}{4}\sin 2t + \frac{1}{2}\cos 2t.$$

44. The Laplace transform of the given equation is

$$\mathscr{L}\{t\} - 2\,\mathscr{L}\{f\} = \mathscr{L}\left\{e^t - e^{-t}\right\}\mathscr{L}\{f\}.$$

Solving for $\mathscr{L}\{f\}$ we obtain

$$\mathscr{L}\{f\} = \frac{s^2-1}{2s^4} = \frac{1}{2}\frac{1}{s^2} - \frac{1}{12}\frac{3!}{s^4}\,.$$

Thus

$$f(t) = \frac{1}{2}t - \frac{1}{12}t^3.$$

45. The Laplace transform of the given equation is

$$s\,\mathscr{L}\{y\} - y(0) = \mathscr{L}\{1\} - \mathscr{L}\{\sin t\} - \mathscr{L}\{1\}\mathscr{L}\{y\}.$$

Solving for $\mathscr{L}\{f\}$ we obtain

$$\mathscr{L}\{y\} = \frac{s^2-s+1}{(s^2+1)^2} = \frac{1}{s^2+1} - \frac{1}{2}\frac{2s}{(s^2+1)^2}\,.$$

Thus

$$y = \sin t - \frac{1}{2}t\sin t.$$

46. The Laplace transform of the given equation is

$$s\,\mathscr{L}\{y\} - y(0) + 6\,\mathscr{L}\{y\} + 9\,\mathscr{L}\{1\}\mathscr{L}\{y\} = \mathscr{L}\{1\}.$$

Solving for $\mathscr{L}\{f\}$ we obtain $\mathscr{L}\{y\} = \dfrac{1}{(s+3)^2}$. Thus, $y = te^{-3t}$.

3.8 Questions with Solutions on Chapter 7.5, Questions-Solutions-Related-Delta-Function

[redacted]

In Problems 1–12 use the Laplace transform to solve the given initial-value problem.

1. $y' - 3y = \delta(t-2), \quad y(0) = 0$

2. $y' + y = \delta(t-1), \quad y(0) = 2$

3. $y'' + y = \delta(t-2\pi), \quad y(0) = 0, y'(0) = 1$

4. $y'' + 16y = \delta(t-2\pi), \quad y(0) = 0, y'(0) = 0$

5. $y'' + y = \delta\left(t-\frac{1}{2}\pi\right) + \delta\left(t-\frac{3}{2}\pi\right),$
$y(0) = 0, y'(0) = 0$

6. $y'' + y = \delta(t-2\pi) + \delta(t-4\pi), \quad y(0) = 1, y'(0) = 0$

7. $y'' + 2y' = \delta(t-1), \quad y(0) = 0, y'(0) = 1$

8. $y'' - 2y' = 1 + \delta(t-2), \quad y(0) = 0, y'(0) = 1$

9. $y'' + 4y' + 5y = \delta(t-2\pi), \quad y(0) = 0, y'(0) = 0$

10. $y'' + 2y' + y = \delta(t-1), \quad y(0) = 0, y'(0) = 0$

11. $y'' + 4y' + 13y = \delta(t-\pi) + \delta(t-3\pi),$
$y(0) = 1, y'(0) = 0$

[redacted]

[redacted]

[redacted]

1. The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{1}{s-3}e^{-2s}$$

so that

$$y = e^{3(t-2)}\mathcal{U}(t-2).$$

2. The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{2}{s+1} + \frac{e^{-s}}{s+1}$$

so that

$$y = 2e^{-t} + e^{-(t-1)}\mathcal{U}(t-1).$$

3. The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{1}{s^2+1}\left(1 + e^{-2\pi s}\right)$$

so that

$$y = \sin t + \sin t\,\mathcal{U}(t-2\pi).$$

4. The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{1}{4}\frac{4}{s^2+16}e^{-2\pi s}$$

so that

$$y = \frac{1}{4}\sin 4(t-2\pi)\mathcal{U}(t-2\pi) = \frac{1}{4}\sin 4t\,\mathcal{U}(t-2\pi).$$

5. The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{1}{s^2+1}\left(e^{-\pi s/2} + e^{-3\pi s/2}\right)$$

so that

$$\begin{aligned} y &= \sin\left(t-\frac{\pi}{2}\right)\mathcal{U}\left(t-\frac{\pi}{2}\right) + \sin\left(t-\frac{3\pi}{2}\right)\mathcal{U}\left(t-\frac{3\pi}{2}\right) \\ &= -\cos t\,\mathcal{U}\left(t-\frac{\pi}{2}\right) + \cos t\,\mathcal{U}\left(t-\frac{3\pi}{2}\right). \end{aligned}$$

6. The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{s}{s^2+1} + \frac{1}{s^2+1}(e^{-2\pi s} + e^{-4\pi s})$$

so that

$$y = \cos t + \sin t[\mathcal{U}(t - 2\pi) + \mathcal{U}(t - 4\pi)].$$

7. The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{1}{s^2 + 2s}(1 + e^{-s}) = \left[\frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2}\right](1 + e^{-s})$$

so that

$$y = \frac{1}{2} - \frac{1}{2}e^{-2t} + \left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\mathcal{U}(t - 1).$$

8. The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{s+1}{s^2(s-2)} + \frac{1}{s(s-2)}e^{-2s} = \frac{3}{4}\frac{1}{s-2} - \frac{3}{4}\frac{1}{s} - \frac{1}{2}\frac{1}{s^2} + \left[\frac{1}{2}\frac{1}{s-2} - \frac{1}{2}\frac{1}{s}\right]e^{-2s}$$

so that

$$y = \frac{3}{4}e^{2t} - \frac{3}{4} - \frac{1}{2}t + \left[\frac{1}{2}e^{2(t-2)} - \frac{1}{2}\right]\mathcal{U}(t - 2).$$

9. The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{1}{(s+2)^2 + 1}e^{-2\pi s}$$

so that

$$y = e^{-2(t-2\pi)}\sin t\,\mathcal{U}(t - 2\pi).$$

10. The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{1}{(s+1)^2}e^{-s}$$

so that

$$y = (t - 1)e^{-(t-1)}\mathcal{U}(t - 1).$$

11. The Laplace transform of the differential equation yields

$$\begin{aligned}\mathscr{L}\{y\} &= \frac{4+s}{s^2 + 4s + 13} + \frac{e^{-\pi s} + e^{-3\pi s}}{s^2 + 4s + 13}\\ &= \frac{2}{3}\frac{3}{(s+2)^2 + 3^2} + \frac{s+2}{(s+2)^2 + 3^2} + \frac{1}{3}\frac{3}{(s+2)^2 + 3^2}\left(e^{-\pi s} + e^{-3\pi s}\right)\end{aligned}$$

so that

$$\begin{aligned}y &= \frac{2}{3}e^{-2t}\sin 3t + e^{-2t}\cos 3t + \frac{1}{3}e^{-2(t-\pi)}\sin 3(t - \pi)\mathcal{U}(t - \pi)\\ &\quad + \frac{1}{3}e^{-2(t-3\pi)}\sin 3(t - 3\pi)\mathcal{U}(t - 3\pi).\end{aligned}$$

3.9 Questions with Solutions on Chapter 7.6, Questions-Solutions-System-Linear-Diff-Equations

[redacted]

In Problems 1–12 use the Laplace transform to solve the given system of differential equations.

1. $$\frac{dx}{dt} = -x + y, \quad \frac{dy}{dt} = 2x, \quad x(0) = 0, \quad y(0) = 1$$

2. $$\frac{dx}{dt} = 2y + e^t, \quad \frac{dy}{dt} = 8x - t, \quad x(0) = 1, \quad y(0) = 1$$

3. $$\frac{dx}{dt} = x - 2y, \quad \frac{dy}{dt} = 5x - y, \quad x(0) = -1, \quad y(0) = 2$$

4. $$\frac{dx}{dt} + 3x + \frac{dy}{dt} = 1, \quad \frac{dx}{dt} - x + \frac{dy}{dt} - y = e^t, \quad x(0) = 0, \quad y(0) = 0$$

5. $$2\frac{dx}{dt} + \frac{dy}{dt} - 2x = 1, \quad \frac{dx}{dt} + \frac{dy}{dt} - 3x - 3y = 2, \quad x(0) = 0, \quad y(0) = 0$$

6. $$\frac{dx}{dt} + x - \frac{dy}{dt} + y = 0, \quad \frac{dx}{dt} + \frac{dy}{dt} + 2y = 0, \quad x(0) = 0, \quad y(0) = 1$$

7. $$\frac{d^2x}{dt^2} + x - y = 0, \quad \frac{d^2y}{dt^2} + y - x = 0, \quad x(0) = 0, \quad x'(0) = -2, \quad y(0) = 0, \quad y'(0) = 1$$

8. $$\frac{d^2x}{dt^2} + \frac{dx}{dt} + \frac{dy}{dt} = 0, \quad \frac{d^2y}{dt^2} + \frac{dy}{dt} - 4\frac{dx}{dt} = 0, \quad x(0) = 1, \quad x'(0) = 0, \quad y(0) = -1, \quad y'(0) = 5$$

9. $$\frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} = t^2, \quad \frac{d^2x}{dt^2} - \frac{d^2y}{dt^2} = 4t, \quad x(0) = 8, \quad x'(0) = 0, \quad y(0) = 0, \quad y'(0) = 0$$

10. $$\frac{dx}{dt} - 4x + \frac{d^3y}{dt^3} = 6\sin t, \quad \frac{dx}{dt} + 2x - 2\frac{d^3y}{dt^3} = 0, \quad x(0) = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0$$

11. $$\frac{d^2x}{dt^2} + 3\frac{dy}{dt} + 3y = 0, \quad \frac{d^2x}{dt^2} + 3y = te^{-t}, \quad x(0) = 0, \quad x'(0) = 2, \quad y(0) = 0$$

12. $$\frac{dx}{dt} = 4x - 2y + 2\mathcal{U}(t-1), \quad \frac{dy}{dt} = 3x - y + \mathcal{U}(t-1), \quad x(0) = 0, \quad y(0) = \tfrac{1}{2}$$

[redacted]

1. Taking the Laplace transform of the system gives

$$s\,\mathscr{L}\{x\} = -\mathscr{L}\{x\} + \mathscr{L}\{y\}$$

$$s\,\mathscr{L}\{y\} - 1 = 2\,\mathscr{L}\{x\}$$

so that

$$\mathscr{L}\{x\} = \frac{1}{(s-1)(s+2)} = \frac{1}{3}\,\frac{1}{s-1} - \frac{1}{3}\,\frac{1}{s+2}$$

and

$$\mathscr{L}\{y\} = \frac{1}{s} + \frac{2}{s(s-1)(s+2)} = \frac{2}{3}\,\frac{1}{s-1} + \frac{1}{3}\,\frac{1}{s+2}.$$

Then

$$x = \frac{1}{3}e^{t} - \frac{1}{3}e^{-2t} \qquad \text{and} \qquad y = \frac{2}{3}e^{t} + \frac{1}{3}e^{-2t}.$$

2. Taking the Laplace transform of the system gives

$$s\,\mathscr{L}\{x\} - 1 = 2\,\mathscr{L}\{y\} + \frac{1}{s-1}$$

$$s\,\mathscr{L}\{y\} - 1 = 8\,\mathscr{L}\{x\} - \frac{1}{s^2}$$

so that

$$\mathscr{L}\{y\} = \frac{s^3 + 7s^2 - s + 1}{s(s-1)(s^2-16)} = \frac{1}{16}\,\frac{1}{s} - \frac{8}{15}\,\frac{1}{s-1} + \frac{173}{96}\,\frac{1}{s-4} - \frac{53}{160}\,\frac{1}{s+4}$$

and

$$y = \frac{1}{16} - \frac{8}{15}e^{t} + \frac{173}{96}e^{4t} - \frac{53}{160}e^{-4t}.$$

Then

$$x = \frac{1}{8}y' + \frac{1}{8}t = \frac{1}{8}t - \frac{1}{15}e^{t} + \frac{173}{192}e^{4t} + \frac{53}{320}e^{-4t}.$$

3. Taking the Laplace transform of the system gives

$$s\,\mathscr{L}\{x\} + 1 = \mathscr{L}\{x\} - 2\,\mathscr{L}\{y\}$$

$$s\,\mathscr{L}\{y\} - 2 = 5\,\mathscr{L}\{x\} - \mathscr{L}\{y\}$$

so that

$$\mathscr{L}\{x\} = \frac{-s-5}{s^2+9} = -\frac{s}{s^2+9} - \frac{5}{3}\,\frac{3}{s^2+9}$$

and

$$x = -\cos 3t - \frac{5}{3}\sin 3t.$$

Then

$$y = \frac{1}{2}x - \frac{1}{2}x' = 2\cos 3t - \frac{7}{3}\sin 3t.$$

4. Taking the Laplace transform of the system gives

$$(s+3)\,\mathscr{L}\{x\} + s\mathscr{L}\{y\} = \frac{1}{s}$$

$$(s-1)\,\mathscr{L}\{x\} + (s-1)\,\mathscr{L}\{y\} = \frac{1}{s-1}$$

so that

$$\mathscr{L}\{y\} = \frac{5s-1}{3s(s-1)^2} = -\frac{1}{3}\frac{1}{s} + \frac{1}{3}\frac{1}{s-1} + \frac{4}{3}\frac{1}{(s-1)^2}$$

and

$$\mathscr{L}\{x\} = \frac{1-2s}{3s(s-1)^2} = \frac{1}{3}\frac{1}{s} - \frac{1}{3}\frac{1}{s-1} - \frac{1}{3}\frac{1}{(s-1)^2}.$$

Then

$$x = \frac{1}{3} - \frac{1}{3}e^t - \frac{1}{3}te^t \quad \text{and} \quad y = -\frac{1}{3} + \frac{1}{3}e^t + \frac{4}{3}te^t.$$

5. Taking the Laplace transform of the system gives

$$(2s-2)\,\mathscr{L}\{x\} + s\,\mathscr{L}\{y\} = \frac{1}{s}$$

$$(s-3)\,\mathscr{L}\{x\} + (s-3)\,\mathscr{L}\{y\} = \frac{2}{s}$$

so that

$$\mathscr{L}\{x\} = \frac{-s-3}{s(s-2)(s-3)} = -\frac{1}{2}\frac{1}{s} + \frac{5}{2}\frac{1}{s-2} - \frac{2}{s-3}$$

and

$$\mathscr{L}\{y\} = \frac{3s-1}{s(s-2)(s-3)} = -\frac{1}{6}\frac{1}{s} - \frac{5}{2}\frac{1}{s-2} + \frac{8}{3}\frac{1}{s-3}.$$

Then

$$x = -\frac{1}{2} + \frac{5}{2}e^{2t} - 2e^{3t} \quad \text{and} \quad y = -\frac{1}{6} - \frac{5}{2}e^{2t} + \frac{8}{3}e^{3t}.$$

6. Taking the Laplace transform of the system gives

$$(s+1)\,\mathscr{L}\{x\} - (s-1)\mathscr{L}\{y\} = -1$$

$$s\,\mathscr{L}\{x\} + (s+2)\,\mathscr{L}\{y\} = 1$$

so that

$$\mathscr{L}\{y\} = \frac{s+1/2}{s^2+s+1} = \frac{s+1/2}{(s+1/2)^2 + (\sqrt{3}/2)^2}$$

and

$$\mathscr{L}\{x\} = \frac{-3/2}{s^2+s+1} = -\sqrt{3}\,\frac{\sqrt{3}/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\,.$$

Then

$$y = e^{-t/2}\cos\frac{\sqrt{3}}{2}t \qquad \text{and} \qquad x = -\sqrt{3}\,e^{-t/2}\sin\frac{\sqrt{3}}{2}t.$$

7. Taking the Laplace transform of the system gives

$$(s^2+1)\,\mathscr{L}\{x\} - \mathscr{L}\{y\} = -2$$

$$-\mathscr{L}\{x\} + (s^2+1)\,\mathscr{L}\{y\} = 1$$

so that

$$\mathscr{L}\{x\} = \frac{-2s^2-1}{s^4+2s^2} = -\frac{1}{2}\frac{1}{s^2} - \frac{3}{2}\frac{1}{s^2+2}$$

and

$$x = -\frac{1}{2}t - \frac{3}{2\sqrt{2}}\sin\sqrt{2}\,t.$$

Then

$$y = x'' + x = -\frac{1}{2}t + \frac{3}{2\sqrt{2}}\sin\sqrt{2}\,t.$$

8. Taking the Laplace transform of the system gives

$$(s+1)\,\mathscr{L}\{x\} + \mathscr{L}\{y\} = 1$$

$$4\mathscr{L}\{x\} - (s+1)\,\mathscr{L}\{y\} = 1$$

so that

$$\mathscr{L}\{x\} = \frac{s+2}{s^2+2s+5} = \frac{s+1}{(s+1)^2+2^2} + \frac{1}{2}\frac{2}{(s+1)^2+2^2}$$

and

$$\mathscr{L}\{y\} = \frac{-s+3}{s^2+2s+5} = -\frac{s+1}{(s+1)^2+2^2} + 2\frac{2}{(s+1)^2+2^2}\,.$$

Then

$$x = e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t \qquad \text{and} \qquad y = -e^{-t}\cos 2t + 2e^{-t}\sin 2t.$$

9. Adding the equations and then subtracting them gives

$$\frac{d^2x}{dt^2} = \frac{1}{2}t^2 + 2t$$

$$\frac{d^2y}{dt^2} = \frac{1}{2}t^2 - 2t.$$

Taking the Laplace transform of the system gives

$$\mathscr{L}\{x\} = 8\frac{1}{s} + \frac{1}{24}\frac{4!}{s^5} + \frac{1}{3}\frac{3!}{s^4}$$

and

$$\mathscr{L}\{y\} = \frac{1}{24}\frac{4!}{s^5} - \frac{1}{3}\frac{3!}{s^4}$$

so that

$$x = 8 + \frac{1}{24}t^4 + \frac{1}{3}t^3 \qquad \text{and} \qquad y = \frac{1}{24}t^4 - \frac{1}{3}t^3.$$

10. Taking the Laplace transform of the system gives

$$(s-4)\,\mathscr{L}\{x\} + s^3\mathscr{L}\{y\} = \frac{6}{s^2+1}$$
$$(s+2)\,\mathscr{L}\{x\} - 2s^3\mathscr{L}\{y\} = 0$$

so that

$$\mathscr{L}\{x\} = \frac{4}{(s-2)(s^2+1)} = \frac{4}{5}\frac{1}{s-2} - \frac{4}{5}\frac{s}{s^2+1} - \frac{8}{5}\frac{1}{s^2+1}$$

and

$$\mathscr{L}\{y\} = \frac{2s+4}{s^3(s-2)(s^2+1)} = \frac{1}{s} - \frac{2}{s^2} - 2\frac{2}{s^3} + \frac{1}{5}\frac{1}{s-2} - \frac{6}{5}\frac{s}{s^2+1} + \frac{8}{5}\frac{1}{s^2+1}.$$

Then

$$x = \frac{4}{5}e^{2t} - \frac{4}{5}\cos t - \frac{8}{5}\sin t$$

and

$$y = 1 - 2t - 2t^2 + \frac{1}{5}e^{2t} - \frac{6}{5}\cos t + \frac{8}{5}\sin t.$$

11. Taking the Laplace transform of the system gives

$$s^2\mathscr{L}\{x\} + 3(s+1)\mathscr{L}\{y\} = 2$$
$$s^2\,\mathscr{L}\{x\} + 3\mathscr{L}\{y\} = \frac{1}{(s+1)^2}$$

so that

$$\mathscr{L}\{x\} = -\frac{2s+1}{s^3(s+1)} = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{2}\frac{2}{s^3} - \frac{1}{s+1}.$$

Then

$$x = 1 + t + \frac{1}{2}t^2 - e^{-t}$$

and

$$y = \frac{1}{3}te^{-t} - \frac{1}{3}x'' = \frac{1}{3}te^{-t} + \frac{1}{3}e^{-t} - \frac{1}{3}.$$

12. Taking the Laplace transform of the system gives

$$(s-4)\,\mathscr{L}\{x\} + 2\mathscr{L}\{y\} = \frac{2e^{-s}}{s}$$
$$-3\,\mathscr{L}\{x\} + (s+1)\,\mathscr{L}\{y\} = \frac{1}{2} + \frac{e^{-s}}{s}$$

3.10 Questions with Solutions, More-Questions-Periodic-Solving-System-LDE

Name [redacted], ID [redacted]

Quiz 3, MTH 205 , Fall 2019

Ayman Badawi

20/20

QUESTION 1. Find $x(t), y(t)$ such that $x(0) = 3, y(0) = 0$ and

$$x'(t) + x(t) - 9y(t) = 0$$

$$y'(t) + x(t) + y(t) = 0$$

$$sX(s) - \overset{3}{\cancel{x(0)}} + X(s) - 9Y(s) = 0$$

$$sY(s) - \underset{0}{\cancel{y(0)}} + X(s) + Y(s) = 0$$

① $X(s)\,(s+1) - 9Y(s) = 3$

② $X(s) + (s+1)\,Y(s) = 0$

✓

$$X(s) = \frac{\begin{vmatrix} 3 & -9 \\ 0 & s+1 \end{vmatrix}}{\begin{array}{cc} X(s) & Y(s) \\ \left|\begin{matrix} s+1 & -9 \\ 1 & s+1 \end{matrix}\right| \end{array}} = \frac{3(s+1) - 0}{\underset{s^2+s+s+1}{(s+1)(s+1)} + 9} = \frac{3\,(s+1)}{s^2+2s+10} = \frac{3(s+1)}{(s+1)^2+9}$$

~~$X(s) = \frac{3s + 3}{s^2+2s+10}$~~

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{3(s+1)}{(s+1)^2+9}\right\}$$

$$\boxed{x(t) = 3e^{-t}\cos(3t)}$$

✓

$$Y(s) = \frac{\begin{vmatrix} s+1 & 3 \\ 1 & 0 \end{vmatrix}}{\begin{array}{cc} X(s) & Y(s) \\ \left|\begin{matrix} s+1 & -9 \\ 1 & s+1 \end{matrix}\right| \end{array}} = \frac{0 - 3}{(s+1)(s+1) + 9} = \frac{-3}{(s+1)^2+9}$$

$$\boxed{y(t) = -e^{-t}\sin(3t)}$$

✓

10/20

Nadeen

QUESTION 2. (8 points) Given $f(t)$ is periodic on the interval $[0, \infty]$. The first period of $f(t)$ is determined by $f(t) = 2$, when $0 \leq t < 4$. Use Laplace-Transformation and find $y(t)$, where $y'' - 4y' + 3y = f(t)$, $y(0) = 0, y'(0) = 0$.

$$s^2Y(s) - \cancel{sy(0)} - \cancel{y'(0)} - 4sY(s) - \cancel{4y(0)} + 3Y(s) = \frac{2-2e^{-4s}}{s(1-e^{-4s})}$$

$$Y(s)(s^2-4s+3) = \frac{2-2e^{-4s}}{s(1-e^{-4s})}$$

$$Y(s) = \frac{2-2e^{-4s}}{s(1-e^{-4s})(s-3)(s-1)} = \frac{2\cancel{(1-e^{-4s})}}{s\cancel{(1-e^{-4s})}(s-3)(s-1)}$$

$$Y(s) = \frac{2}{s(s-3)(s-1)} = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s-1}$$

$s=0$ $\quad s=3$ $\quad s=1$

$A = 2/3$ $\quad B = \frac{1}{3}$ $\quad C = -1$

$$Y(s) = \frac{2/3}{s} + \frac{1/3}{s-3} - \frac{1}{s-1}$$

$$y(t) = \frac{2}{3} + \frac{1}{3}e^{3t} - e^{t}$$

✓

$$\int_0^4 e^{-st}(2)\,dt$$

$f(t) = 2[u_0 - u_4]$

$= 2u_0 - 2u_4$

$$\frac{2e^{0}}{s} - \frac{2e^{-4s}}{s}$$

$$= \frac{\frac{2}{s} - \frac{2e^{-4s}}{s}}{1-e^{-4s}}$$

$s^2\ 9 - s^2 - 1$

$$\frac{1}{s^2+1} - \frac{1}{s^2-9}$$

QUESTION 3. (8 points) let $f(t) = \int_0^t cos(u)\,du$, where $0 \leq t < \infty$. Use Laplace-Transformation and find $y(t)$, where $y'' - 9y = f(t)$, $y(0) = 0, y'(0) = 0$.

$$s^2Y(s) - \cancel{sy(0)} - \cancel{y'(0)} - 9Y(s) = \frac{1}{s^2+1}$$ ✓

$$Y(s)(s^2-9) = \frac{1}{s^2+1}$$

$$\mathcal{L}\int_0^t \cos(u)\,du \quad f(t)$$

$$\frac{\frac{\cancel{s}}{s^2+1}}{\cancel{s}} = \frac{1}{s^2+1}$$

$s^2 - 9 - s^2 - 1$

$$Y(s) = \frac{1}{(s^2+1)(s^2-9)} = \frac{1}{(s^2+1)(s-3)(s+3)} = \frac{A}{s-3} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1} = \frac{1}{\frac{(s-3)(s+3)}{(s^2+1)}}$$

$$\frac{1}{s^2+1} - \frac{1}{s^2-9} = \frac{-10}{(s^2+1)(s^2-9)}$$

$$\frac{1}{-10}\left[\frac{1}{s^2+1} - \frac{1}{s^2-9}\right] \quad \mathcal{L}^{-1}Y(s) = \mathcal{L}^{-1}\frac{-1}{10}\left[\frac{1}{s^2+1} - \frac{1}{s^2-9}\right]$$

$$y(t) = \frac{-1}{10}\sin t + \frac{1}{30}\sinh(3t)$$

✓

QUESTION 2. (8 points) Use Laplace to solve the differential equation :

$$y'(t) = e^{3t} + \int_0^t 4y(u)\,du,\ y(0) = 0$$

$\int 4y(u)\,du$

$4 * y(t)$

$y'(t) = e^{3t} + 4 * y(t)$

$\mathcal{L}(y'(t)) = \mathcal{L}(e^{3t}) + \mathcal{L}(4 * y(t))$

$sY(s) - y(0) = \frac{1}{s-3} + \frac{4Y(s)}{s}$

$sY(s) - \frac{4}{s}Y(s) = \frac{1}{s-3}$

$Y(s)\left[s - \frac{4}{s}\right] = \frac{1}{s-3}$

$Y(s) = \frac{1}{(s-3)} \times \frac{s}{(s^2-4)}$

$Y(s) = \frac{s}{(s-3)(s-2)(s+2)}$

$\frac{s}{(s-3)(s-2)(s+2)} = \frac{A}{s-3} + \frac{B}{s-2} + \frac{C}{s+2}$

$s = 3 \quad s = 2 \quad s = -2$

$A = \frac{3}{5} \quad B = \frac{-1}{2} \quad C = -\frac{1}{10}$

$y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left\{\frac{3/5}{s-3} - \frac{1/2}{s-2} - \frac{1/10}{s+2}\right\}$

$y(t) = \frac{3}{5}e^{3t} - \frac{1}{2}e^{2t} - \frac{1}{10}e^{-2t}$

QUESTION 6. (10 points) Use Laplace and solve the following system of Linear Diff. Equations:

$$x'(t) - y(t) = 0, x(0) = 2$$

$$y'(t) - x(t) = -t, y(0) = 1$$

$$sX(s) - x(0) - Y(s) = 0$$

$$sX(s) - Y(s) = 2 \text{ -- } ①$$

$$sY(s) - y(0) - X(s) = -\frac{1}{s^2}$$

$$-X(s) + sY(s) = -\frac{1}{s^2} + 1 \rightarrow \frac{s^2-1}{s^2} \text{ -- } ②$$

$$X(s) = \frac{\begin{vmatrix} 2 & -1 \\ \frac{s^2-1}{s^2} & s \end{vmatrix}}{\begin{vmatrix} s & -1 \\ -1 & s \end{vmatrix}} = \frac{2s + \frac{s^2-1}{s^2}}{s^2-1}$$

$$X(s) = \frac{2s^3+s^2-1}{s^2(s^2-1)} = \frac{2s^{\cancel{3}}}{\cancel{s^2}(s^2-1)} + \frac{\cancel{s^2-1}}{s^2\cancel{(s^2-1)}}$$

$$X(s) = \frac{2s}{s^2-1} + \frac{1}{s^2}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{2s}{s^2-1} + \frac{1}{s^2}\right\}$$

$$x(t) = 2\cosh(t) + t$$ ✓

$$Y(s) = \frac{\begin{vmatrix} s & 2 \\ -1 & \frac{s^2-1}{s^2} \end{vmatrix}}{\begin{vmatrix} s & -1 \\ -1 & s \end{vmatrix}} = \frac{\frac{s(s^2-1)}{s^2} + 2}{s^2-1} = \frac{s(s^2-1)+2s^2}{s^2(s^2-1)}$$

$$Y(s) = \frac{\cancel{s}\cancel{(s^2-1)}}{s^{\cancel{2}}\cancel{(s^2-1)}} + \frac{2\cancel{s^2}}{\cancel{s^2}(s^2-1)}$$

$$Y(s) = \frac{1}{s} + \frac{2}{s^2-1}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{2}{s^2-1}\right\} \Rightarrow y(t) = 1 + 2\sinh(t)$$ ✓

Name Rana Sakhnini, ID 52000

100 Excellent

Exam I, MTH 205, Fall 2014

Ayman Badawi

(−∞ , 12)

12 − x > 0   x ≠ 4

12 > x

x < 12

QUESTION 1. (6 points) Find the largest interval around x so that the LDE: $\frac{\sqrt{x-4}}{\sqrt{12-x}}y^{(3)} + \frac{x-1}{x-7}y' + 3y = x^2 + 13$, $y^{(2)}(5) = y'(5) = 7$, and $y(5) = -6$ has a unique solution.

$\frac{\sqrt{x-4}}{\sqrt{12-x}}$

12 − x > 0

12 > x

x < 12

(−∞ , 12)

(−∞ , 4) ∪ (4 , 12)

x − 7

(−∞ , 7) ∪ (7, ∞)

ℝ

$\boxed{I = (4, 7)}$

$\int_0^x 2 \sin u \, du$

$(-2\cos u)_0^x$

$-2\cos x + 2$

QUESTION 2. (10 points) Solve for $x(t), y(t)$

$x'(t) - y(t) = 2$

$x(t) + y'(t) = 2$, where $x(0) = 2$, $y(0) = -1$, $x'(0) = 1$, $y'(0) = 0$

$x'(t) = \cos t$

$\boxed{\cos t - 2 = y(t)}$ ✓

$sX(s) - x(0) - Y(s) = \frac{2}{s}$

$sX(s) - 2 - Y(s) = \frac{2}{s}$

$\boxed{sX(s) - Y(s) = \frac{2+2s}{s}}$

$X(s) + sY(s) + 1 = \frac{2}{s}$

$\boxed{X(s) + sY(s) = \frac{2-s}{s}}$

$sX(s) - \frac{2+2s}{s} = Y(s)$

$X(s) + s^2X(s) - 2 - 2s = \frac{2-s}{s}$

$X(s)(1+s^2) = \frac{2-s}{s} + 2 + 2s$

$= \frac{2 - s + 2s + 2s^2}{s}$

$X(s) = \frac{2s^2 + s + 2}{s(1+s^2)}$

$= \frac{2s}{(s^2+1)} + \frac{1}{(s^2+1)} + \frac{2}{s(s^2+1)}$

$x(t) = 2\cos t + \sin t + 2 * \sin t$

$= 2\cos t + \sin t - 2\cos t + 2 = \boxed{\sin t + 2}$ ✓

Bana

(iii) $y^{(2)} + \int_0^x y(r)e^{x-r}\,dr = \int_0^x (x-r)e^{r}\,dr, y(0) = 0, y'(0) = 1$

$$s^2Y(s) - \cancel{sy(0)} - \cancel{y'(0)}^{1} + Y(s)\left(\frac{1}{s-1}\right) = \left(\frac{1}{s^2}\right)\left(\frac{1}{s-1}\right)$$

$$Y(s)\left(\frac{1}{s-1} + s^2\right) - 1 = \frac{1}{s^2(s-1)}$$

$$Y(s)\left(\frac{1+s^2(s-1)}{\cancel{(s-1)}}\right) = \frac{1+s^2(s-1)}{s^2\cancel{(s-1)}}$$

$$Y(s) = \frac{\cancel{(1+s^2(s-1))}}{s^2\cancel{(1+s^2(s-1))}} = \frac{1}{s^2}$$

$$y(x) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = x$$

2

(iv) $y^{(2)} + 2y' + 2y = xe^{-x}, y(0) = 0$ and $y'(0) = 1$. [Hint: note that by completing the square method we have $s^2 + bs + c = (s+b/2)^2 + c - b^2/4$ and $\frac{e}{f} + d = \frac{e+fd}{f}$]

$$s^2Y(s) - \cancel{sy(0)} - y'(0) + 2sY(s) - \cancel{2y(0)} + 2Y(s) = \frac{1}{(s+1)^2}$$

$$Y(s)\left[s^2 + 2s + 2\right] - 1 = \frac{1 + (s+1)^2}{(s+1)^2}$$

$$Y(s)\left[s^2 + 2s + 1 - 1 + 2\right] = \frac{1 + (s+1)^2}{(s+1)^2}$$

$$Y(s)\left[\cancel{(s+1)^2 + 1}\right] = \frac{1 \cancel{+ (s+1)^2}}{(s+1)^2}$$

$$Y(s) = \frac{1}{(s+1)^2}$$

2

$$y(x) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = xe^{-x}$$

3.11 Questions with Solutions on Chapter 4.4, Questions-Solutions-Undetermined-Coefficient-Method

Note: Solve the INITIAL VALUE Problem, (means , conditions are given at the SAME X-VALUE, i.e., y(0) = ..., y^\(0) = ... (here x = 0) or y(1) = ... , y^\(1) = ..., y^\\(1) = ...(here x = 1)(see 27-31)

Solve the boundary value problem: (means , The given conditions NEED not be the same x; i.e y(0) = ..., y^\prime (1) = ..., (here the conditions are given at x = 0 and at x = 1), see 37-- 40

In Problems 1–26 solve the given differential equation by undetermined coefficients.

1. $y'' + 3y' + 2y = 6$

2. $4y'' + 9y = 15$

3. $y'' - 10y' + 25y = 30x + 3$

4. $y'' + y' - 6y = 2x$

5. $\frac{1}{4}y'' + y' + y = x^2 - 2x$

6. $y'' - 8y' + 20y = 100x^2 - 26xe^x$

7. $y'' + 3y = -48x^2e^{3x}$

8. $4y'' - 4y' - 3y = \cos 2x$

9. $y'' - y' = -3$

10. $y'' + 2y' = 2x + 5 - e^{-2x}$

11. $y'' - y' + \frac{1}{4}y = 3 + e^{x/2}$

12. $y'' - 16y = 2e^{4x}$

13. $y'' + 4y = 3 \sin 2x$

14. $y'' - 4y = (x^2 - 3) \sin 2x$

15. $y'' + y = 2x \sin x$

16. $y'' - 5y' = 2x^3 - 4x^2 - x + 6$

In Problems 27–36 solve the given initial-value problem.

27. $y'' + 4y = -2, \quad y\left(\frac{\pi}{8}\right) = \frac{1}{2}, y'\left(\frac{\pi}{8}\right) = 2$

28. $2y'' + 3y' - 2y = 14x^2 - 4x - 11, \quad y(0) = 0, y'(0) = 0$

29. $5y'' + y' = -6x, \quad y(0) = 0, y'(0) = -10$

30. $y'' + 4y' + 4y = (3 + x)e^{-2x}, \quad y(0) = 2, y'(0) = 5$

31. $y'' + 4y' + 5y = 35e^{-4x}, \quad y(0) = -3, y'(0) = 1$

[redacted]

In Problems 37–40 solve the given boundary-value problem.

37. $y'' + y = x^2 + 1, \quad y(0) = 5, y(1) = 0$

38. $y'' - 2y' + 2y = 2x - 2, \quad y(0) = 0, y(\pi) = \pi$

39. $y'' + 3y = 6x, \quad y(0) = 0, y(1) + y'(1) = 0$

40. $y'' + 3y = 6x, \quad y(0) + y'(0) = 0, y(1) = 0$

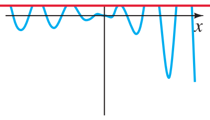
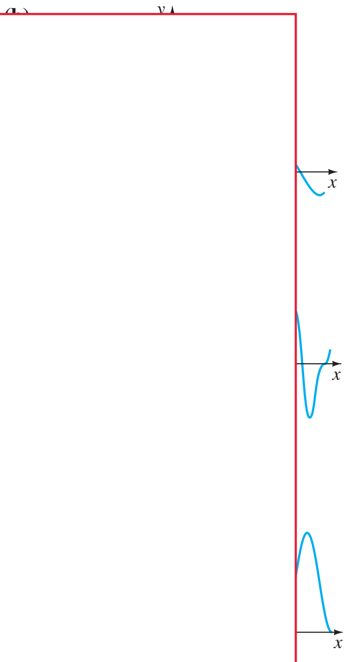


FIGURE 4.4.1 Solution curve

[redacted]



r solution of the given
an aid in carrying out
lgebra.

os $2x$
$1)e^{2x}$ sin $2x$

n x

Exercises 4.4

Undetermined Coefficients – Superposition Approach

1. From $m^2+3m+2=0$ we find $m_1=-1$ and $m_2=-2$. Then $y_c=c_1e^{-x}+c_2e^{-2x}$ and we assum $y_p=A$. Substituting into the differential equation we obtain $2A=6$. Then $A=3$, $y_p=3$ and

$$y=c_1e^{-x}+c_2e^{-2x}+3.$$

2. From $4m^2+9=0$ we find $m_1=-\frac{3}{2}i$ and $m_2=\frac{3}{2}i$. Then $y_c=c_1\cos\frac{3}{2}x+c_2\sin\frac{3}{2}x$ and we assum $y_p=A$. Substituting into the differential equation we obtain $9A=15$. Then $A=\frac{5}{3}$, $y_p=\frac{5}{3}$ and

$$y=c_1\cos\frac{3}{2}x+c_2\sin\frac{3}{2}x+\frac{5}{3}.$$

3. From $m^2-10m+25=0$ we find $m_1=m_2=5$. Then $y_c=c_1e^{5x}+c_2xe^{5x}$ and we assum $y_p=Ax+B$. Substituting into the differential equation we obtain $25A=30$ and $-10A+25B=$ Then $A=\frac{6}{5}$, $B=\frac{3}{5}$, $y_p=\frac{6}{5}x+\frac{3}{5}$, and

$$y=c_1e^{5x}+c_2xe^{5x}+\frac{6}{5}x+\frac{3}{5}.$$

4. From $m^2+m-6=0$ we find $m_1=-3$ and $m_2=2$. Then $y_c=c_1e^{-3x}+c_2e^{2x}$ and we assu $y_p=Ax+B$. Substituting into the differential equation we obtain $-6A=2$ and $A-6B=0$. Th $A=-\frac{1}{3}$, $B=-\frac{1}{18}$, $y_p=-\frac{1}{3}x-\frac{1}{18}$, and

$$y=c_1e^{-3x}+c_2e^{2x}-\frac{1}{3}x-\frac{1}{18}.$$

5. From $\frac{1}{4}m^2+m+1=0$ we find $m_1=m_2=-2$. Then $y_c=c_1e^{-2x}+c_2xe^{-2x}$ and we assu $y_p=Ax^2+Bx+C$. Substituting into the differential equation we obtain $A=1$, $2A+B=-$ and $\frac{1}{2}A+B+C=0$. Then $A=1$, $B=-4$, $C=\frac{7}{2}$, $y_p=x^2-4x+\frac{7}{2}$, and

$$y=c_1e^{-2x}+c_2xe^{-2x}+x^2-4x+\frac{7}{2}.$$

6. From $m^2-8m+20=0$ we find $m_1=4+2i$ and $m_2=4-2i$. Then $y_c=e^{4x}(c_1\cos 2x+c_2\sin 2$ and we assume $y_p=Ax^2+Bx+C+(Dx+E)e^x$. Substituting into the differential equation

obtain

$$2A - 8B + 20C = 0$$

$$-6D + 13E = 0$$

$$-16A + 20B = 0$$

$$13D = -26$$

$$20A = 100.$$

Then $A = 5$, $B = 4$, $C = \frac{11}{10}$, $D = -2$, $E = -\frac{12}{13}$, $y_p = 5x^2 + 4x + \frac{11}{10} + \left(-2x - \frac{12}{13}\right)e^x$ and

$$y = e^{4x}(c_1 \cos 2x + c_2 \sin 2x) + 5x^2 + 4x + \frac{11}{10} + \left(-2x - \frac{12}{13}\right)e^x.$$

7. From $m^2 + 3 = 0$ we find $m_1 = \sqrt{3}\,i$ and $m_2 = -\sqrt{3}\,i$. Then $y_c = c_1 \cos \sqrt{3}\,x + c_2 \sin \sqrt{3}\,x$ and we assume $y_p = (Ax^2 + Bx + C)e^{3x}$. Substituting into the differential equation we obtain $2A + 6B + 12C = 0$, $12A + 12B = 0$, and $12A = -48$. Then $A = -4$, $B = 4$, $C = -\frac{4}{3}$, $y_p = \left(-4x^2 + 4x - \frac{4}{3}\right)e^{3x}$ and

$$y = c_1 \cos \sqrt{3}\,x + c_2 \sin \sqrt{3}\,x + \left(-4x^2 + 4x - \frac{4}{3}\right)e^{3x}.$$

8. From $4m^2 - 4m - 3 = 0$ we find $m_1 = \frac{3}{2}$ and $m_2 = -\frac{1}{2}$. Then $y_c = c_1 e^{3x/2} + c_2 e^{-x/2}$ and we assume $y_p = A \cos 2x + B \sin 2x$. Substituting into the differential equation we obtain $-19 - 8B = 1$ and $8A - 19B = 0$. Then $A = -\frac{19}{425}$, $B = -\frac{8}{425}$, $y_p = -\frac{19}{425} \cos 2x - \frac{8}{425} \sin 2x$, and

$$y = c_1 e^{3x/2} + c_2 e^{-x/2} - \frac{19}{425} \cos 2x - \frac{8}{425} \sin 2x.$$

9. From $m^2 - m = 0$ we find $m_1 = 1$ and $m_2 = 0$. Then $y_c = c_1 e^x + c_2$ and we assume $y_p = Ax$. Substituting into the differential equation we obtain $-A = -3$. Then $A = 3$, $y_p = 3x$ and $y = c_1 e^x + c_2 + 3x$.

10. From $m^2 + 2m = 0$ we find $m_1 = -2$ and $m_2 = 0$. Then $y_c = c_1 e^{-2x} + c_2$ and we assume $y_p = Ax^2 + Bx + Cxe^{-2x}$. Substituting into the differential equation we obtain $2A + 2B = 5$, $4A = 2$, and $-2C = -1$. Then $A = \frac{1}{2}$, $B = 2$, $C = \frac{1}{2}$, $y_p = \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}$, and

$$y = c_1 e^{-2x} + c_2 + \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}.$$

11. From $m^2 - m + \frac{1}{4} = 0$ we find $m_1 = m_2 = \frac{1}{2}$. Then $y_c = c_1 e^{x/2} + c_2 xe^{x/2}$ and we assume $y_p = A + Bx^2 e^{x/2}$. Substituting into the differential equation we obtain $\frac{1}{4}A = 3$ and $2B = 1$. Then $A = 12$, $B = \frac{1}{2}$, $y_p = 12 + \frac{1}{2}x^2 e^{x/2}$, and

$$y = c_1 e^{x/2} + c_2 xe^{x/2} + 12 + \frac{1}{2}x^2 e^{x/2}.$$

12. From $m^2 - 16 = 0$ we find $m_1 = 4$ and $m_2 = -4$. Then $y_c = c_1e^{4x} + c_2e^{-4x}$ and we ass $y_p = Axe^{4x}$. Substituting into the differential equation we obtain $8A = 2$. Then $A = \frac{1}{4}$, $y_p = \frac{1}{4}$ and

$$y = c_1e^{4x} + c_2e^{-4x} + \frac{1}{4}xe^{4x}.$$

13. From $m^2 + 4 = 0$ we find $m_1 = 2i$ and $m_2 = -2i$. Then $y_c = c_1 \cos 2x + c_2 \sin 2x$ and we as $y_p = Ax \cos 2x + Bx \sin 2x$. Substituting into the differential equation we obtain $4B = 0$ $-4A = 3$. Then $A = -\frac{3}{4}$, $B = 0$, $y_p = -\frac{3}{4}x \cos 2x$, and

$$y = c_1 \cos 2x + c_2 \sin 2x - \frac{3}{4}x \cos 2x.$$

14. From $m^2 - 4 = 0$ we find $m_1 = 2$ and $m_2 = -2$. Then $y_c = c_1e^{2x} + c_2e^{-2x}$ and we assume $y_p = (Ax^2 + Bx + C) \cos 2x + (Dx^2 + Ex + F) \sin 2x$. Substituting into the differential equati obtain

$$-8A = 0$$

$$-8B + 8D = 0$$

$$2A - 8C + 4E = 0$$

$$-8D = 1$$

$$-8A - 8E = 0$$

$$-4B + 2D - 8F = -3.$$

Then $A = 0$, $B = -\frac{1}{8}$, $C = 0$, $D = -\frac{1}{8}$, $E = 0$, $F = \frac{13}{32}$, so $y_p = -\frac{1}{8}\,x \cos 2x + \left(-\frac{1}{8}\,x^2 + \frac{13}{32}\right)$ and

$$y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{8}\,x \cos 2x + \left(-\frac{1}{8}\,x^2 + \frac{13}{32}\right) \sin 2x.$$

15. From $m^2 + 1 = 0$ we find $m_1 = i$ and $m_2 = -i$. Then $y_c = c_1 \cos x + c_2 \sin x$ and we as $y_p = (Ax^2 + Bx) \cos x + (Cx^2 + Dx) \sin x$. Substituting into the differential equation we $4C = 0$, $2A + 2D = 0$, $-4A = 2$, and $-2B + 2C = 0$. Then $A = -\frac{1}{2}$, $B = 0$, $C = 0$, $D =$ $y_p = -\frac{1}{2}x^2 \cos x + \frac{1}{2}x \sin x$, and

$$y = c_1 \cos x + c_2 \sin x - \frac{1}{2}x^2 \cos x + \frac{1}{2}x \sin x.$$

16. From $m^2 - 5m = 0$ we find $m_1 = 5$ and $m_2 = 0$. Then $y_c = c_1e^{5x} + c_2$ and we $y_p = Ax^4 + Bx^3 + Cx^2 + Dx$. Substituting into the differential equation we obtain -20 $12A - 15B = -4$, $6B - 10C = -1$, and $2C - 5D = 6$. Then $A = -\frac{1}{10}$, $B = \frac{14}{75}$, $C =$ $D = -\frac{697}{625}$, $y_p = -\frac{1}{10}x^4 + \frac{14}{75}x^3 + \frac{53}{250}x^2 - \frac{697}{625}x$, and

$$y = c_1e^{5x} + c_2 - \frac{1}{10}x^4 + \frac{14}{75}x^3 + \frac{53}{250}x^2 - \frac{697}{625}x.$$

Exercises 4.4 Undetermined Coefficients – Superposition Approach

17. From $m^2-2m+5=0$ we find $m_1=1+2i$ and $m_2=1-2i$. Then $y_c=e^x(c_1\cos 2x+c_2\sin 2x)$ and we assume $y_p=Axe^x\cos 2x+Bxe^x\sin 2x$. Substituting into the differential equation we obtain $4B=1$ and $-4A=0$. Then $A=0$, $B=\frac{1}{4}$, $y_p=\frac{1}{4}xe^x\sin 2x$, and

$$y=e^x(c_1\cos 2x+c_2\sin 2x)+\frac{1}{4}xe^x\sin 2x.$$

18. From $m^2-2m+2=0$ we find $m_1=1+i$ and $m_2=1-i$. Then $y_c=e^x(c_1\cos x+c_2\sin x)$ and we assume $y_p=Ae^{2x}\cos x+Be^{2x}\sin x$. Substituting into the differential equation we obtain $A+2B=1$ and $-2A+B=-3$. Then $A=\frac{7}{5}$, $B=-\frac{1}{5}$, $y_p=\frac{7}{5}e^{2x}\cos x-\frac{1}{5}e^{2x}\sin x$ and

$$y=e^x(c_1\cos x+c_2\sin x)+\frac{7}{5}e^{2x}\cos x-\frac{1}{5}e^{2x}\sin x.$$

27. We have $y_c = c_1 \cos 2x + c_2 \sin 2x$ and we assume $y_p = A$. Substituting into the differential equa we find $A = -\frac{1}{2}$. Thus $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{2}$. From the initial conditions we obtain $c_1 =$ and $c_2 = \sqrt{2}$, so $y = \sqrt{2}\,\sin 2x - \frac{1}{2}$.

28. We have $y_c = c_1 e^{-2x} + c_2 e^{x/2}$ and we assume $y_p = Ax^2 + Bx + C$. Substituting into the differe equation we find $A = -7$, $B = -19$, and $C = -37$. Thus $y = c_1 e^{-2x} + c_2 e^{x/2} - 7x^2 - 19x -$ From the initial conditions we obtain $c_1 = -\frac{1}{5}$ and $c_2 = \frac{186}{5}$, so

$$y = -\frac{1}{5}e^{-2x} + \frac{186}{5}e^{x/2} - 7x^2 - 19x - 37.$$

29. We have $y_c = c_1 e^{-x/5} + c_2$ and we assume $y_p = Ax^2 + Bx$. Substituting into the differential equ we find $A = -3$ and $B = 30$. Thus $y = c_1 e^{-x/5} + c_2 - 3x^2 + 30x$. From the initial conditio obtain $c_1 = 200$ and $c_2 = -200$, so

$$y = 200e^{-x/5} - 200 - 3x^2 + 30x.$$

30. We have $y_c = c_1 e^{-2x} + c_2 x e^{-2x}$ and we assume $y_p = (Ax^3 + Bx^2)e^{-2x}$. Substituting int differential equation we find $A = \frac{1}{6}$ and $B = \frac{3}{2}$. Thus $y = c_1 e^{-2x} + c_2 x e^{-2x} + \left(\frac{1}{6}x^3 + \frac{3}{2}x^2\right)$ From the initial conditions we obtain $c_1 = 2$ and $c_2 = 9$, so

$$y = 2e^{-2x} + 9xe^{-2x} + \left(\frac{1}{6}x^3 + \frac{3}{2}x^2\right)e^{-2x}.$$

37. We have $y_c = c_1 \cos x + c_2 \sin x$ and we assume $y_p = Ax^2 + Bx + C$. Substituting into the differen equation we find $A = 1$. $B = 0$, and $C = -1$. Thus $y = c_1 \cos x + c_2 \sin x + x^2 - 1$. From $y(0) =$ and $y(1) = 0$ we obtain

$$c_1 - 1 = 5$$

$$(\cos 1)c_1 + (\sin 1)c_2 = 0.$$

Solving this system we find $c_1 = 6$ and $c_2 = -6 \cot 1$. The solution of the boundary-value prob is

$$y = 6 \cos x - 6(\cot 1) \sin x + x^2 - 1.$$

38. We have $y_c = e^x(c_1 \cos x + c_2 \sin x)$ and we assume $y_p = Ax + B$. Substituting into the differe equation we find $A = 1$ and $B = 0$. Thus $y = e^x(c_1 \cos x + c_2 \sin x) + x$. From $y(0) = 0$ and $y(\pi) =$ we obtain

$$c_1 = 0$$

$$\pi - e^{\pi} c_1 = \pi.$$

Solving this system we find $c_1 = 0$ and c_2 is any real number. The solution of the boundary-v problem is

$$y = c_2 e^x \sin x + x.$$

39. The general solution of the differential equation $y'' + 3y = 6x$ is $y = c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x -$ The condition $y(0) = 0$ implies $c_1 = 0$ and so $y = c_2 \sin \sqrt{3}x + 2x$. The condition $y(1) + y'(1 =$ implies $c_2 \sin \sqrt{3} + 2 + c_2\sqrt{3} \cos \sqrt{3} + 2 = 0$ so $c_2 = -4/(\sin \sqrt{3} + \sqrt{3} \cos \sqrt{3})$. The solution is

$$y = \frac{-4 \sin \sqrt{3}x}{\sin \sqrt{3} + \sqrt{3} \cos \sqrt{3}} + 2x.$$

40. Using the general solution $y = c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x + 2x$, the boundary conditions $y(0) + y'(0 =$ $y(1) = 0$ yield the system

$$c_1 + \sqrt{3}c_2 + 2 = 0$$

$$c_1 \cos \sqrt{3} + c_2 \sin \sqrt{3} + 2 = 0.$$

Solving gives

$$c_1 = \frac{2(-\sqrt{3} + \sin \sqrt{3})}{\sqrt{3} \cos \sqrt{3} - \sin \sqrt{3}} \quad \text{and} \quad c_2 = \frac{2(1 - \cos \sqrt{3})}{\sqrt{3} \cos \sqrt{3} - \sin \sqrt{3}}.$$

3.12 Questions with Solutions on Chapter 4.7, Questions-Solutions-Cauchy-Euler

EXERCISES 4.7

Answers to selected odd-numbered problems begin on page ANS-5.

In Problems 1–18 solve the given differential equation.

1. $x^2y'' - 2y = 0$
2. $4x^2y'' + y = 0$
3. $xy'' + y' = 0$
4. $xy'' - 3y' = 0$
5. $x^2y'' + xy' + 4y = 0$
6. $x^2y'' + 5xy' + 3y = 0$
7. $x^2y'' - 3xy' - 2y = 0$
8. $x^2y'' + 3xy' - 4y = 0$
9. $25x^2y'' + 25xy' + y = 0$
10. $4x^2y'' + 4xy' - y = 0$
11. $x^2y'' + 5xy' + 4y = 0$
12. $x^2y'' + 8xy' + 6y = 0$
13. $3x^2y'' + 6xy' + y = 0$
14. $x^2y'' - 7xy' + 41y = 0$
15. $x^3y''' - 6y = 0$
16. $x^3y''' + xy' - y = 0$
17. $xy^{(4)} + 6y''' = 0$
18. $x^4y^{(4)} + 6x^3y''' + 9x^2y'' + 3xy' + y = 0$

[redacted]

In Problems 25–30 solve the given initial-value problem. Use a graphing utility to graph the solution curve.

25. $x^2y'' + 3xy' = 0, \quad y(1) = 0, y'(1) = 4$
26. $x^2y'' - 5xy' + 8y = 0, \quad y(2) = 32, y'(2) = 0$
27. $x^2y'' + xy' + y = 0, \quad y(1) = 1, y'(1) = 2$
28. $x^2y'' - 3xy' + 4y = 0, \quad y(1) = 5, y'(1) = 3$

[redacted]

[redacted]

Solutions

1. The auxiliary equation is $m^2 - m - 2 = (m+1)(m-2) = 0$ so that $y = c_1x^{-1} + c_2x^2$.
2. The auxiliary equation is $4m^2 - 4m + 1 = (2m-1)^2 = 0$ so that $y = c_1x^{1/2} + c_2x^{1/2}\ln x$.
3. The auxiliary equation is $m^2 = 0$ so that $y = c_1 + c_2\ln x$.
4. The auxiliary equation is $m^2 - 4m = m(m-4) = 0$ so that $y = c_1 + c_2x^4$.
5. The auxiliary equation is $m^2 + 4 = 0$ so that $y = c_1\cos(2\ln x) + c_2\sin(2\ln x)$.
6. The auxiliary equation is $m^2 + 4m + 3 = (m+1)(m+3) = 0$ so that $y = c_1x^{-1} + c_2x^{-3}$.
7. The auxiliary equation is $m^2 - 4m - 2 = 0$ so that $y = c_1x^{2-\sqrt{6}} + c_2x^{2+\sqrt{6}}$.
8. The auxiliary equation is $m^2 + 2m - 4 = 0$ so that $y = c_1x^{-1+\sqrt{5}} + c_2x^{-1-\sqrt{5}}$.
9. The auxiliary equation is $25m^2 + 1 = 0$ so that $y = c_1\cos\left(\frac{1}{5}\ln x\right) + c_2\sin\left(\frac{1}{5}\ln x\right)$.
10. The auxiliary equation is $4m^2 - 1 = (2m-1)(2m+1) = 0$ so that $y = c_1x^{1/2} + c_2x^{-1/2}$.
11. The auxiliary equation is $m^2 + 4m + 4 = (m+2)^2 = 0$ so that $y = c_1x^{-2} + c_2x^{-2}\ln x$.
12. The auxiliary equation is $m^2 + 7m + 6 = (m+1)(m+6) = 0$ so that $y = c_1x^{-1} + c_2x^{-6}$.
13. The auxiliary equation is $3m^2 + 3m + 1 = 0$ so that

$$y = x^{-1/2}\left[c_1\cos\left(\frac{\sqrt{3}}{6}\ln x\right) + c_2\sin\left(\frac{\sqrt{3}}{6}\ln x\right)\right].$$

14. The auxiliary equation is $m^2 - 8m + 41 = 0$ so that $y = x^4\left[c_1\cos(5\ln x) + c_2\sin(5\ln x)\right]$.

15. Assuming that $y = x^m$ and substituting into the differential equation we obtain

$$m(m-1)(m-2)-6 = m^3-3m^2+2m-6 = (m-3)(m^2+2) = 0.$$

Thus

$$y = c_1x^3 + c_2\cos\left(\sqrt{2}\ln x\right) + c_3\sin\left(\sqrt{2}\ln x\right).$$

16. Assuming that $y = x^m$ and substituting into the differential equation we obtain

$$m(m-1)(m-2)+m-1 = m^3-3m^2+3m-1 = (m-1)^3 = 0.$$

Thus

$$y = c_1x + c_2x\ln x + c_3x(\ln x)^2.$$

17. Assuming that $y = x^m$ and substituting into the differential equation we obtain

$$m(m-1)(m-2)(m-3)+6m(m-1)(m-2) = m^4-7m^2+6m = m(m-1)(m-2)(m+3) = 0.$$

Thus

$$y = c_1 + c_2x + c_3x^2 + c_4x^{-3}.$$

18. Assuming that $y = x^m$ and substituting into the differential equation we obtain

$$m(m-1)(m-2)(m-3)+6m(m-1)(m-2)+9m(m-1)+3m+1 = m^4+2m^2+1 = (m^2+1)^2 = 0.$$

Thus

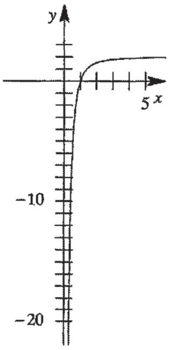
$$y = c_1\cos(\ln x) + c_2\sin(\ln x) + c_3(\ln x)\cos(\ln x) + c_4(\ln x)\sin(\ln x).$$

The auxiliary equation is $m^2 + 2m = m(m + 2) = 0$, so that $y = c_1 + c_2x^{-2}$ and $y' = -2c_2x^{-3}$. The initial conditions imply

$$c_1 + c_2 = 0$$

$$-2c_2 = 4.$$

Thus, $c_1 = 2$, $c_2 = -2$, and $y = 2 - 2x^{-2}$. The graph is given to the right.



The auxiliary equation is $m^2 - 6m + 8 = (m - 2)(m - 4) = 0$, so that

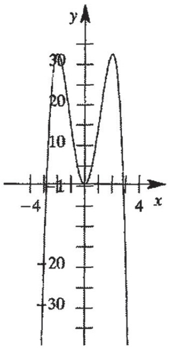
$$y = c_1x^2 + c_2x^4 \quad \text{and} \quad y' = 2c_1x + 4c_2x^3.$$

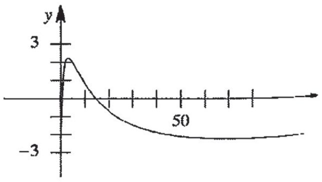
The initial conditions imply

$$4c_1 + 16c_2 = 32$$

$$4c_1 + 32c_2 = 0.$$

Thus, $c_1 = 16$, $c_2 = -2$, and $y = 16x^2 - 2x^4$. The graph is given to the right.





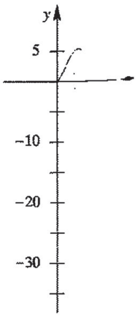
27. The auxiliary equation is $m^2+1=0$, so that

$$y=c_1\cos(\ln x)+c_2\sin(\ln x)$$

and

$$y'=-c_1\frac{1}{x}\sin(\ln x)+c_2\frac{1}{x}\cos(\ln x).$$

The initial conditions imply $c_1=1$ and $c_2=2$. Thus $y=\cos(\ln x)+2\sin(\ln x)$. The graph is given to the right.



28. The auxiliary equation is $m^2-4m+4=(m-2)^2=0$, so that

$$y=c_1x^2+c_2x^2\ln x \quad \text{and} \quad y'=2c_1x+c_2(x+2x\ln x).$$

The initial conditions imply $c_1=5$ and $c_2+10=3$. Thus $y=5x^2-7x^2\ln x$. The graph is given to the right.

3.13 Questions with Solutions on Chapter 4.6, Variation With Contant Coef. LDE

Variation and undetermined method

EXERCISES 4.6

Answers to selected odd-numbered problems begin on page ANS-5.

In Problems 1–18 solve each differential equation by variation of parameters.

1. $y'' + y = \sec x$

2. $y'' + y = \tan x$

3. $y'' + y = \sin x$

4. $y'' + y = \sec \theta \tan \theta$

5. $y'' + y = \cos^2 x$

6. $y'' + y = \sec^2 x$

7. $y'' - y = \cosh x$

8. $y'' - y = \sinh 2x$

9. $y'' - 4y = \dfrac{e^{2x}}{x}$

10. $y'' - 9y = \dfrac{9x}{e^{3x}}$

11. $y'' + 3y' + 2y = \dfrac{1}{1 + e^x}$

12. $y'' - 2y' + y = \dfrac{e^x}{1 + x^2}$

Exercises 4.6 Variation of Parameters

The particular solution, $y_p = u_1y_1 + u_2y_2$, *in the following problems can take on a variety of especially where trigonometric functions are involved. The validity of a particular form can checked by substituting it back into the differential equation.*

1. The auxiliary equation is $m^2 + 1 = 0$, so $y_c = c_1 \cos x + c_2 \sin x$ and

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

Identifying $f(x) = \sec x$ we obtain

$$u_1' = -\frac{\sin x \sec x}{1} = -\tan x$$

$$u_2' = \frac{\cos x \sec x}{1} = 1.$$

Then $u_1 = \ln|\cos x|$, $u_2 = x$, and

$$y = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x.$$

2. The auxiliary equation is $m^2 + 1 = 0$, so $y_c = c_1 \cos x + c_2 \sin x$ and

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

Identifying $f(x) = \tan x$ we obtain

$$u_1' = -\sin x \tan x = \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x$$

$$u_2' = \sin x.$$

Then $u_1 = \sin x - \ln|\sec x + \tan x|$, $u_2 = -\cos x$, and

$$y = c_1 \cos x + c_2 \sin x + \cos x\,(\sin x - \ln|\sec x + \tan x|) - \cos x \sin x$$

$$= c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|.$$

3. The auxiliary equation is $m^2 + 1 = 0$, so $y_c = c_1 \cos x + c_2 \sin x$ and

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

Identifying $f(x) = \sin x$ we obtain

$$\begin{aligned} u_1' &= -\sin^2 x \\ u_2' &= \cos x \sin x. \end{aligned}$$

Then

$$\begin{aligned} u_1 &= \frac{1}{4}\sin 2x - \frac{1}{2}x = \frac{1}{2}\sin x\cos x - \frac{1}{2}x \\ u_2 &= -\frac{1}{2}\cos^2 x. \end{aligned}$$

and

$$\begin{aligned} y &= c_1\cos x + c_2\sin x + \frac{1}{2}\sin x\cos^2 x - \frac{1}{2}x\cos x - \frac{1}{2}\cos^2 x\sin x \\ &= c_1\cos x + c_2\sin x - \frac{1}{2}x\cos x. \end{aligned}$$

4. The auxiliary equation is $m^2+1=0$, so $y_c = c_1\cos x + c_2\sin x$ and

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

Identifying $f(x) = \sec x\tan x$ we obtain

$$\begin{aligned} u_1' &= -\sin x(\sec x\tan x) = -\tan^2 x = 1 - \sec^2 x \\ u_2' &= \cos x(\sec x\tan x) = \tan x. \end{aligned}$$

Then $u_1 = x - \tan x$, $u_2 = -\ln|\cos x|$, and

$$\begin{aligned} y &= c_1\cos x + c_2\sin x + x\cos x - \sin x - \sin x\ln|\cos x| \\ &= c_1\cos x + c_3\sin x + x\cos x - \sin x\ln|\cos x|. \end{aligned}$$

5. The auxiliary equation is $m^2+1=0$, so $y_c = c_1\cos x + c_2\sin x$ and

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

Identifying $f(x) = \cos^2 x$ we obtain

$$\begin{aligned} u_1' &= -\sin x\cos^2 x \\ u_2' &= \cos^3 x = \cos x\left(1 - \sin^2 x\right). \end{aligned}$$

Then $u_1 = \frac{1}{3}\cos^3 x$, $u_2 = \sin x - \frac{1}{3}\sin^3 x$, and

$$\begin{aligned} y &= c_1 \cos x + c_2 \sin x + \frac{1}{3}\cos^4 x + \sin^2 x - \frac{1}{3}\sin^4 x \\ &= c_1 \cos x + c_2 \sin x + \frac{1}{3}\left(\cos^2 x + \sin^2 x\right)\left(\cos^2 x - \sin^2 x\right) + \sin^2 x \\ &= c_1 \cos x + c_2 \sin x + \frac{1}{3}\cos^2 x + \frac{2}{3}\sin^2 x \\ &= c_1 \cos x + c_2 \sin x + \frac{1}{3} + \frac{1}{3}\sin^2 x. \end{aligned}$$

6. The auxiliary equation is $m^2 + 1 = 0$, so $y_c = c_1 \cos x + c_2 \sin x$ and

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

Identifying $f(x) = \sec^2 x$ we obtain

$$u_1' = -\frac{\sin x}{\cos^2 x}$$

$$u_2' = \sec x.$$

Then

$$u_1 = -\frac{1}{\cos x} = -\sec x$$

$$u_2 = \ln|\sec x + \tan x|$$

and

$$y = c_1 \cos x + c_2 \sin x - \cos x \sec x + \sin x \ln|\sec x + \tan x|$$

$$= c_1 \cos x + c_2 \sin x - 1 + \sin x \ln|\sec x + \tan x|.$$

7. The auxiliary equation is $m^2 - 1 = 0$, so $y_c = c_1 e^x + c_2 e^{-x}$ and

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2.$$

Identifying $f(x) = \cosh x = \frac{1}{2}(e^{-x} + e^x)$ we obtain

$$u_1' = \frac{1}{4}e^{-2x} + \frac{1}{4}$$

$$u_2' = -\frac{1}{4} - \frac{1}{4}e^{2x}.$$

Then

$$u_1 = -\frac{1}{8}e^{-2x} + \frac{1}{4}x$$

$$u_2 = -\frac{1}{8}e^{2x} - \frac{1}{4}x$$

and

$$\begin{aligned} y &= c_1e^x + c_2e^{-x} - \frac{1}{8}e^{-x} + \frac{1}{4}xe^x - \frac{1}{8}e^x - \frac{1}{4}xe^{-x} \\ &= c_3e^x + c_4e^{-x} + \frac{1}{4}x(e^x - e^{-x}) \\ &= c_3e^x + c_4e^{-x} + \frac{1}{2}x\sinh x. \end{aligned}$$

The auxiliary equation is $m^2 - 1 = 0$, so $y_c = c_1e^x + c_2e^{-x}$ and

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2.$$

Identifying $f(x) = \sinh 2x$ we obtain

$$\begin{aligned} u_1' &= -\frac{1}{4}e^{-3x} + \frac{1}{4}e^x \\ u_2' &= \frac{1}{4}e^{-x} - \frac{1}{4}e^{3x}. \end{aligned}$$

Then

$$\begin{aligned} u_1 &= \frac{1}{12}e^{-3x} + \frac{1}{4}e^x \\ u_2 &= -\frac{1}{4}e^{-x} - \frac{1}{12}e^{3x}. \end{aligned}$$

and

$$\begin{aligned} y &= c_1e^x + c_2e^{-x} + \frac{1}{12}e^{-2x} + \frac{1}{4}e^{2x} - \frac{1}{4}e^{-2x} - \frac{1}{12}e^{2x} \\ &= c_1e^x + c_2e^{-x} + \frac{1}{6}\left(e^{2x} - e^{-2x}\right) \\ &= c_1e^x + c_2e^{-x} + \frac{1}{3}\sinh 2x. \end{aligned}$$

The auxiliary equation is $m^2 - 4 = 0$, so $y_c = c_1e^{2x} + c_2e^{-2x}$ and

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4.$$

Identifying $f(x) = e^{2x}/x$ we obtain $u_1' = 1/4x$ and $u_2' = -e^{4x}/4x$. Then

$$\begin{aligned} u_1 &= \frac{1}{4}\ln|x|, \\ u_2 &= -\frac{1}{4}\int_{x_0}^{x}\frac{e^{4t}}{t}\,dt \end{aligned}$$

and

$$y = c_1e^{2x} + c_2e^{-2x} + \frac{1}{4}\left(e^{2x}\ln|x| - e^{-2x}\int_{x_0}^{x}\frac{e^{4t}}{t}\,dt\right), \qquad x_0 > 0.$$

10. The auxiliary equation is $m^2 - 9 = 0$, so $y_c = c_1e^{3x} + c_2e^{-3x}$ and

$$W = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6.$$

Identifying $f(x) = 9x/e^{3x}$ we obtain $u_1' = \frac{3}{2}xe^{-6x}$ and $u_2' = -\frac{3}{2}x$. Then

$$u_1 = -\frac{1}{24}e^{-6x} - \frac{1}{4}xe^{-6x},$$

$$u_2 = -\frac{3}{4}x^2$$

and

$$y = c_1e^{3x} + c_2e^{-3x} - \frac{1}{24}e^{-3x} - \frac{1}{4}xe^{-3x} - \frac{3}{4}x^2e^{-3x}$$

$$= c_1e^{3x} + c_3e^{-3x} - \frac{1}{4}xe^{-3x}(1 - 3x).$$

11. The auxiliary equation is $m^2 + 3m + 2 = (m + 1)(m + 2) = 0$, so $y_c = c_1e^{-x} + c_2e^{-2x}$ and

$$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}.$$

Identifying $f(x) = 1/(1 + e^x)$ we obtain

$$u_1' = \frac{e^x}{1 + e^x}$$

$$u_2' = -\frac{e^{2x}}{1 + e^x} = \frac{e^x}{1 + e^x} - e^x.$$

Then $u_1 = \ln(1 + e^x)$, $u_2 = \ln(1 + e^x) - e^x$, and

$$y = c_1e^{-x} + c_2e^{-2x} + e^{-x}\ln(1 + e^x) + e^{-2x}\ln(1 + e^x) - e^{-x}$$

$$= c_3e^{-x} + c_2e^{-2x} + (1 + e^{-x})e^{-x}\ln(1 + e^x).$$

12. The auxiliary equation is $m^2 - 2m + 1 = (m - 1)^2 = 0$, so $y_c = c_1e^x + c_2xe^x$ and

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^{2x}.$$

Identifying $f(x) = e^x/\left(1 + x^2\right)$ we obtain

$$u_1' = -\frac{xe^xe^x}{e^{2x}\left(1 + x^2\right)} = -\frac{x}{1 + x^2}$$

$$u_2' = \frac{e^xe^x}{e^{2x}\left(1 + x^2\right)} = \frac{1}{1 + x^2}.$$

3.14 Questions with Solutions on Chapter 4.7, Variation with Cauchy Euler LDE

Variation Method and Cauchy-Euler Equations

In Problems 19–24 solve the given differential equation by variation of parameters.

19. $xy'' - 4y' = x^4$

20. $2x^2y'' + 5xy' + y = x^2 - x$

21. $x^2y'' - xy' + y = 2x$ **22.** $x^2y'' - 2xy' + 2y = x^4e^x$

23. $x^2y'' + xy' - y = \ln x$ **24.** $x^2y'' + xy' - y = \dfrac{1}{x + 1}$

19 The auxiliary equation is $m^2 - 5m = m(m-5) = 0$ so that $y_c = c_1 + c_2x^5$ and

$$W(1, x^5) = \begin{vmatrix} 1 & x^5 \\ 0 & 5x^4 \end{vmatrix} = 5x^4.$$

Identifying $f(x) = x^3$ we obtain $u_1' = -\frac{1}{5}x^4$ and $u_2' = 1/5x$. Then $u_1 = -\frac{1}{25}x^5$, $u_2 = \frac{1}{5}\ln x$, and

$$y = c_1 + c_2x^5 - \frac{1}{25}x^5 + \frac{1}{5}x^5 \ln x = c_1 + c_3x^5 + \frac{1}{5}x^5 \ln x.$$

26 The auxiliary equation is $2m^2 + 3m + 1 = (2m+1)(m+1) = 0$ so that $y_c = c_1x^{-1} + c_2x^{-1/2}$ and

$$W(x^{-1}, x^{-1/2}) = \begin{vmatrix} x^{-1} & x^{-1/2} \\ -x^{-2} & -\frac{1}{2}x^{-3/2} \end{vmatrix} = \frac{1}{2}x^{-5/2}.$$

Identifying $f(x) = \frac{1}{2} - \frac{1}{2x}$ we obtain $u_1' = x - x^2$ and $u_2' = x^{3/2} - x^{1/2}$. Then $u_1 = \frac{1}{2}x^2 - \frac{1}{3}x^3$, $= \frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2}$, and

$$y = c_1x^{-1} + c_2x^{-1/2} + \frac{1}{2}x - \frac{1}{3}x^2 + \frac{2}{5}x^2 - \frac{2}{3}x = c_1x^{-1} + c_2x^{-1/2} - \frac{1}{6}x + \frac{1}{15}x^2.$$

21. The auxiliary equation is $m^2 - 2m + 1 = (m - 1)^2 = 0$ so that $y_c = c_1x + c_2x\ln x$ and

$$W(x, x\ln x) = \begin{vmatrix} x & x\ln x \\ 1 & 1 + \ln x \end{vmatrix} = x.$$

Identifying $f(x) = 2/x$ we obtain $u_1' = -2\ln x/x$ and $u_2' = 2/x$. Then $u_1 = -(\ln x)^2$, $u_2 = 2$ and

$$y = c_1x + c_2x\ln x - x(\ln x)^2 + 2x(\ln x)^2$$

$$= c_1x + c_2x\ln x + x(\ln x)^2, \qquad x > 0.$$

22. The auxiliary equation is $m^2 - 3m + 2 = (m - 1)(m - 2) = 0$ so that $y_c = c_1x + c_2x^2$ and

$$W(x, x^2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2.$$

Identifying $f(x) = x^2e^x$ we obtain $u_1' = -x^2e^x$ and $u_2' = xe^x$. Then $u_1 = -x^2e^x + 2xe^x -$ $u_2 = xe^x - e^x$, and

$$y = c_1x + c_2x^2 - x^3e^x + 2x^2e^x - 2xe^x + x^3e^x - x^2e^x$$

$$= c_1x + c_2x^2 + x^2e^x - 2xe^x.$$

23. The auxiliary equation $m(m - 1) + m - 1 = m^2 - 1 = 0$ has roots $m_1 = -1$, $m_2 =$ $y_c = c_1x^{-1} + c_2x$. With $y_1 = x^{-1}$, $y_2 = x$, and the identification $f(x) = \ln x/x^2$, we get

$$W = 2x^{-1}, \qquad W_1 = -\ln x/x, \qquad \text{and} \qquad W_2 = \ln x/x^3.$$

Then $u_1' = W_1/W = -(\ln x)/2$, $u_2' = W_2/W = (\ln x)/2x^2$, and integration by parts gives

$$u_1 = \frac{1}{2}x - \frac{1}{2}x\ln x$$

$$u_2 = -\frac{1}{2}x^{-1}\ln x - \frac{1}{2}x^{-1},$$

so

$$y_p = u_1y_1 + u_2y_2 = \left(\frac{1}{2}x - \frac{1}{2}x\ln x\right)x^{-1} + \left(-\frac{1}{2}x^{-1}\ln x - \frac{1}{2}x^{-1}\right)x = -\ln x$$

and

$$y = y_c + y_p = c_1x^{-1} + c_2x - \ln x, \qquad x > 0.$$

24. The auxiliary equation $m(m - 1) + m - 1 = m^2 - 1 = 0$ has roots $m_1 = -1$, $m_2 =$ $y_c = c_1x^{-1} + c_2x$. With $y_1 = x^{-1}$, $y_2 = x$, and the identification $f(x) = 1/x^2(x + 1)$, we get

$$W = 2x^{-1}, \qquad W_1 = -1/x(x + 1), \qquad \text{and} \qquad W_2 = 1/x^3(x + 1).$$

Then $u_1' = W_1/W = -1/2(x+1)$, $u_2' = W_2/W = 1/2x^2(x+1)$, and integration (by partial fractions for u_2') gives

$$\begin{aligned} u_1 &= -\frac{1}{2}\ln(x+1) \\ u_2 &= -\frac{1}{2}x^{-1} - \frac{1}{2}\ln x + \frac{1}{2}\ln(x+1), \end{aligned}$$

so

$$\begin{aligned} y_p = u_1y_1 + u_2y_2 &= \left[-\frac{1}{2}\ln(x+1)\right]x^{-1} + \left[-\frac{1}{2}x^{-1} - \frac{1}{2}\ln x + \frac{1}{2}\ln(x+1)\right]x \\ &= -\frac{1}{2} - \frac{1}{2}x\ln x + \frac{1}{2}x\ln(x+1) - \frac{\ln(x+1)}{2x} = -\frac{1}{2} + \frac{1}{2}x\ln\left(1+\frac{1}{x}\right) - \frac{\ln(x+1)}{2x} \end{aligned}$$

and

$$y = y_c + y_p = c_1x^{-1} + c_2x - \frac{1}{2} + \frac{1}{2}x\ln\left(1+\frac{1}{x}\right) - \frac{\ln(x+1)}{2x}, \qquad x > 0.$$

3.15 Questions with Solutions on Chapter 2.3, Questions-Solutions-on-First-Order-LDE

EXERCISES 2.3

Answers to selected odd-numbered problems begin on page ANS-2.

In Problems 1–24 find the general solution of the given differential equation. Give the largest interval *I* over which the general solution is defined. Determine whether there are any transient terms in the general solution.

1. $\dfrac{dy}{dx} = 5y$

2. $\dfrac{dy}{dx} + 2y = 0$

3. $\dfrac{dy}{dx} + y = e^{3x}$

4. $3\dfrac{dy}{dx} + 12y = 4$

5. $y' + 3x^2y = x^2$

6. $y' + 2xy = x^3$

7. $x^2y' + xy = 1$

8. $y' = 2y + x^2 + 5$

9. $x\dfrac{dy}{dx} - y = x^2 \sin x$

10. $x\dfrac{dy}{dx} + 2y = 3$

11. $x\dfrac{dy}{dx} + 4y = x^3 - x$

12. $(1 + x)\dfrac{dy}{dx} - xy = x + x^2$

13. $x^2y' + x(x + 2)y = e^x$

✓ **14.** $xy' + (1 + x)y = e^{-x} \sin 2x$

✓ **15.** $y\,dx - 4(x + y^6)\,dy = 0$

Exercises 2.3

Linear Equations

1. For $y' - 5y = 0$ an integrating factor is $e^{-\int 5\,dx} = e^{-5x}$ so that $\dfrac{d}{dx}\left[e^{-5x}y\right] = 0$ and $y = ce^{5x}$ for $-\infty < x < \infty$. There is no transient term.

2. For $y' + 2y = 0$ an integrating factor is $e^{\int 2\,dx} = e^{2x}$ so that $\dfrac{d}{dx}\left[e^{2x}y\right] = 0$ and $y = ce^{-2x}$ for $-\infty < x < \infty$. The transient term is ce^{-2x}.

3. For $y' + y = e^{3x}$ an integrating factor is $e^{\int dx} = e^{x}$ so that $\dfrac{d}{dx}\left[e^{x}y\right] = e^{4x}$ and $y = \frac{1}{4}e^{3x} + ce^{-x}$ for $-\infty < x < \infty$. The transient term is ce^{-x}.

4. For $y' + 4y = \frac{4}{3}$ an integrating factor is $e^{\int 4\,dx} = e^{4x}$ so that $\dfrac{d}{dx}\left[e^{4x}y\right] = \frac{4}{3}e^{4x}$ and $y = \frac{1}{3} + ce^{-4x}$ for $-\infty < x < \infty$. The transient term is ce^{-4x}.

5. For $y'+3x^2y = x^2$ an integrating factor is $e^{\int 3x^2\,dx} = e^{x^3}$ so that $\frac{d}{dx}\left[e^{x^3}y\right] = x^2e^{x^3}$ and $y = \frac{1}{3}+c$ for $-\infty < x < \infty$. The transient term is ce^{-x^3}.

6. For $y' + 2xy = x^3$ an integrating factor is $e^{\int 2x\,dx} = e^{x^2}$ so that $\frac{d}{dx}\left[e^{x^2}y\right] = x^3e^{x^2}$ and $=$ $\frac{1}{2}x^2 - \frac{1}{2} + ce^{-x^2}$ for $-\infty < x < \infty$. The transient term is ce^{-x^2}.

7. For $y' + \frac{1}{x}y = \frac{1}{x^2}$ an integrating factor is $e^{\int(1/x)dx} = x$ so that $\frac{d}{dx}[xy] = \frac{1}{x}$ and $y = \frac{1}{x}\ln x -$ for $0 < x < \infty$. The entire solution is transient.

8. For $y' - 2y = x^2 + 5$ an integrating factor is $e^{-\int 2\,dx} = e^{-2x}$ so that $\frac{d}{dx}\left[e^{-2x}y\right] = x^2e^{-2x} + 5e$ and $y = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{11}{4} + ce^{2x}$ for $-\infty < x < \infty$. There is no transient term.

9. For $y' - \frac{1}{x}y = x\sin x$ an integrating factor is $e^{-\int(1/x)dx} = \frac{1}{x}$ so that $\frac{d}{dx}\left[\frac{1}{x}y\right] = \sin x$ $y = cx - x\cos x$ for $0 < x < \infty$. There is no transient term.

10. For $y' + \frac{2}{x}y = \frac{3}{x}$ an integrating factor is $e^{\int(2/x)dx} = x^2$ so that $\frac{d}{dx}\left[x^2y\right] = 3x$ and $y = \frac{3}{2} + c$ for $0 < x < \infty$. The transient term is cx^{-2}.

11. For $y' + \frac{4}{x}y = x^2 - 1$ an integrating factor is $e^{\int(4/x)dx} = x^4$ so that $\frac{d}{dx}\left[x^4y\right] = x^6 - x^4$ $y = \frac{1}{7}x^3 - \frac{1}{5}x + cx^{-4}$ for $0 < x < \infty$. The transient term is cx^{-4}.

12. For $y' - \frac{x}{(1+x)}y = x$ an integrating factor is $e^{-\int[x/(1+x)]dx} = (x+1)e^{-x}$ so that $\frac{d}{dx}\left[(x+1)e^{-x}y\right] =$ $x(x+1)e^{-x}$ and $y = -x - \frac{2x+3}{x+1} + \frac{ce^x}{x+1}$ for $-1 < x < \infty$. There is no transient term.

13. For $y' + \left(1 + \frac{2}{x}\right)y = \frac{e^x}{x^2}$ an integrating factor is $e^{\int[1+(2/x)]dx} = x^2e^x$ so that $\frac{d}{dx}\left[x^2e^xy\right] = e^{2x}$ $y = \frac{1}{2}\frac{e^x}{x^2} + \frac{ce^{-x}}{x^2}$ for $0 < x < \infty$. The transient term is $\frac{ce^{-x}}{x^2}$.

14. For $y' + \left(1 + \frac{1}{x}\right)y = \frac{1}{x}e^{-x}\sin 2x$ an integrating factor is $e^{\int[1+(1/x)]dx} = xe^x$ so that $\frac{d}{dx}\left[xe^xy\right] =$ $\sin 2x$ and $y = -\frac{1}{2x}e^{-x}\cos 2x + \frac{ce^{-x}}{x}$ for $0 < x < \infty$. The entire solution is transient.

15. For $\frac{dx}{dy} - \frac{4}{y}x = 4y^5$ an integrating factor is $e^{-\int(4/y)dy} = e^{\ln y^{-4}} = y^{-4}$ so that $\frac{d}{dy}\left[y^{-4}x\right] = 4y$ $x = 2y^6 + cy^4$ for $0 < y < \infty$. There is no transient term.

3.16 Questions with Solutions on Chapter 2.5, Bernoulli AND Substitution

EXERCISES 2.5

Answers to selected odd-numbered problems begin on page ANS-2.

Each DE in Problems 1–14 is homogeneous.

In Problems 1–10 solve the given differential equation by using an appropriate substitution.

[redacted]

Each DE in Problems 15–22 is a Bernoulli equation.

In Problems 15–20 solve the given differential equation by using an appropriate substitution.

15. $x\dfrac{dy}{dx} + y = \dfrac{1}{y^2}$

16. $\dfrac{dy}{dx} - y = e^x y^2$

17. $\dfrac{dy}{dx} = y(xy^3 - 1)$

18. $x\dfrac{dy}{dx} - (1 + x)y = xy^2$

19. $t^2\dfrac{dy}{dt} + y^2 = ty$

20. $3(1 + t^2)\dfrac{dy}{dt} = 2ty(y^3 - 1)$

In Problems 21 and 22 solve the given initial-value problem.

21. $x^2\dfrac{dy}{dx} - 2xy = 3y^4, \quad y(1) = \frac{1}{2}$

22. $y^{1/2}\dfrac{dy}{dx} + y^{3/2} = 1, \quad y(0) = 4$

Each DE in Problems 23–30 is of the form given in (5).

In Problems 23–28 solve the given differential equation by using an appropriate substitution.

23. $\dfrac{dy}{dx} = (x + y + 1)^2$

24. $\dfrac{dy}{dx} = \dfrac{1 - x - y}{x + y}$

25. $\dfrac{dy}{dx} = \tan^2(x + y)$

26. $\dfrac{dy}{dx} = \sin(x + y)$

27. $\dfrac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$

28. $\dfrac{dy}{dx} = 1 + e^{y-x+5}$

In Problems 29 and 30 solve the given initial-value problem.

29. $\dfrac{dy}{dx} = \cos(x + y), \quad y(0) = \pi/4$

30. $\dfrac{dy}{dx} = \dfrac{3x + 2y}{3x + 2y + 2}, \quad y(-1) = -1$

Discussion Problems

[redacted]

Note for Bernoulli
I used v = y^(1-n)/ here they use w = y^(1-n)

so w^\prime = (1-n)y^(-n) X y^\prime
so y^\prime + a_0(t)y = f(t)y^n, n not = 1.
by substitution....
w^\prime + (1-n)a_0(t)w = (1-n)f(t). Find w/ then
y = w^{1/(1-n)}
Note 23-30 can be done (but i explain in class)

16. From $y' - y = e^x y^2$ and $w = y^{-1}$ we obtain $\frac{dw}{dx} + w = -e^x$. An integrating factor is e^x so that $e^x w = -\frac{1}{2}e^{2x} + c$ or $y^{-1} = -\frac{1}{2}e^x + ce^{-x}$.

17. From $y' + y = xy^4$ and $w = y^{-3}$ we obtain $\frac{dw}{dx} - 3w = -3x$. An integrating factor is e^{-3x} so that $e^{-3x}w = xe^{-3x} + \frac{1}{3}e^{-3x} + c$ or $y^{-3} = x + \frac{1}{3} + ce^{3x}$.

18. From $y' - \left(1 + \frac{1}{x}\right)y = y^2$ and $w = y^{-1}$ we obtain $\frac{dw}{dx} + \left(1 + \frac{1}{x}\right)w = -1$. An integrating factor is xe^x so that $xe^x w = -xe^x + e^x + c$ or $y^{-1} = -1 + \frac{1}{x} + \frac{c}{x}e^{-x}$.

19. From $y' - \frac{1}{t}y = -\frac{1}{t^2}y^2$ and $w = y^{-1}$ we obtain $\frac{dw}{dt} + \frac{1}{t}w = \frac{1}{t^2}$. An integrating factor is t so that $tw = \ln t + c$ or $y^{-1} = \frac{1}{t}\ln t + \frac{c}{t}$. Writing this in the form $\frac{t}{y} = \ln t + c$, we see that the solution can also be expressed in the form $e^{t/y} = c_1 t$.

20. From $y' + \frac{2}{3\left(1+t^2\right)}y = \frac{2t}{3\left(1+t^2\right)}y^4$ and $w = y^{-3}$ we obtain $\frac{dw}{dt} - \frac{2t}{1+t^2}w = \frac{-2t}{1+t^2}$. An integrating factor is $\frac{1}{1+t^2}$ so that $\frac{w}{1+t^2} = \frac{1}{1+t^2} + c$ or $y^{-3} = 1 + c\left(1+t^2\right)$.

21. From $y' - \frac{2}{x}y = \frac{3}{x^2}y^4$ and $w = y^{-3}$ we obtain $\frac{dw}{dx} + \frac{6}{x}w = -\frac{9}{x^2}$. An integrating factor is x^6 so tha

$x^6 w = -\frac{9}{5}x^5 + c$ or $y^{-3} = -\frac{9}{5}x^{-1} + cx^{-6}$. If $y(1) = \frac{1}{2}$ then $c = \frac{49}{5}$ and $y^{-3} = -\frac{9}{5}x^{-1} + \frac{49}{5}x^{-6}$.

22. From $y' + y = y^{-1/2}$ and $w = y^{3/2}$ we obtain $\frac{dw}{dx} + \frac{3}{2}w = \frac{3}{2}$. An integrating factor is $e^{3x/2}$ so tha $e^{3x/2}w = e^{3x/2} + c$ or $y^{3/2} = 1 + ce^{-3x/2}$. If $y(0) = 4$ then $c = 7$ and $y^{3/2} = 1 + 7e^{-3x/2}$.

23. Let $u = x + y + 1$ so that $du/dx = 1 + dy/dx$. Then $\frac{du}{dx} - 1 = u^2$ or $\frac{1}{1+u^2}\,du = dx$. Thu $\tan^{-1} u = x + c$ or $u = \tan(x + c)$, and $x + y + 1 = \tan(x + c)$ or $y = \tan(x + c) - x - 1$.

24. Let $u = x + y$ so that $du/dx = 1 + dy/dx$. Then $\frac{du}{dx} - 1 = \frac{1-u}{u}$ or $u\,du = dx$. Thus $\frac{1}{2}u^2 = x +$ or $u^2 = 2x + c_1$, and $(x + y)^2 = 2x + c_1$.

25. Let $u = x + y$ so that $du/dx = 1 + dy/dx$. Then $\frac{du}{dx} - 1 = \tan^2 u$ or $\cos^2 u\,du = dx$. Thu $\frac{1}{2}u + \frac{1}{4}\sin 2u = x + c$ or $2u + \sin 2u = 4x + c_1$, and $2(x+y) + \sin 2(x+y) = 4x + c_1$ or $2y + \sin 2(x+y) = 2x + c_1$.

26. Let $u = x + y$ so that $du/dx = 1 + dy/dx$. Then $\frac{du}{dx} - 1 = \sin u$ or $\frac{1}{1+\sin u}\,du = dx$. Multiplyin by $(1 - \sin u)/(1 - \sin u)$ we have $\frac{1-\sin u}{\cos^2 u}\,du = dx$ or $(\sec^2 u - \sec u \tan u)du = dx$. Thu $\tan u - \sec u = x + c$ or $\tan(x + y) - \sec(x + y) = x + c$.

27. Let $u = y - 2x + 3$ so that $du/dx = dy/dx - 2$. Then $\frac{du}{dx} + 2 = 2 + \sqrt{u}$ or $\frac{1}{\sqrt{u}}\,du = dx$. Thu $2\sqrt{u} = x + c$ and $2\sqrt{y - 2x + 3} = x + c$.

28. Let $u = y - x + 5$ so that $du/dx = dy/dx - 1$. Then $\frac{du}{dx} + 1 = 1 + e^u$ or $e^{-u}du = dx$. Thu $-e^{-u} = x + c$ and $-e^{y-x+5} = x + c$.

29. Let $u = x + y$ so that $du/dx = 1 + dy/dx$. Then $\frac{du}{dx} - 1 = \cos u$ and $\frac{1}{1+\cos u}\,du = dx$. Now

$$\frac{1}{1+\cos u} = \frac{1-\cos u}{1-\cos^2 u} = \frac{1-\cos u}{\sin^2 u} = \csc^2 u - \csc u \cot u$$

so we have $\int(\csc^2 u - \csc u \cot u)du = \int dx$ and $-\cot u + \csc u = x + c$. Thus $-\cot(x+y) + \csc(x+y) = x + c$. Setting $x = 0$ and $y = \pi/4$ we obtain $c = \sqrt{2} - 1$. The solution is

$$\csc(x + y) - \cot(x + y) = x + \sqrt{2} - 1.$$

30. Let $u = 3x + 2y$ so that $du/dx = 3 + 2\,dy/dx$. Then $\frac{du}{dx} = 3 + \frac{2u}{u+2} = \frac{5u+6}{u+2}$ and $\frac{u+2}{5u+6}\,du = d$ Now by long division

$$\frac{u+2}{5u+6} = \frac{1}{5} + \frac{4}{25u+30}$$

3.17 Questions with Solutions on Chapter 2.4, Exact Nonlinear DE

EXERCISES 2.4

Answers to selected odd-numbered problems begin on page ANS-2.

In Problems 1–20 determine whether the given differential equation is exact. If it is exact, solve it.

1. $(2x - 1)\,dx + (3y + 7)\,dy = 0$

2. $(2x + y)\,dx - (x + 6y)\,dy = 0$

3. $(5x + 4y)\,dx + (4x - 8y^3)\,dy = 0$

4. $(\sin y - y \sin x)\,dx + (\cos x + x \cos y - y)\,dy = 0$

5. $(2xy^2 - 3)\,dx + (2x^2y + 4)\,dy = 0$

6. $\left(2y - \dfrac{1}{x} + \cos 3x\right)\dfrac{dy}{dx} + \dfrac{y}{x^2} - 4x^3 + 3y \sin 3x = 0$

7. $(x^2 - y^2)\,dx + (x^2 - 2xy)\,dy = 0$

8. $\left(1 + \ln x + \dfrac{y}{x}\right)dx = (1 - \ln x)\,dy$

9. $(x - y^3 + y^2 \sin x)\,dx = (3xy^2 + 2y \cos x)\,dy$

10. $(x^3 + y^3)\,dx + 3xy^2\,dy = 0$

11. $(y \ln y - e^{-xy})\,dx + \left(\dfrac{1}{y} + x \ln y\right)dy = 0$

12. $(3x^2y + e^y)\,dx + (x^3 + xe^y - 2y)\,dy = 0$

13. $x\dfrac{dy}{dx} = 2xe^x - y + 6x^2$

14. $\left(1 - \dfrac{3}{y} + x\right)\dfrac{dy}{dx} + y = \dfrac{3}{x} - 1$

15. $\left(x^2y^3 - \dfrac{1}{1 + 9x^2}\right)\dfrac{dx}{dy} + x^3y^2 = 0$

16. $(5y - 2x)y' - 2y = 0$

17. $(\tan x - \sin x \sin y)\,dx + \cos x \cos y\,dy = 0$

18. $(2y \sin x \cos x - y + 2y^2e^{xy^2})\,dx$
$= (x - \sin^2 x - 4xye^{xy^2})\,dy$

19. $(4t^3y - 15t^2 - y)\,dt + (t^4 + 3y^2 - t)\,dy = 0$

20. $\left(\dfrac{1}{t} + \dfrac{1}{t^2} - \dfrac{y}{t^2 + y^2}\right)dt + \left(ye^y + \dfrac{t}{t^2 + y^2}\right)dy = 0$

In Problems 21–26 solve the given initial-value problem.

21. $(x + y)^2\, dx + (2xy + x^2 - 1)\, dy = 0, \quad y(1) = 1$ ✓

22. $(e^x + y)\, dx + (2 + x + ye^y)\, dy = 0, \quad y(0) = 1$ ✓

23. $(4y + 2t - 5)\, dt + (6y + 4t - 1)\, dy = 0, \quad y(-1) = 2$ ✓

24. $\left(\dfrac{3y^2 - t^2}{y^5}\right)\dfrac{dy}{dt} + \dfrac{t}{2y^4} = 0, \quad y(1) = 1$ ✓

$M_y = f_{yx}$ }

Note

$N_x = f_x$

Exercises 2.4

Exact Equations

1. Let $M = 2x-1$ and $N = 3y+7$ so that $M_y = 0 = N_x$. From $f_x = 2x-1$ we obtain $f = x^2-x+h(y$ $h'(y) = 3y+7$, and $h(y) = \frac{3}{2}y^2 + 7y$. A solution is $x^2 - x + \frac{3}{2}y^2 + 7y = c$.

2. Let $M = 2x + y$ and $N = -x - 6y$. Then $M_y = 1$ and $N_x = -1$, so the equation is not exact.

3. Let $M = 5x + 4y$ and $N = 4x - 8y^3$ so that $M_y = 4 = N_x$. From $f_x = 5x + 4y$ we obta $f = \frac{5}{2}x^2 + 4xy + h(y)$, $h'(y) = -8y^3$, and $h(y) = -2y^4$. A solution is $\frac{5}{2}x^2 + 4xy - 2y^4 = c$.

4. Let $M = \sin y - y\sin x$ and $N = \cos x + x\cos y - y$ so that $M_y = \cos y - \sin x = N_x$. Fro $f_x = \sin y - y\sin x$ we obtain $f = x\sin y + y\cos x + h(y)$, $h'(y) = -y$, and $h(y) = -\frac{1}{2}y^2$. A solutio is $x\sin y + y\cos x - \frac{1}{2}y^2 = c$.

5. Let $M = 2y^2x - 3$ and $N = 2yx^2 + 4$ so that $M_y = 4xy = N_x$. From $f_x = 2y^2x - 3$ we obta $f = x^2y^2 - 3x + h(y)$, $h'(y) = 4$, and $h(y) = 4y$. A solution is $x^2y^2 - 3x + 4y = c$.

6. Let $M = 4x^3 - 3y\sin 3x - y/x^2$ and $N = 2y - 1/x + \cos 3x$ so that $M_y = -3\sin 3x - 1/x^2$ ar $N_x = 1/x^2 - 3\sin 3x$. The equation is not exact.

7. Let $M = x^2 - y^2$ and $N = x^2 - 2xy$ so that $M_y = -2y$ and $N_x = 2x - 2y$. The equation is n exact.

8. Let $M = 1 + \ln x + y/x$ and $N = -1 + \ln x$ so that $M_y = 1/x = N_x$. From $f_y = -1 + \ln x$ we obtain $f = -y + y\ln x + h(y)$, $h'(x) = 1 + \ln x$, and $h(y) = x\ln x$. A solution is $-y + y\ln x + x\ln x = c$.

9. Let $M = y^3 - y^2\sin x - x$ and $N = 3xy^2 + 2y\cos x$ so that $M_y = 3y^2 - 2y\sin x = N_x$. From $f_x = y^3 - y^2\sin x - x$ we obtain $f = xy^3 + y^2\cos x - \frac{1}{2}x^2 + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution is $xy^3 + y^2\cos x - \frac{1}{2}x^2 = c$.

10. Let $M = x^3 + y^3$ and $N = 3xy^2$ so that $M_y = 3y^2 = N_x$. From $f_x = x^3 + y^3$ we obtain $f = \frac{1}{4}x^4 + xy^3 + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution is $\frac{1}{4}x^4 + xy^3 = c$.

11. Let $M = y\ln y - e^{-xy}$ and $N = 1/y + x\ln y$ so that $M_y = 1 + \ln y + xe^{-xy}$ and $N_x = \ln y$. The equation is not exact.

12. Let $M = 3x^2y + e^y$ and $N = x^3 + xe^y - 2y$ so that $M_y = 3x^2 + e^y = N_x$. From $f_x = 3x^2y + e^y$ we obtain $f = x^3y + xe^y + h(y)$, $h'(y) = -2y$, and $h(y) = -y^2$. A solution is $x^3y + xe^y - y^2 = c$.

13. Let $M = y - 6x^2 - 2xe^x$ and $N = x$ so that $M_y = 1 = N_x$. From $f_x = y - 6x^2 - 2xe^x$ we obtain $f = xy - 2x^3 - 2xe^x + 2e^x + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution is $xy - 2x^3 - 2xe^x + 2e^x = c$.

14. Let $M = 1 - 3/x + y$ and $N = 1 - 3/y + x$ so that $M_y = 1 = N_x$. From $f_x = 1 - 3/x + y$ we obtain $f = x - 3\ln|x| + xy + h(y)$, $h'(y) = 1 - \dfrac{3}{y}$, and $h(y) = y - 3\ln|y|$. A solution is $x + y + xy - 3\ln|xy| = c$.

15. Let $M = x^2y^3 - 1/\left(1 + 9x^2\right)$ and $N = x^3y^2$ so that $M_y = 3x^2y^2 = N_x$. From $f_x = x^2y^3 - 1/\left(1 + 9x^2\right)$ we obtain $f = \frac{1}{3}x^3y^3 - \frac{1}{3}\arctan(3x) + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution is $x^3y^3 - \arctan(3x) = c$.

16. Let $M = -2y$ and $N = 5y - 2x$ so that $M_y = -2 = N_x$. From $f_x = -2y$ we obtain $f = -2xy + h(y)$, $h'(y) = 5y$, and $h(y) = \frac{5}{2}y^2$. A solution is $-2xy + \frac{5}{2}y^2 = c$.

17. Let $M = \tan x - \sin x\sin y$ and $N = \cos x\cos y$ so that $M_y = -\sin x\cos y = N_x$. From $f_x = \tan x - \sin x\sin y$ we obtain $f = \ln|\sec x| + \cos x\sin y + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution is $\ln|\sec x| + \cos x\sin y = c$.

18. Let $M = 2y\sin x\cos x - y + 2y^2e^{xy^2}$ and $N = -x + \sin^2 x + 4xye^{xy^2}$ so that

$$M_y = 2\sin x\cos x - 1 + 4xy^3e^{xy^2} + 4ye^{xy^2} = N_x.$$

From $f_x = 2y\sin x\cos x - y + 2y^2e^{xy^2}$ we obtain $f = y\sin^2 x - xy + 2e^{xy^2} + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution is $y\sin^2 x - xy + 2e^{xy^2} = c$.

19. Let $M = 4t^3y - 15t^2 - y$ and $N = t^4 + 3y^2 - t$ so that $M_y = 4t^3 - 1 = N_t$. From $f_t = 4t^3y - 15t^2 - y$ we obtain $f = t^4y - 5t^3 - ty + h(y)$, $h'(y) = 3y^2$, and $h(y) = y^3$. A solution is $t^4y - 5t^3 - ty + y^3 = c$.

20. Let $M = 1/t + 1/t^2 - y/\left(t^2 + y^2\right)$ and $N = ye^y + t/\left(t^2 + y^2\right)$ so that $M_y = \left(y^2 - t^2\right)/\left(t^2 + y^2\right)^2 = N_t$. From $f_t = 1/t + 1/t^2 - y/\left(t^2 + y^2\right)$ we obtain $f = \ln|t| - \dfrac{1}{t} - \arctan\left(\dfrac{t}{y}\right) + h(y)$, $h'(y) = ye^y$,

and $h(y) = ye^y - e^y$. A solution is

$$\ln|t| - \frac{1}{t} - \arctan\left(\frac{t}{y}\right) + ye^y - e^y = c.$$

21. Let $M = x^2+2xy+y^2$ and $N = 2xy+x^2-1$ so that $M_y = 2(x+y) = N_x$. From $f_x = x^2+2xy+y^2$ w obtain $f = \frac{1}{3}x^3+x^2y+xy^2+h(y)$, $h'(y) = -1$, and $h(y) = -y$. The solution is $\frac{1}{3}x^3+x^2y+xy^2-y =$ If $y(1) = 1$ then $c = 4/3$ and a solution of the initial-value problem is $\frac{1}{3}x^3 + x^2y + xy^2 - y = \frac{4}{3}$.

22. Let $M = e^x + y$ and $N = 2 + x + ye^y$ so that $M_y = 1 = N_x$. From $f_x = e^x + y$ we obtai $f = e^x + xy + h(y)$, $h'(y) = 2 + ye^y$, and $h(y) = 2y + ye^y - y$. The solution i $e^x + xy + 2y + ye^y - e^y = c$. If $y(0) = 1$ then $c = 3$ and a solution of the initial-value problem is $e^x + xy + 2y + ye^y - e^y = 3$.

23. Let $M = 4y + 2t - 5$ and $N = 6y + 4t - 1$ so that $M_y = 4 = N_t$. From $f_t = 4y + 2t - 5$ we obtai $f = 4ty+t^2-5t+h(y)$, $h'(y) = 6y-1$, and $h(y) = 3y^2-y$. The solution is $4ty+t^2-5t+3y^2-y =$ If $y(-1) = 2$ then $c = 8$ and a solution of the initial-value problem is $4ty + t^2 - 5t + 3y^2 - y = 8$.

24. Let $M = t/2y^4$ and $N = \left(3y^2 - t^2\right)/y^5$ so that $M_y = -2t/y^5 = N_t$. From $f_t = t/2y^4$ we obtai $f = \dfrac{t^2}{4y^4} + h(y)$, $h'(y) = \dfrac{3}{y^3}$, and $h(y) = -\dfrac{3}{2y^2}$. The solution is $\dfrac{t^2}{4y^4} - \dfrac{3}{2y^2} = c$. If $y(1) = 1$ the $c = -5/4$ and a solution of the initial-value problem is $\dfrac{t^2}{4y^4} - \dfrac{3}{2y^2} = -\dfrac{5}{4}$.

3.18 Questions with Solutions on Chapter 2.2, Separable DE

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Answers to selected odd-numbered problems begin on page ANS-1.

1. $\dfrac{dy}{dx} = \sin 5x$

2. $\dfrac{dy}{dx} = (x + 1)^2$

3. $dx + e^{3x}dy = 0$

4. $dy - (y - 1)^2 dx = 0$

5. $x\dfrac{dy}{dx} = 4y$

6. $\dfrac{dy}{dx} + 2xy^2 = 0$

7. $\dfrac{dy}{dx} = e^{3x+2y}$

8. $e^x y\dfrac{dy}{dx} = e^{-y} + e^{-2x-y}$

9. $y \ln x \dfrac{dx}{dy} = \left(\dfrac{y+1}{x}\right)^2$

10. $\dfrac{dy}{dx} = \left(\dfrac{2y+3}{4x+5}\right)^2$

11. $\csc y\, dx + \sec^2 x\, dy = 0$

12. $\sin 3x\, dx + 2y \cos^3 3x\, dy = 0$

13. $(e^y + 1)^2 e^{-y}\, dx + (e^x + 1)^3 e^{-x}\, dy = 0$

14. $x(1 + y^2)^{1/2}\, dx = y(1 + x^2)^{1/2}\, dy$

15. $\dfrac{dS}{dr} = kS$

16. $\dfrac{dQ}{dt} = k(Q - 70)$

17. $\dfrac{dP}{dt} = P - P^2$

18. $\dfrac{dN}{dt} + N = Nte^{t+2}$

19. $\dfrac{dy}{dx} = \dfrac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$

20. $\dfrac{dy}{dx} = \dfrac{xy + 2y - x - 2}{xy - 3y + x - 3}$

21. $\dfrac{dy}{dx} = x\sqrt{1 - y^2}$

22. $(e^x + e^{-x})\dfrac{dy}{dx} = y^2$

In Problems 23–28 find an explicit solution of the given initial-value problem.

23. $\dfrac{dx}{dt} = 4(x^2 + 1), \quad x(\pi/4) = 1$

24. $\dfrac{dy}{dx} = \dfrac{y^2 - 1}{x^2 - 1}, \quad y(2) = 2$

25. $x^2\dfrac{dy}{dx} = y - xy, \quad y(-1) = -1$

26. $\dfrac{dy}{dt} + 2y = 1, \quad y(0) = \frac{5}{2}$

27. $\sqrt{1 - y^2}\, dx - \sqrt{1 - x^2}\, dy = 0, \quad y(0) = \dfrac{\sqrt{3}}{2}$

28. $(1 + x^4)\, dy + x(1 + 4y^2)\, dx = 0, \quad y(1) = 0$

In Problems 29 and 30 proceed as in Example 5 and find an explicit solution of the given initial-value problem.

29. $\dfrac{dy}{dx} = ye^{-x^2}, \quad y(4) = 1$

30. $\dfrac{dy}{dx} = y^2 \sin x^2, \quad y(-2) = \frac{1}{3}$

31. **(a)** Find a solution of the initial-value problem consisting of the differential equation in Example 3 and the initial conditions $y(0) = 2$, $y(0) = -2$, and $y\left(\frac{1}{4}\right) = 1$.

1. From $dy = \sin 5x\, dx$ we obtain $y = -\frac{1}{5}\cos 5x + c$.

2. From $dy = (x+1)^2\, dx$ we obtain $y = \frac{1}{3}(x+1)^3 + c$.

3. From $dy = -e^{-3x}\, dx$ we obtain $y = \frac{1}{3}e^{-3x} + c$.

4. From $\dfrac{1}{(y-1)^2}\, dy = dx$ we obtain $-\dfrac{1}{y-1} = x + c$ or $y = 1 - \dfrac{1}{x+c}$.

5. From $\dfrac{1}{y}\, dy = \dfrac{4}{x}\, dx$ we obtain $\ln|y| = 4\ln|x| + c$ or $y = c_1x^4$.

6. From $\dfrac{1}{y^2}\, dy = -2x\, dx$ we obtain $-\dfrac{1}{y} = -x^2 + c$ or $y = \dfrac{1}{x^2 + c_1}$.

7. From $e^{-2y}dy = e^{3x}dx$ we obtain $3e^{-2y} + 2e^{3x} = c$.

8. From $ye^y dy = \left(e^{-x} + e^{-3x}\right) dx$ we obtain $ye^y - e^y + e^{-x} + \dfrac{1}{3}e^{-3x} = c$.

9. From $\left(y + 2 + \dfrac{1}{y}\right) dy = x^2 \ln x\, dx$ we obtain $\dfrac{y^2}{2} + 2y + \ln|y| = \dfrac{x^3}{3}\ln|x| - \dfrac{1}{9}x^3 + c$.

10. From $\dfrac{1}{(2y+3)^2}\, dy = \dfrac{1}{(4x+5)^2}\, dx$ we obtain $\dfrac{2}{2y+3} = \dfrac{1}{4x+5} + c$.

11. From $\dfrac{1}{\csc y}\, dy = -\dfrac{1}{\sec^2 x}\, dx$ or $\sin y\, dy = -\cos^2 x\, dx = -\frac{1}{2}(1 + \cos 2x)\, dx$ we obtain
$-\cos y = -\frac{1}{2}x - \frac{1}{4}\sin 2x + c$ or $4\cos y = 2x + \sin 2x + c_1$.

12. From $2y\, dy = -\dfrac{\sin 3x}{\cos^3 3x}\, dx$ or $2y\, dy = -\tan 3x \sec^2 3x\, dx$ we obtain $y^2 = -\frac{1}{6}\sec^2 3x + c$.

13. From $\dfrac{e^y}{(e^y+1)^2}\, dy = \dfrac{-e^x}{(e^x+1)^3}\, dx$ we obtain $-(e^y+1)^{-1} = \frac{1}{2}(e^x+1)^{-2} + c$.

14. From $\dfrac{y}{(1+y^2)^{1/2}}\, dy = \dfrac{x}{(1+x^2)^{1/2}}\, dx$ we obtain $\left(1+y^2\right)^{1/2} = \left(1+x^2\right)^{1/2} + c$.

15. From $\dfrac{1}{S}\, dS = k\, dr$ we obtain $S = ce^{kr}$.

16. From $\dfrac{1}{Q-70}\, dQ = k\, dt$ we obtain $\ln|Q - 70| = kt + c$ or $Q - 70 = c_1e^{kt}$.

17. From $\frac{1}{P-P^2}dP = \left(\frac{1}{P}+\frac{1}{1-P}\right)dP = dt$ we obtain $\ln|P| - \ln|1-P| = t+c$ so that $\ln\left|\frac{P}{1-P}\right| = t+c$ or $\frac{P}{1-P} = c_1e^t$. Solving for P we have $P = \frac{c_1e^t}{1+c_1e^t}$.

18. From $\frac{1}{N}dN = \left(te^{t+2}-1\right)dt$ we obtain $\ln|N| = te^{t+2} - e^{t+2} - t + c$ or $N = c_1e^{te^{t+2}-e^{t+2}-t}$.

19. From $\frac{y-2}{y+3}dy = \frac{x-1}{x+4}dx$ or $\left(1-\frac{5}{y+3}\right)dy = \left(1-\frac{5}{x+4}\right)dx$ we obtain $y - 5\ln|y+3| = x - 5\ln|x+4| + c$ or $\left(\frac{x+4}{y+3}\right)^5 = c_1e^{x-y}$.

20. From $\frac{y+1}{y-1}dy = \frac{x+2}{x-3}dx$ or $\left(1+\frac{2}{y-1}\right)dy = \left(1+\frac{5}{x-3}\right)dx$ we obtain $y + 2\ln|y-1| = x + 5\ln|x-3| + c$ or $\frac{(y-1)^2}{(x-3)^5} = c_1e^{x-y}$.

21. From $x\,dx = \frac{1}{\sqrt{1-y^2}}dy$ we obtain $\frac{1}{2}x^2 = \sin^{-1}y + c$ or $y = \sin\left(\frac{x^2}{2}+c_1\right)$.

22. From $\frac{1}{y^2}dy = \frac{1}{e^x+e^{-x}}dx = \frac{e^x}{(e^x)^2+1}dx$ we obtain $-\frac{1}{y} = \tan^{-1}e^x + c$ or $y = -\frac{1}{\tan^{-1}e^x+c}$.

23. From $\frac{1}{x^2+1}dx = 4\,dt$ we obtain $\tan^{-1}x = 4t+c$. Using $x(\pi/4) = 1$ we find $c = -3\pi/4$. The solution of the initial-value problem is $\tan^{-1}x = 4t - \frac{3\pi}{4}$ or $x = \tan\left(4t-\frac{3\pi}{4}\right)$.

24. From $\frac{1}{y^2-1}dy = \frac{1}{x^2-1}dx$ or $\frac{1}{2}\left(\frac{1}{y-1}-\frac{1}{y+1}\right)dy = \frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx$ we obtain $\ln|y-1| - \ln|y+1| = \ln|x-1| - \ln|x+1| + \ln c$ or $\frac{y-1}{y+1} = \frac{c(x-1)}{x+1}$. Using $y(2) = 2$ we find $c = 1$. A solution of the initial-value problem is $\frac{y-1}{y+1} = \frac{x-1}{x+1}$ or $y = x$.

25. From $\frac{1}{y}dy = \frac{1-x}{x^2}dx = \left(\frac{1}{x^2}-\frac{1}{x}\right)dx$ we obtain $\ln|y| = -\frac{1}{x} - \ln|x| = c$ or $xy = c_1e^{-1/x}$. Using $y(-1) = -1$ we find $c_1 = e^{-1}$. The solution of the initial-value problem is $xy = e^{-1-1/x}$ or $y = e^{-(1+1/x)}/x$.

26. From $\frac{1}{1-2y}dy = dt$ we obtain $-\frac{1}{2}\ln|1-2y| = t+c$ or $1-2y = c_1e^{-2t}$. Using $y(0) = 5/2$ we find $c_1 = -4$. The solution of the initial-value problem is $1-2y = -4e^{-2t}$ or $y = 2e^{-2t} + \frac{1}{2}$.

27. Separating variables and integrating we obtain

$$\frac{dx}{\sqrt{1-x^2}} - \frac{dy}{\sqrt{1-y^2}} = 0 \quad \text{and} \quad \sin^{-1}x - \sin^{-1}y = c.$$

Setting $x = 0$ and $y = \sqrt{3}/2$ we obtain $c = -\pi/3$. Thus, an implicit solution of the initial-value problem is $\sin^{-1} x - \sin^{-1} y = -\pi/3$. Solving for y and using an addition formula from trigonometry, we get

$$y = \sin\left(\sin^{-1} x + \frac{\pi}{3}\right) = x \cos\frac{\pi}{3} + \sqrt{1-x^2}\sin\frac{\pi}{3} = \frac{x}{2} + \frac{\sqrt{3}\sqrt{1-x^2}}{2}.$$

28. From $\dfrac{1}{1+(2y)^2}\,dy = \dfrac{-x}{1+(x^2)^2}\,dx$ we obtain

$$\frac{1}{2}\tan^{-1} 2y = -\frac{1}{2}\tan^{-1} x^2 + c \quad \text{or} \quad \tan^{-1} 2y + \tan^{-1} x^2 = c_1.$$

Using $y(1) = 0$ we find $c_1 = \pi/4$. Thus, an implicit solution of the initial-value problem is $\tan^{-1} 2y + \tan^{-1} x^2 = \pi/4$. Solving for y and using a trigonometric identity we get

$$\begin{aligned} 2y &= \tan\left(\frac{\pi}{4} - \tan^{-1} x^2\right) \\ y &= \frac{1}{2}\tan\left(\frac{\pi}{4} - \tan^{-1} x^2\right) \\ &= \frac{1}{2}\,\frac{\tan\frac{\pi}{4} - \tan(\tan^{-1} x^2)}{1 + \tan\frac{\pi}{4}\tan(\tan^{-1} x^2)} \\ &= \frac{1}{2}\,\frac{1-x^2}{1+x^2}. \end{aligned}$$

29. Separating variables, integrating from 4 to x, and using t as a dummy variable of integration gives

$$\begin{aligned} \int_4^x \frac{1}{y}\frac{dy}{dt}\,dt &= \int_4^x e^{-t^2}\,dt \\ \ln y(t)\Big|_4^x &= \int_4^x e^{-t^2}\,dt \\ \ln y(x) - \ln y(4) &= \int_4^x e^{-t^2}\,dt \end{aligned}$$

Using the initial condition we have

$$\ln y(x) = \ln y(4) + \int_4^x e^{-t^2}\,dt = \ln 1 + \int_4^x e^{-t^2}\,dt = \int_4^x e^{-t^2}\,dt.$$

Thus,

$$y(x) = e^{\int_4^x e^{-t^2}\,dt}.$$

3.19 Questions with Solutions on Chapter 3.1, Cooling Warming and Mixture Applications

Using this age, determine what percentage of the original amount of C-14 remained in the cloth as of 1988.

Newton's Law of Cooling/Warming

15. A small metal bar, whose initial temperature was 20° C, is dropped into a large container of boiling water. How long will it take the bar to reach 90° C if it is known that its temperature increases 2° in 1 second? How long will it take the bar to reach 98° C?

16. Two large containers A and B of the same size are filled with different fluids. The fluids in containers A and B are maintained at 0° C and 100° C, respectively. A small metal bar, whose initial temperature is 100° C, is lowered into container A. After 1 minute the temperature of the bar is 90° C. After 2 minutes the bar is removed and instantly transferred to the other container. After 1 minute in container B the temperature of the bar rises 10°. How long, measured from the start of the entire process, will it take the bar to reach 99.9° C?

17. A thermometer reading 70° F is placed in an oven preheated to a constant temperature. Through a glass window in the oven door, an observer records that the thermometer reads 110° F after $\frac{1}{2}$ minute and 145° F after 1 minute. How hot is the oven?

18. At $t = 0$ a sealed test tube containing a chemical is immersed in a liquid bath. The initial temperature of the chemical in the test tube is 80° F. The liquid bath has a controlled temperature (measured in degrees Fahrenheit) given by $T_m(t) = 100 - 40e^{-0.1t}$, $t \geq 0$, where t is measured in minutes.

(a) Assume that $k = -0.1$ in (2). Before solving the IVP, describe in words what you expect the temperature $T(t)$ of the chemical to be like in the short term. In the long term.

(b) Solve the initial-value problem. Use a graphing utility to plot the graph of $T(t)$ on time intervals of various lengths. Do the graphs agree with your predictions in part (a)?

19. A dead body was found within a closed room of a house where the temperature was a constant 70° F. At the time of discovery the core temperature of the body was determined to be 85° F. One hour later a second measurement showed that the core temperature of the body was 80° F. Assume that the time of death corresponds to $t = 0$ and that the core temperature at that time was 98.6° F. Determine how many hours elapsed before the body was found. [*Hint*: Let $t_1 > 0$ denote the time that the body was discovered.]

20. The rate at which a body cools also depends on its exposed surface area S. If S is a constant, then a modification of (2) is

$$\frac{dT}{dt} = kS(T - T_m),$$

where $k < 0$ and T_m is a constant. Suppose that two cups A and B are filled with coffee at the same time. Initially, the temperature of the coffee is 150° F. The exposed surface area of the coffee in cup B is twice the surface area of the coffee in cup A. After 30 min the temperature of the coffee in cup A is 100° F. If $T_m = 70°$ F, then what is the temperature of the coffee in cup B after 30 min?

Mixtures

21. A tank contains 200 liters of fluid in which 30 grams of salt is dissolved. Brine containing 1 gram of salt per liter is then pumped into the tank at a rate of 4 L/min; the well-mixed solution is pumped out at the same rate. Find the number $A(t)$ of grams of salt in the tank at time t.

22. Solve Problem 21 assuming that pure water is pumped into the tank.

23. A large tank is filled to capacity with 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped into the tank at a rate of 5 gal/min. The well-mixed solution is pumped out at the same rate. Find the number $A(t)$ of pounds of salt in the tank at time t.

24. In Problem 23, what is the concentration $c(t)$ of the salt in the tank at time t? At $t = 5$ min? What is the concentration of the salt in the tank after a long time, that is, as $t \to \infty$? At what time is the concentration of the salt in the tank equal to one-half this limiting value?

25. Solve Problem 23 under the assumption that the solution is pumped out at a faster rate of 10 gal/min. When is the tank empty?

26. Determine the amount of salt in the tank at time t in Example 5 if the concentration of salt in the inflow is variable and given by $c_{in}(t) = 2 + \sin(t/4)$ lb/gal. Without actually graphing, conjecture what the solution curve of the IVP should look like. Then use a graphing utility to plot the graph of the solution on the interval [0, 300]. Repeat for the interval [0, 600] and compare your graph with that in Figure 3.1.4(a).

27. A large tank is partially filled with 100 gallons of fluid in which 10 pounds of salt is dissolved. Brine containing

15. We use the fact that the boiling temperature for water is 100° C. Now assume that $dT/dt = k(T - 100)$ so that $T = 100 + ce^{kt}$. If $T(0) = 20°$ and $T(1) = 22°$, then $c = -80$ and $k = \ln(39/40) \approx -0.0253$. Then $T(t) = 100 - 80e^{-0.0253t}$, and when $T = 90$, $t = 82.1$ seconds. $T(t) = 98°$ then $t = 145.7$ seconds.

16. The differential equation for the first container is $dT_1/dt = k_1(T_1 - 0) = k_1T_1$, whose solution $T_1(t) = c_1e^{k_1t}$. Since $T_1(0) = 100$ (the initial temperature of the metal bar), we have $100 = c_1$ a $T_1(t) = 100e^{k_1t}$. After 1 minute, $T_1(1) = 100e^{k_1} = 90°\text{C}$, so $k_1 = \ln 0.9$ and $T_1(t) = 100e^{t\ln 0.}$ After 2 minutes, $T_1(2) = 100e^{2\ln 0.9} = 100(0.9)^2 = 81°\text{C}$.

The differential equation for the second container is $dT_2/dt = k_2(T_2 - 100)$, whose solution $T_2(t) = 100+c_2e^{k_2t}$. When the metal bar is immersed in the second container, its initial temperatu is $T_2(0) = 81$, so

$$T_2(0) = 100 + c_2e^{k_2(0)} = 100 + c_2 = 81$$

and $c_2 = -19$. Thus, $T_2(t) = 100 - 19e^{k_2t}$. After 1 minute in the second tank, the temperature the metal bar is 91°C, so

$$\begin{aligned} T_2(1) &= 100 - 19e^{k_2} = 91 \\ e^{k_2} &= \frac{9}{19} \\ k_2 &= \ln\frac{9}{19} \end{aligned}$$

and $T_2(t) = 100 - 19e^{t\ln(9/19)}$. Setting $T_2(t) = 99.9$ we have

$$\begin{aligned} 100 - 19e^{t\ln(9/19)} &= 99.9 \\ e^{t\ln(9/19)} &= \frac{0.1}{19} \\ t &= \frac{\ln(0.1/19)}{\ln(9/19)} \approx 7.02. \end{aligned}$$

Thus, from the start of the "double dipping" process, the total time until the bar reaches 99. in the second container is approximately 9.02 minutes.

17. Using separation of variables to solve $dT/dt = k(T - T_m)$ we get $T(t) = T_m + ce^{kt}$. Using $T(0) =$ we find $c = 70 - T_m$, so $T(t) = T_m + (70 - T_m)e^{kt}$. Using the given observations, we obtain

$$\begin{aligned} T\left(\tfrac{1}{2}\right) &= T_m + (70 - T_m)e^{k/2} = 110 \\ T(1) &= T_m + (70 - T_m)e^{k} = 145. \end{aligned}$$

Then, from the first equation, $e^{k/2} = (110 - T_m)/(70 - T_m)$ and

$$e^k = (e^{k/2})^2 = \left(\frac{110 - T_m}{70 - T_m}\right)^2 = \frac{145 - T_m}{70 - T_m}$$

$$\frac{(110 - T_m)^2}{70 - T_m} = 145 - T_m$$

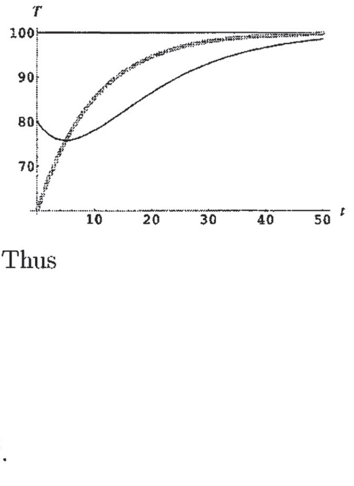
$$12100 - 220T_m + T_m^2 = 10150 - 215T_m + T_m^2$$

$$T_m = 390.$$

The temperature in the oven is 390°.

18. (a) The initial temperature of the bath is $T_m(0) = 60°$, so in the short term the temperature of the chemical, which starts at 80°, should decrease or cool. Over time, the temperature of the bath will increase toward 100° since $e^{-0.1t}$ decreases from 1 toward 0 as t increases from 0. Thus, in the long term, the temperature of the chemical should increase or warm toward 100°.

(b) Adapting the model for Newton's law of cooling, we have



$$\frac{dT}{dt} = -0.1(T - 100 + 40e^{-0.1t}), \quad T(0) = 80.$$

Writing the differential equation in the form

$$\frac{dT}{dt} + 0.1T = 10 - 4e^{-0.1t}$$

we see that it is linear with integrating factor $e^{\int 0.1\,dt} = e^{0.1t}$. Thus

$$\frac{d}{dt}[e^{0.1t}T] = 10e^{0.1t} - 4$$

$$e^{0.1t}T = 100e^{0.1t} - 4t + c$$

and

$$T(t) = 100 - 4te^{-0.1t} + ce^{-0.1t}.$$

Now $T(0) = 80$ so $100 + c = 80$, $c = -20$ and

$$T(t) = 100 - 4te^{-0.1t} - 20e^{-0.1t} = 100 - (4t + 20)e^{-0.1t}.$$

The thinner curve verifies the prediction of cooling followed by warming toward 100°. The wider curve shows the temperature T_m of the liquid bath.

Identifying $T_m = 70$, the differential equation is $dT/dt = k(T - 70)$. Assuming $T(0) = 98.6$ and separating variables we find $T(t) = 70 + 28.9e^{kt}$. If $t_1 > 0$ is the time of discovery of the body, then

$$T(t_1) = 70 + 28.6e^{kt_1} = 85 \quad \text{and} \quad T(t_1 + 1) = 70 + 28.6e^{k(t_1+1)} = 80.$$

Therefore $e^{kt_1} = 15/28.6$ and $e^{k(t_1+1)} = 10/28.6$. This implies

$$e^k = \frac{10}{28.6}e^{-kt_1} = \frac{10}{28.6}\cdot\frac{28.6}{15} = \frac{2}{3},$$

so $k = \ln\frac{2}{3} \approx -0.405465108$. Therefore

$$t_1 = \frac{1}{k}\ln\frac{15}{28.6} \approx 1.5916 \approx 1.6.$$

Death took place about 1.6 hours prior to the discovery of the body.

20. Solving the differential equation $dT/dt = kS(T - T_m)$ subject to $T(0) = T_0$ gives

$$T(t) = T_m + (T_0 - T_m)e^{kSt}.$$

The temperatures of the coffee in cups A and B are, respectively,

$$T_A(t) = 70 + 80e^{kSt} \quad\text{and}\quad T_B(t) = 70 + 80e^{2kSt}.$$

Then $T_A(30) = 70 + 80e^{30kS} = 100$, which implies $e^{30kS} = \frac{3}{8}$. Hence

$$\begin{aligned} T_B(30) &= 70 + 80e^{60kS} = 70 + 80\left(e^{30kS}\right)^2 \\ &= 70 + 80\left(\frac{3}{8}\right)^2 = 70 + 80\left(\frac{9}{64}\right) = 81.25°\text{F}. \end{aligned}$$

21. From $dA/dt = 4 - A/50$ we obtain $A = 200 + ce^{-t/50}$. If $A(0) = 30$ then $c = -170$ $A = 200 - 170e^{-t/50}$.

22. From $dA/dt = 0 - A/50$ we obtain $A = ce^{-t/50}$. If $A(0) = 30$ then $c = 30$ and $A = 30e^{-t/50}$.

23. From $dA/dt = 10 - A/100$ we obtain $A = 1000 + ce^{-t/100}$. If $A(0) = 0$ then $c = -1000$ $A(t) = 1000 - 1000e^{-t/100}$.

24. From Problem 23 the number of pounds of salt in the tank at time t is $A(t) = 1000 - 1000e^{-t}$ The concentration at time t is $c(t) = A(t)/500 = 2 - 2e^{-t/100}$. Therefore $c(5) = 2 - 2e^{-1/2}$ $0.0975\,\text{lb/gal}$ and $\lim_{t\to\infty} c(t) = 2$. Solving $c(t) = 1 = 2 - 2e^{-t/100}$ for t we obtain $t = 100\ln$ 69.3 min.

25. From

$$\frac{dA}{dt} = 10 - \frac{10A}{500 - (10 - 5)t} = 10 - \frac{2A}{100 - t}$$

we obtain $A = 1000 - 10t + c(100 - t)^2$. If $A(0) = 0$ then $c = -\frac{1}{10}$. The tank is empty in minutes.

26. With $c_{in}(t) = 2 + \sin(t/4)\,\text{lb/gal}$, the initial-value problem is

$$\frac{dA}{dt} + \frac{1}{100}A = 6 + 3\sin\frac{t}{4}, \quad A(0) = 50.$$

The differential equation is linear with integrating factor $e^{\int dt/100} = e^{t/100}$, so

$$\frac{d}{dt}[e^{t/100}A(t)] = \left(6 + 3\sin\frac{t}{4}\right)e^{t/100}$$

$$e^{t/100}A(t) = 600e^{t/100} + \frac{150}{313}e^{t/100}\sin\frac{t}{4} - \frac{3750}{313}e^{t/100}\cos\frac{t}{4} + c,$$

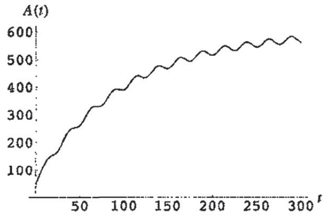
and

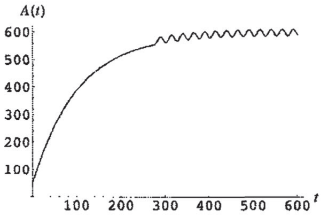
$$A(t) = 600 + \frac{150}{313}\sin\frac{t}{4} - \frac{3750}{313}\cos\frac{t}{4} + ce^{-t/100}.$$

Letting $t = 0$ and $A = 50$ we have $600 - 3750/313 + c = 50$ and $c = -168400/313$. Then

$$A(t) = 600 + \frac{150}{313}\sin\frac{t}{4} - \frac{3750}{313}\cos\frac{t}{4} - \frac{168400}{313}e^{-t/100}.$$

The graphs on $[0, 300]$ and $[0, 600]$ below show the effect of the sine function in the input when compared with the graph in Figure 3.1.4(a) in the text.





[illegible] From

$$\frac{dA}{dt} = 3 - \frac{4A}{100 + (6 - 4)t} = 3 - \frac{2A}{50 + t}$$

we obtain $A = 50 + t + c(50 + t)^{-2}$. If $A(0) = 10$ then $c = -100{,}000$ and $A(30) = 64.38$ pounds.

[illegible] (a) Initially the tank contains 300 gallons of solution. Since brine is pumped in at a rate of 3 gal/min and the mixture is pumped out at a rate of 2 gal/min, the net change is an increase of 1 gal/min. Thus, in 100 minutes the tank will contain its capacity of 400 gallons.

(b) The differential equation describing the amount of salt in the tank is $A'(t) = 6 - 2A/(300 + t)$ with solution

$$A(t) = 600 + 2t - (4.95 \times 10^7)(300 + t)^{-2}, \qquad 0 \le t \le 100,$$

as noted in the discussion following Example 5 in the text. Thus, the amount of salt in the tank when it overflows is

$$A(100) = 800 - (4.95 \times 10^7)(400)^{-2} = 490.625 \text{ lbs.}$$

When the tank is overflowing the amount of salt in the tank is governed by the differential

3.20 Questions with Solutions on Chapter 4.3, Reduction of order

Note that you will laugh

The questions on Reduction from1-14/ y_1 is not needed :))))) since you can do them using undetermined method or cauchy-euler.

The book is doing reduction before undetermined and before Cauchy-Euler

Question 15/Yes y1 is needed

Question 16/ y_1 is not needed.
Anyway/ Practice using reduction

use y^\\ + q(x) y^\ + p(x)y = 0/ given y_1

First find L = e^{integral -q(x) dx}

y_2 = y_1 (Integral (L / y_1^2) dx)

In Problems 1–16 the indicated function $y_1(x)$ is a solution of the given differential equation. Use reduction of order or formula (5), as instructed, to find a second solution $y_2(x)$.

1. $y'' - 4y' + 4y = 0; \quad y_1 = e^{2x}$

2. $y'' + 2y' + y = 0; \quad y_1 = xe^{-x}$

3. $y'' + 16y = 0; \quad y_1 = \cos 4x$

4. $y'' + 9y = 0; \quad y_1 = \sin 3x$

5. $y'' - y = 0; \quad y_1 = \cosh x$

6. $y'' - 25y = 0; \quad y_1 = e^{5x}$

7. $9y'' - 12y' + 4y = 0; \quad y_1 = e^{2x/3}$

8. $6y'' + y' - y = 0; \quad y_1 = e^{x/3}$

9. $x^2y'' - 7xy' + 16y = 0; \quad y_1 = x^4$

10. $x^2y'' + 2xy' - 6y = 0; \quad y_1 = x^2$

11. $xy'' + y' = 0; \quad y_1 = \ln x$

12. $4x^2y'' + y = 0; \quad y_1 = x^{1/2} \ln x$

13. $x^2y'' - xy' + 2y = 0; \quad y_1 = x \sin(\ln x)$

14. $x^2y'' - 3xy' + 5y = 0; \quad y_1 = x^2 \cos(\ln x)$

15. $(1 - 2x - x^2)y'' + 2(1 + x)y' - 2y = 0; \quad y_1 = x + 1$

16. $(1 - x^2)y'' + 2xy' = 0; \quad y_1 = 1$

[redacted]

[redacted]

2.

3.

[redacted]

3.21 Questions with Solutions, Review Exam-One

Name, ID

Nadeen Tarek

Exam I, MTH 205, Fall 2019

Ayman Badawi

Aerror $\ell\{\ \} = \frac{1}{2}e^{-3t}t^2 - \frac{1}{2}e^{-3t}t$

Total = $\frac{80}{80}$

QUESTION 1. (12 points)

(i) $\ell^{-1}\{\frac{s}{(s+3)^4}\}$

Note $\frac{s}{(s+3)^4} = \frac{s+3-3}{(s+3)^4} = \frac{1}{(s+3)^3} - \frac{3}{(s+3)^4}$

$$\frac{A}{S+3} + \frac{B}{(S+3)^2} + \frac{C}{(S+3)^3} + \frac{D}{(S+3)^4} = \frac{S}{(S+3)^4}$$

$$A(S+3)^3 + B(S+3)^2 + C(S+3) + D = S$$

$S=-1$

$8A + 4B + 2C + D = -1$

$\boxed{8A + 4B + 2C = 2}$

$S=-3$

$\boxed{D=-3}$

$S=0$

$27A + 9B + 3C + D = 0$

$\boxed{27A + 9B + 3C = 3}$

$S=1$

$64A + 16B + 4C - 3 = 1$

$\boxed{64A + 16B + 4C = 4}$

$A=0 \quad C=1$

$B=0$

$$\frac{1}{(S+3)^3} - \frac{3}{(S+3)^4}$$

$$= \frac{1}{2}e^{-3t}t^2 - \frac{3}{6}e^{-3t}t^3$$

✓

(ii) $\ell^{-1}\{\frac{e^{-2s}}{s^2+4s+13}\}$

$\frac{4}{2} = (2)^2$

$$= \frac{1/e^{-2s}}{(S+2)^2+9} = U_2 f(t-2)$$

F(S)

$$f(t) = \ell^{-1}\frac{1}{(S+2)^2+9} = \frac{1}{3}e^{-2t}\sin(3t)$$

$$= \boxed{U_2 \frac{1}{3}e^{-2(t-2)}\sin(3(t-2))}$$

✓

iii) $\ell^{-1}\{\frac{8s}{(s^2+16)^2}\}$

$t \sin 4t$

✓

$\ell \sin x = \frac{1}{S^2+1}$ $\quad \ell \sin 4x = \frac{4}{S^2+16}$

$$\frac{Vu' - uV'}{(S^2+16)^2} = \frac{(S^2+16)(0) - (4 \cdot 2S)}{(S^2+16)^2}$$

$$= \frac{-8S}{(S^2+16)^2}$$

$\Rightarrow \ell^{-1}\{\frac{-8s}{(s^2+16)^2}\} =$

$-t\ f(t)$

$$\frac{Vu' - uV'}{(S^2+16)^2} = \frac{(S^2+16)(0) -}{}$$

Nadeen

QUESTION 2. (8 points) Given $f(t)$ is periodic on the interval $[0, \infty]$. The first period of $f(t)$ is determined by $f(t) = 2$, when $0 \le t < 4$. Use Laplace-Transformation and find $y(t)$, where $y'' - 4y' + 3y = f(t)$, $y(0) = 0, y'(0) = 0$.

$$S^2Y(S) - \cancel{Sy(0)} - \cancel{y'(0)} - 4SY(S) - \cancel{4y(0)} + 3Y(S) = \frac{2-2e^{-4S}}{S(1-e^{-4S})}$$

$$Y(S)\,(S^2-4S+3) = \frac{2-2e^{-4S}}{S(1-e^{-4S})}$$

$$Y(S) = \frac{2-2e^{-4S}}{S(1-e^{-4S})(S-3)(S-1)} = \frac{2\cancel{(1-e^{-4S})}}{S\cancel{(1-e^{-4S})}(S-3)(S-1)}$$

$$Y(S) = \frac{2}{S(S-3)(S-1)} = \frac{A}{S} + \frac{B}{S-3} + \frac{C}{S-1}$$

$0 \qquad S=3 \qquad S=1$

$A = 2/3 \quad B = \frac{1}{3} \quad C = -1$

$$Y(S) = \frac{2/3}{S} + \frac{1/3}{S-3} - \frac{1}{S-1}$$

$$y(t) = \frac{2}{3} + \frac{1}{3}e^{3t} - e^{t}$$

✓

$$\int_0^4 e^{-St}(2)\,dt$$

$f(t) = 2\,[u_0 - u_4]$

$= 2u_0 - 2u_4$

$$\frac{2e^{0}}{S} - \frac{2e^{-4S}}{S}$$

$$= \frac{\frac{2}{S} - \frac{2e^{-4S}}{S}}{1-e^{-4S}}$$

QUESTION 3. (8 points) let $f(t) = \int_0^t cos(u)\,du$, where $0 \le t < \infty$. Use Laplace-Transformation and find $y(t)$, where $y'' - 9y = f(t)$, $y(0) = 0, y'(0) = 0$.

$$S^2Y(S) - \cancel{Sy(0)} - \cancel{y'(0)} - 9Y(S) = \frac{1}{S^2+1}$$

✓

$$\mathcal{L}\int_0^t \cos(u)\,du \quad f(t)$$

$$\frac{\frac{\cancel{S}}{S^2+1}}{\cancel{S}} = \frac{1}{S^2+1}$$

$$Y(S)\,(S^2-9) = \frac{1}{S^2+1}$$

S^2-9-S^2-1

$$Y(S) = \frac{1}{(S^2+1)(S^2-9)} = \frac{1}{(S^2+1)(S-3)(S+3)} = \frac{A}{S-3} + \frac{B}{S+3} + \frac{CS+D}{S^2+1} = \frac{1}{\frac{(S-3)(S+3)}{(S^2+1)}}$$

$$\frac{1}{S^2+1} - \frac{1}{S^2-9} = \frac{-10}{(S^2+1)(S^2-9)}$$

$$\frac{1}{-10}\left[\frac{1}{S^2+1} - \frac{1}{S^2-9}\right] \qquad \mathcal{L}^{-1}Y(S) = \mathcal{L}^{-1}\frac{-1}{10}\left[\frac{1}{S^2+1} - \frac{1}{S^2-9}\right]$$

$$y(t) = \frac{-1}{10}\sin t + \frac{1}{30}\sinh(3t)$$

✓

Nadeen

QUESTION 4. (8 points) Use Laplace-Transformation and find $y(t)$, where $y''' + 2y' = U_5(t)$, $y(0) = 0, y'(0) = y''(0) = 0$.

$$s^3Y(s) - \underset{0}{\cancel{s^2y(0)}} - \underset{0}{\cancel{sy'(0)}} - \underset{0}{\cancel{y''(0)}} + 2sY(s) - \underset{0}{\cancel{2y(0)}} = \frac{e^{-5s}}{s}$$

$$Y(s)\,(s^3+2s) = \frac{e^{-5s}}{s}$$

$$\mathcal{L}^{-1}\; Y(s) = \frac{e^{-5s}}{s\,(s^3+2s)} = \mathcal{L}^{-1}\, \boxed{\frac{1}{s^2(s^2+2)}}\,e^{-5s} \quad F(s)$$

$$y(t) = U_5 f(t-5)$$

$$\mathcal{L}^{-1}\cdot \frac{1}{2}\left[\frac{1}{s^2} - \frac{1}{s^2+2}\right] = \frac{1}{2}\left[t - \frac{1}{\sqrt{2}}\sin\sqrt{2}t\right]$$

$$y(t) = \frac{1}{2}U_5\left[(t-5) - \frac{1}{\sqrt{2}}\sin\sqrt{2}(t-5)\right]$$

✓

$$\mathcal{L}\; U_5(t)$$

$$\frac{e^{-5s}}{s}$$

$$\cancel{s^2} + 2 - \cancel{s^2}$$

$$\frac{1}{s^2} - \frac{1}{s^2+2} = \frac{2}{s^2(s^2+2)}$$

$$\frac{1}{2}$$

QUESTION 5. (8 points) Use Laplace-Transformation and find $y(t)$, where $y''' - 6y'' + 5y' = 0$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 20$.

$$s^3Y(s) - \underset{0}{\cancel{s^2y(0)}} - \underset{0}{\cancel{sy'(0)}} - y''(0) - 6s^2Y(s) - \underset{0}{\cancel{6sy(0)}} - \underset{0}{\cancel{6y'(0)}} + 5sY(s) - \underset{0}{\cancel{5y(0)}} = 0$$

$$s^3Y(s) - 20 - 6s^2Y(s) + 5sY(s) = 0$$

$$Y(s)\,(s^3 - 6s^2 + 5s) = 20$$

$$Y(s) = \frac{20}{s(s^2-6s+5)} = \frac{20}{s(s-5)(s-1)} = \frac{A}{s} + \frac{B}{s-5} + \frac{C}{s-1}$$

$s = 0$ $s = 5$ $s = 1$

$A = 4$ $B = 1$ $C = -5$

$$Y(s) = \frac{4}{s} + \frac{1}{s-5} - \frac{5}{s-1}$$

$$y(t) = 4 + e^{5t} - 5e^{t}$$

✓

QUESTION 6. **(8 points)** Solve for $x(t), y(t)$, where

$X(0)=0 ,\ X'(0)=1$

$$x''(t)+y(t)=0,\ x(0)=x'(0)=1$$

$$x'(t)+y'(t)=0,\ y(0)=1$$

$$s^2X(s) - s\cancel{x(0)} - x'(0) + Y(s) = 0 \quad ①$$

$$sX(s) - \cancel{x(0)} + sY(s) - y(0) = 0 \quad ②$$

$$s^2X(s) + Y(s) = 1 \quad ①$$

$$sX(s) + sY(s) = 1 \quad ②$$

$$\frac{1}{s} - \frac{1}{s+1} = \frac{1}{s(s+1)} \qquad s+1-s$$

$$X(s) = \frac{\begin{vmatrix}1 & 1\\ 1 & s\end{vmatrix}}{\begin{vmatrix}s^2 & 1\\ s & s\end{vmatrix}} = \frac{s-1}{s^3-s} = \frac{s-1}{s(s^2-1)}$$

$$X(s) = \frac{\cancel{s-1}}{s\cancel{(s-1)}(s+1)} = \frac{1}{s(s+1)}$$

$$\mathcal{L}^{-1}X(s) = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right]$$

$$\boxed{x(t) = 1 - e^{-t}}$$

$$Y(s) = \frac{\begin{vmatrix}s^2 & 1\\ s & 1\end{vmatrix}}{\begin{vmatrix}s^2 & 1\\ s & s\end{vmatrix}} = \frac{s^2-s}{s^3-s} = \frac{\cancel{s}(s-1)}{\cancel{s}(s^2-1)} = \frac{\cancel{s-1}}{\cancel{(s-1)}(s+1)} = \frac{1}{s+1}$$

$$\mathcal{L}^{-1}Y(s) = \mathcal{L}^{-1}\frac{1}{s+1}$$

$$\boxed{y(t) = e^{-t}}$$

QUESTION 7. **(4 points)** Find the general solution of $y(t)$, where $y'' - 6y' + 18y = 0$

$-\frac{6}{2} = -3$

$m^2 - 6m + 18 = 0$

$(m-3)^2 + 9 = 0$

$(m-3)^2 = -9$

$m - 3 = \pm 3i$

$m = 3 \pm 3i$

✓

$y_h = e^{3t}\left[c_1 \cos(3t) + c_2 \sin(3t)\right]$

QUESTION 8. **(8 points)** Find the general solution of $y(t)$, where $y''' + 9y' = sin(t) + 4$.

$m^3 + 9m = m(m^2+9)$

$m(m^2+9) = 0$

$m = 0 \quad m = \pm 3i$

$y_g = y_h + y_p$

$y_h = C_1 + C_2 \cos(3t) + C_3 \sin(3t)$

$y_p = a \sin(t) + b \cos(t) + At$

$y_p' = a \cos(t) - b \sin(t) + A$

$y_p'' = -a \sin(t) - b \cos(t)$

$y_p''' = -a \cos(t) + b \sin(t)$

$-a \cos(t) + b \sin(t) + 9a \cos(t) - 9b \sin(t) + 9A = \sin(t) + 4$

$(b - 9b) \sin(t) + (9a - a) \cos(t) + 9A = \sin(t) + 4$

$-8b = 1 \quad 8a = 0 \quad \boxed{A = \frac{4}{9}}$

$b = \frac{-1}{8} \quad a = 0$

$y_p = \frac{-1}{8} \cos t + \frac{4}{9} t$

$y_g = C_1 + C_2 \cos(3t) + C_3 \sin(3t) - \frac{1}{8} \cos(t) + \frac{4}{9} t$

✓

QUESTION 9. (8 points) Find the general solution of $y(t)$, where $y''' + 5y'' = t$.

$m^3 + 5m^2 = 0$

$m^2(m+5) = 0$

$m = 0$ $m = -5$

2 times

$y_h = C_1 + C_2 t + C_3 e^{-5t}$

$y_p = (a_3 t^3 + a_2 t^2 + a_1 t + a_0)$

$y_p' = 3a_3 t^2 + 2a_2 t$

$y_p'' = 6a_3 t + 2a_2$

$y_p''' = 6a_3$

$6a_3 + 5(6a_3 t + 2a_2) = t$

$6a_3 + 30a_3 t + 10a_2 = t$

$30a_3 = 1$

$\boxed{a_3 = \frac{1}{30}}$

$6a_3 + 10a_2 = 0$

$\frac{1}{5} + 10a_2 = 0$

$10a_2 = -\frac{1}{5}$

$\boxed{a_2 = -\frac{1}{50}}$

$\boxed{y_p = \frac{1}{30}t^3 - \frac{1}{50}t^2}$

$\boxed{y_g = C_1 + C_2 t + C_3 e^{-5t} + \frac{1}{30}t^3 - \frac{1}{50}t^2}$ ✓

QUESTION 10. (8 points) Find the general solution of $y(t)$, where $y' + 3y = te^{2t}$

$m + 3 = 0$

$m = -3$

$y_h = C_1 e^{-3t}$

$y_p = (a_1 t + a_0)e^{2t}$

$y_p = (a_1 e^{2t} t + a_0 e^{2t})$

$y_p' = 2a_1 t e^{2t} + a_1 e^{2t} + 2a_0 e^{2t}$

$2a_1 t e^{2t} + a_1 e^{2t} + 2a_0 e^{2t} + 3a_1 t e^{2t} + 3a_0 e^{2t} = te^{2t}$

$[2a_1 t + 3a_1 t]e^{2t} + [a_1 + 2a_0 + 3a_0]e^{2t} = te^{2t}$

$[5a_1 t]e^{2t} + [a_1 + 5a_0]e^{2t} = te^{2t}$

$5a_1 = 1$

$\boxed{a_1 = \frac{1}{5}}$

$a_1 + 5a_0 = 0$

$\frac{1}{5} + 5a_0 = 0$

$5a_0 = -\frac{1}{5}$

$\boxed{a_0 = -\frac{1}{25}}$

$\underbrace{a_1 t}_{u}\underbrace{e^{2t}}_{v}$

$uv' + vu'$

$2a_1 t e^{2t} + e^{2t}(a_1)$

Faculty information

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$y_p = \left(\frac{1}{5}t - \frac{1}{25}\right)e^{2t}$

$\boxed{y_g = C_1 e^{-3t} + \left(\frac{1}{5}t - \frac{1}{25}\right)e^{2t}}$ ✓

Name Rana Sakhnini, ID 52000

100 Excellent

Exam I, MTH 205, Fall 2014

Ayman Badawi

$(-\infty, 12)$

$12 - x > 0 \quad x \neq 4$

$12 > x$

$x < 12$

QUESTION 1. (6 points) Find the largest interval around x so that the LDE: $\frac{\sqrt{x-4}}{\sqrt{12-x}} y^{(3)} + \frac{x-1}{x-7} y' + 3y = x^2 + 13$, $y^{(2)}(5) = y'(5) = 7$, and $y(5) = -6$ has a unique solution.

$\frac{\sqrt{x-4}}{\sqrt{12-x}}$

$12 - x > 0$

$12 > x$

$x < 12$

$(-\infty, 12)$

$(-\infty, 4) \cup (4, 12)$

$x - 7$

$(-\infty, 7) \cup (7, \infty)$

$\mathbb{R}$

$\boxed{I = (4, 7)}$

$\int_0^x 2 \sin u \, du.$

$(-2\cos u)_0^x$

$-2\cos x + 2$

QUESTION 2. (10 points) Solve for $x(t), y(t)$

$x'(t) - y(t) = 2$

$x(t) + y'(t) = 2$, where $x(0) = 2$, $y(0) = -1$, $x'(0) = 1$, $y'(0) = 0$

$sX(s) - x(0) - Y(s) = \frac{2}{s}$

$sX(s) - 2 - Y(s) = \frac{2}{s}$

$\boxed{sX(s) - Y(s) = \frac{2 + 2s}{s}}$

$X(s) + sY(s) + 1 = \frac{2}{s}$

$\boxed{X(s) + sY(s) = \frac{2 - s}{s}}$

$sX(s) - \frac{2 + 2s}{s} = Y(s)$

$X(s) + s^2 X(s) - 2 - 2s = \frac{2 - s}{s}$

$X(s)(1 + s^2) = \frac{2 - s}{s} + 2 + 2s$

$= \frac{2 - s + 2s + 2s^2}{s}$

$X(s) = \frac{2s^2 + s + 2}{s(1 + s^2)}$

$= \frac{2s}{(s^2+1)} + \frac{1}{(s^2+1)} + \frac{2}{s(s^2+1)}$

$x(t) = 2\cos t + \sin t + 2 * \sin t$

$= 2\cos t + \sin t - 2\cos t + 2 = \boxed{\sin t + 2}$

$x'(t) = \cos t$

$\boxed{\cos t - 2 = y(t)}$

QUESTION 3. (30 points, each is 6 points)

(i) Find $\ell^{-1}\{\frac{s^3+24}{s^5}\}$ $\ell^{-1}\left\{\frac{1}{s^2}+\frac{24}{s^5}\right\}$

$= x + \frac{24}{4!}x^4$ $= x + x^4$

(ii) Find $\ell^{-1}\{\frac{e^{-2s}}{(s+4)^2+4}\}$ $= \ell^{-1}\left\{e^{-2s}\left(\frac{1}{(s+4)^2+4}\right)\right\} = \frac{1}{2}u(x-2)\sin 2(x-2)e^{-4(x-2)}$

$f(x+2) = \frac{1}{2}\ell^{-1}\left\{\frac{2}{(s+4)^2+4}\right\} = \frac{1}{2}\sin 2x\, e^{-4x}$

$f(x) = \frac{1}{2}\sin 2(x-2)\, e^{-4(x-2)}$ ✓

(iii) Find $\ell\{u(x-1)e^{(x-1)}sin(x-1)\}$ $= e^{-s}\,\ell\{e^x \sin x\} =$

$= e^{-s}\frac{1}{(s-1)^2+1}$ ✓

(iv) Find $\ell^{-1}\{\frac{s+2}{s^2+4s+5}\}$ $\ell^{-1}\left\{\frac{s+2}{(s^2+4s+4)-4+5}\right\}$ $\ell^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\}$

$= e^{-2x}\cos x$ ✓

$x-r$

$e^{2x-2r}e^{r}$

(v) Find $\ell\{\int_0^x e^{2x-r}sin(r)\,dr\}$ $= \ell\left\{\int_0^x e^{2(x-r)}(\sin r e^{r})dr\right\}$

$= \ell\left\{\left(e^{2x} * (\sin x)e^{x}\right)\right\}$

$= \left(\frac{1}{s-2}\right)\left(\frac{1}{(s-1)^2+1}\right)$ ✓

Sana

QUESTION 4. (54 points, each is 9 points) Use any method you want to solve for $y(x)$:

(i) $y^{(2)} - 2y' + y = u(x-1)e^{(x-1)}$ [Here you need to find y_g]. $y_h = c_1e^x + c_2xe^x$

$\underline{y_h}$ $m^2 - 2m + 1 = 0$ $m=1$

$m=1$

$\underline{y_p}$ $Y(s)\left[(s-1)^2\right] = e^{-s}\ell\{e^x\} = \frac{e^{-s}}{(s-1)}$

$Y(s) = \frac{e^{-s}}{(s-1)^3}$ | $y_p = \ell^{-1}\left\{e^{-s}\left(\frac{1}{(s-1)^3}\right)\right\}$

$\boxed{=\frac{1}{2}u(x-1)(x-1)^2e^{x-1}}$

$f(x+1) = \frac{1}{2!}\ell^{-1}\left\{\frac{2!}{(s-1)^3}\right\} = \frac{1}{2}x^2e^{+x}$

$f(x) = \frac{1}{2}(x-1)^2e^{x-1}$

$y_g = c_1e^x + c_2xe^x + \frac{1}{2}u(x-1)(x-1)^2e^{x-1}$

(ii) $y^{(6)} + 5y^{(5)} + 4y^{(4)} = 30e^{-4x}$ [here you need to find y_g].

$\underline{y_h}$

$m^6 + 5m^5 + 4m^4 = 0$

$m^4(m^2 + 5m + 4) = 0$

$y_h = C_1 + C_2x + C_3x^2 + C_4x^3 + c_5e^{-x} + c_6e^{-4x}$

$m=0$

$m=0$

$m=0$

$m=0$

$m=-1$

$m=-4$

$Y(s)\left[s^4(s+1)(s+4)\right] = \frac{30}{s+4}$

$Y(s) = \frac{30}{(s+4)^2(s+1)s^4}$

$= \cancel{\frac{a}{(s+4)}} + \boxed{\frac{b}{(s+4)^2}} + \cancel{\frac{c}{(s+1)}} + \cancel{\frac{d}{s}} + \cancel{\frac{e}{s^2}} + \cancel{\frac{g}{s^3}} + \cancel{\frac{h}{s^4}}$

$b = -\frac{5}{128}$

$y_p = \ell^{-1}\left\{-\frac{5}{128}\frac{1}{(s+4)^2}\right\}$

$= -\frac{5}{128}xe^{-4x}$

$y_g = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5e^{-x} + c_6e^{-4x} - \frac{5}{128}xe^{-4x}$

Bana

(iii) $y^{(2)} + \int_0^x y(r)e^{x-r}\,dr = \int_0^x (x-r)e^{r}\,dr, y(0) = 0, y'(0) = 1$

$$s^2 Y(s) - \cancel{s y(0)} - \cancel{y'(0)}^{1} + Y(s)\left(\frac{1}{s-1}\right) = \left(\frac{1}{s^2}\right)\left(\frac{1}{s-1}\right)$$

$$Y(s)\left(\frac{1}{s-1} + s^2\right) - 1 = \frac{1}{s^2(s-1)}$$

$$Y(s)\left(\frac{1+s^2(s-1)}{\cancel{(s-1)}}\right) = \frac{1+s^2(s-1)}{s^2\cancel{(s-1)}}$$

$$Y(s) = \frac{\cancel{(1+s^2(s-1))}}{s^2\cancel{(1+s^2(s-1))}} = \frac{1}{s^2}$$

$$y(x) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = x$$

✓

(iv) $y^{(2)} + 2y' + 2y = xe^{-x}, y(0) = 0$ and $y'(0) = 1$. [Hint: note that by completing the square method we have $s^2 + bs + c = (s+b/2)^2 + c - b^2/4$ and $\frac{e}{f} + d = \frac{e+fd}{f}$]

$$s^2 Y(s) - \cancel{s y(0)} - y'(0) + 2sY(s) - \cancel{2y(0)} + 2Y(s) = \frac{1}{(s+1)^2}$$

$$Y(s)\left[s^2 + 2s + 2\right] - 1 = \frac{1+(s+1)^2}{(s+1)^2}$$

$$Y(s)\left[s^2 + 2s + 1 - 1 + 2\right] = \frac{1+(s+1)^2}{(s+1)^2}$$

$$Y(s)\left[\cancel{(s+1)^2 + 1}\right] = \frac{1\cancel{+(s+1)^2}}{(s+1)^2}$$

$$Y(s) = \frac{1}{(s+1)^2}$$

✓

$$y(x) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = xe^{-x}$$

(v) $y^{(3)} - 4y^{(2)} + 5y' = 10$ [here you need to find y_g]

$\underline{y_h}$ $m^3 - 4m^2 + 5m = 0$

$m(m^2 - 4m + 5) = 0$

$m = 0$

$m = 2 + i$

$m = 2 - i$

$y_h = c_1 + c_2 e^{2x} \cos x$
$+ c_3 e^{2x} \sin x$

$\underline{y_p}$ $Y(s)[s(s^2 - 4s + 5)] = \frac{10}{s}$

$Y(s) = \cancel{\frac{a}{s}} + \frac{b}{s^2} + \frac{c}{(s^2 - 4s + 5)}$

$b = 2$

$y(x) = 2x$

$y_g = c_1 + c_2 e^{2x} \cos x$
$+ c_3 e^{2x} \sin x$
$+ 2x$

(vi) Let $k(x) = 4xe^{3x}$. Consider the LDE: $y^{(2)} + ay' + by = k(x)$. Find a, b so that $y(x) = k(x) = 4xe^{3x}$ is the unique solution to the given LDE. [Hint: If you want to use Laplace, then since y(x) is given, you should be able to find $y(0)$ and $y'(0)$, anyway it is clear that $y(0) = 0, y'(0) = 4$].

$s^2 Y(s) - \cancel{s y(0)} - y'(0) + as Y(s) - \cancel{a y(0)} + b Y(s) = \frac{4}{(s-3)^2}$

$Y(s)[s^2 + as + b] = \frac{4}{(s-3)^2} + \frac{4(s-3)^2}{} = \frac{4((s-3)^2 + 1)}{(s-3)^2 (s^2 + as + b)}$

$\frac{(s-3)^2 + 1}{(s^2 + as + b)} = 1$

$s^2 - 6s + 10 = s^2 + as + b$

$a = -6$

$b = 10$

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Name Khaled El-Awadi, ID 35531

Exam ONE, MTH 205, Summer 2010

Ayman Badawi

$$\frac{100}{\text{Excellent!!}}$$

QUESTION 1. (20 points) Let

$$f(x) = \begin{cases} 1 & if \quad 0 \le x < 5 \\ -1 & if \quad 5 \le x < 7 \\ 0 & if \quad 7 \le x < \infty \end{cases}$$

a) Write $f(x)$ in terms of unit step functions.

$$f(x) = 1\left[\overset{1}{\cancel{u(x-0)}} - u(x-5)\right] - 1\left[u(x-5) - u(x-7)\right] + 0$$

$$= 1 - u(x-5) - u(x-5) + u(x-7)$$

$$= 1 - 2u(x-5) + u(x-7) \checkmark$$

b) Solve the D.E : $y^{(2)} - 2y' - 3y = f(x)$, $y(0) = y'(0) = 0$

Good

$$\ell\{y^{(2)}\} - 2\ell\{y'\} - 3\ell\{y\} = \ell\{1\} - 2\ell\{u(x-5)\} + \ell\{u$$

$$s^2Y(s) - 2sY(s) - 3Y(s) = \frac{1}{s} - \frac{2e^{-5s}}{s} + \frac{e^{-7s}}{s}$$

$$Y(s)(s^2 - 2s - 3) = \frac{1}{s} - \frac{2e^{-5s}}{s} + \frac{e^{-7s}}{s}$$

$$Y(s) = \frac{1}{s(s-3)(s+1)} - \frac{2e^{-5s}}{s(s-3)(s+1)} + \frac{e^{-7s}}{s(s-3)(s+1)}$$

$$\ell^{-1}\left\{\frac{1}{s(s-3)(s+1)}\right\} = \ell^{-1}\left\{\frac{\cancel{x}\,\frac{1}{-3}}{\underset{s=0}{s}} + \frac{\cancel{x}\,\frac{1}{12}}{\underset{s=3}{s-3}} + \frac{\overset{1/4}{\cancel{x}}}{\underset{s=-1}{s+1}}\right\}$$

$$= -\frac{1}{3} + \frac{1}{12}e^{3x} + \frac{1}{4}e^{-x}$$

$$y(x) = \ell^{-1}\left\{\frac{1}{s(s-3)(s+1)}\right\} - 2\ell^{-1}\left\{\frac{e^{-5s}}{s(s-3)(s+$$

$$+ \ell^{-1}\left\{\frac{e^{-7s}}{s(s-3)(s+1)}\right\}$$

$$= -\frac{1}{3} + \frac{1}{12}e^{3x} + \frac{1}{4}e^{-x} - 2\left[u(x-5)\left(-\frac{1}{3} + \frac{1}{12}e^{3(x-5)} + \frac{1}{4}e^{-(x-5)}\right)\right] =$$

$$u(x-7)\left[-\frac{1}{} + \frac{1}{}e^{3(x+7)} \quad \frac{1}{}e^{-(x-7)}\right.$$

$$= -\frac{1}{3} + \frac{1}{12}e^{3x} + \frac{1}{4}e^{-x} - 2u(x-5)\left(-\frac{1}{3} + \frac{1}{12}e^{3(x-5)}\right.$$

$$\left. + \frac{1}{4}e^{-(x-5)}\right) + u(x-7)\left(-\frac{1}{3} + \frac{1}{12}e^{3(x-7)} + \frac{1}{4}e^{-(x-7)}\right)$$

QUESTION 2. (20 points) Given $f(x)$ is periodic with period $T = 4$ and defined on $[0, \infty)$. Also given that the first period of $f(x)$ is determined by

$$\begin{cases} 1 & if \quad 0 \le x < 2 \\ 0 & if \quad 2 \le x < 4 \end{cases}$$

a) Find $\ell\{f(x)\}$. [hint: you must simplify your answer, hence note that $1 - e^{-4s} = (1 - e^{-2s})(1 + e^{-2s})$].

$$\ell\{f(x)\} = \frac{1}{(1-e^{-2s})(1+e^{-2s})}\left(\int_0^2 e^{-sx}\,dx + 0\right)$$

$$= \frac{1}{(1-e^{-2s})(1+e^{-2s})}\left(\left.\frac{e^{-sx}}{-s}\right|_0^2\right)$$

$$= \frac{1}{\cancel{(1-e^{-2s})}(1+e^{-2s})}\left(\frac{\cancel{1-e^{-2s}}}{s}\right) = \frac{1}{s(1+e^{-2s})}$$

✓

b) Find $y(x)$ such that $\int_0^x f(r)y(x-r)\,dr - \int_0^x sin(r)\,dr = \int_0^x re^r\,dr$

$$\ell\left\{\int_0^x f(r)\,y(x-r)\,dr\right\} - \ell\left\{\int_0^x \sin(r)\,dr\right\} = \ell\left\{\int_0^x re^r\,dr\right\}$$

$$= \ell\{f(x) * y(x)\} - \ell\{1 * \sin(x)\} = \ell\{1 * xe^x\}$$

$$\frac{1}{s(1+e^{-2s})}Y(s) - \frac{1}{s(s^2+1)} = \frac{1}{s}\left(\frac{1}{(s-1)^2}\right)$$

Good

$$Y(s) = \left(\frac{1}{\cancel{s}(s-1)^2} + \frac{1}{\cancel{s}(s^2+1)}\right)\cancel{s}(1+e^{-2s})$$

$$Y(s) = \frac{1}{(s-1)^2} + \frac{e^{-2s}}{(s-1)^2} + \frac{1}{s^2+1} + \frac{e^{-2s}}{(s^2+1)}$$

$$y(x) = \ell^{-1}\left\{\frac{1}{(s-1)^2}\right\} + \ell^{-1}\left\{\frac{e^{-2s}}{(s-1)^2}\right\} + \ell^{-1}\left\{\frac{1}{s^2+1}\right\} + \ell^{-1}\left\{\frac{e^{-2s}}{(s^2+1)}\right\}$$

$$= xe^x + (x-2)\,u(x-2)\,e^{(x-2)} + \sin x + u(x-2)\sin(x-$$

✓ ✓

QUESTION 3. (18 points)

(i) find $\ell\{3^{2x} + cos(4x) - e^{x+5}\}$

$$= \ell\{3^{2x}\} + \ell\{\cos(4x)\} - \ell\{e^x \cdot e^5\}$$

$$= \ell\{e^{(2\ln 3)x}\} + \ell\{\cos(4x)\} - e^5 \ell\{e^x\}$$

$$= \frac{1}{s - 2\ln 3} + \frac{s}{s^2+16} - \frac{e^5}{s-1}$$

(ii) Find $\ell\{x\boxed{e^{3x}sin(x)}\}$ $\overset{F(x)}{}$

$$= (-1)^1 F^{(1)}(s)$$

$$= -\left(\frac{-2(s-3)}{((s-3)^2+1)^2}\right)$$

$$= \frac{2(s-3)}{((s-3)^2+1)^2} ✓$$

$$F(s) = \frac{1}{(s-3)^2+1}$$

$$= ((s-3)^2+1)$$

$$F'(s) = \frac{-(2(s-3))}{((s-3)^2+1}$$

(iii) Find $\ell\{\int_0^x e^{(x+3r)} r^3\, dr\}$

$$\ell\left\{\int_0^x e^{(x+3r)} r^3\, dr\right\} = \ell\left\{\int_0^x e^{(x-r)} \cdot e^{4r} r^3\, dr\right\}$$

$$= \ell\{e^x * e^{4x} x^3\}$$

$$= \left(\frac{1}{s-1}\right)\left(\frac{6}{(s-4)^4}\right)$$

$$= \frac{6}{(s-1)^{5}(x-4)^4}$$

QUESTION 4. (18 points)

(i) find $\ell^{-1}\{\frac{1}{s(s-4)^2}\}$ $= \ell^{-1}\left\{\frac{1}{s} \cdot \frac{1}{(s-4)^2}\right\}$

$$= \int_0^x 1 \cdot e^{4r} r\,dr = \int_0^x e^{4r} r\,dr$$

r	$\oplus\ e^{4r}$
1	$\ominus\ \frac{e^{4r}}{4}$
0	$\frac{e^{4r}}{16}$

$$= \left(\frac{re^{4r}}{4} - \frac{e^{4r}}{16} \right)_0^x$$

✓ $$= \frac{xe^{4x}}{4} - \frac{e^{4x}}{16} + \frac{1}{16}$$

(ii) find $\ell^{-1}\{\frac{se^{-2s}}{(s-5)^2}\}$ $= u(x-2)\ f(x-2)$

$$= u(x-2)\left(e^{5(x-2)} + 5e^{5(x-2)}(x-2) \right)$$

✓

$$\ell^{-1}\left\{\frac{\cancel{(s-5)}}{(s-5)^{\cancel{2}}}\right\} + 5\ell^{-1}\left\{\frac{1}{(s-5)^2}\right\}$$

$$= e^{5x} + 5e^{5x}x$$

(iii) find $\ell^{-1}\{\frac{s+4}{(s-1)^2+1}\}$

$$\ell^{-1}\left\{\frac{(s-1)}{(s-1)^2+1}\right\} + 5\ell^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}$$

$$= e^x \cos(x) + 5e^x \sin(x)$$ ✓

✓

QUESTION 5. (10 points) Find the largest interval around x = 4 such that

$$(\sqrt{8-x})y^{(2)} + \frac{3}{x+5}y' + y = \frac{5}{x-3}, y(4) = 0, y'(4) = -1$$

has a unique solution.

$a_2(x) = \sqrt{8-x} \neq 0$ & continous at $(-\infty, 8)$

$a_1(x) = \frac{3}{x+5}$ is continous at $(-\infty, -5) \cup (-5, \infty)$

$a_0(x) = 1$ " " " $(-\infty, \infty)$

$K(x) = \frac{5}{x-3}$ " " " $(-\infty, 3) \cup (3, \infty)$

$\Rightarrow$ I is $(3, 8)$ ✓

QUESTION 6. (14 points) Solve the D.E: $y^{(2)} - 6y' + 9y = x^3e^{3x}$, $y(0) = y'(0) = 0$.

$$\mathcal{L}\{y^{(2)}\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{x^3e^{3x}\}$$

$$s^2Y(s) - \cancel{sy(0)} - \cancel{y'(0)} - 6\left(sY(s) - \cancel{y(0)}\right) + 9Y(s) = \frac{6}{(s-3)}$$

$$(s^2 - 6s + 9)(Y(s)) = \frac{6}{(s-3)^4}$$

$$(s-3)(s-3)\ Y(s) = \frac{6}{(s-3)^4}$$

$$Y(s) = \frac{6}{(s-3)^6}$$ ✓

$$y(x) = \frac{6}{5!}\mathcal{L}^{-1}\left\{\frac{5!}{(s-3)^6}\right\} = \frac{6}{5!}e^{3x}x^5$$

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3.22 Questions with Solutions, Review Exam II

Name [redacted], ID [redacted]

Nadeen Tarek

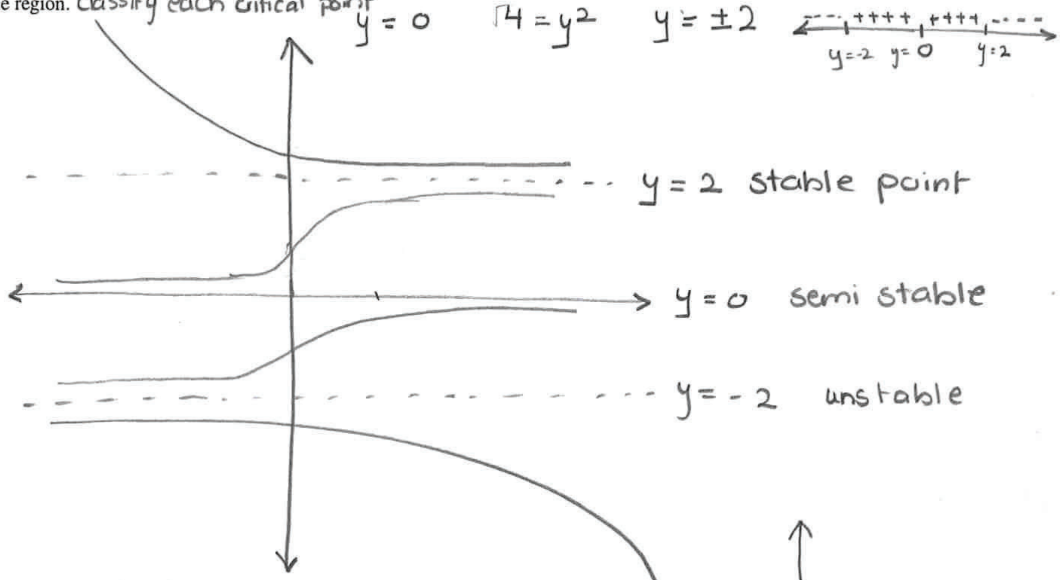
Exam II, MTH 205, Fall 2019

Ayman Badawi

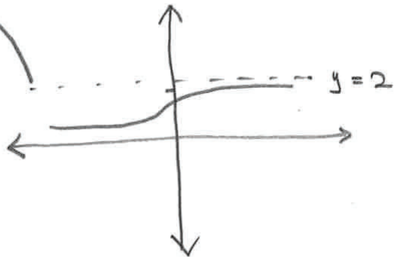
Total = $\frac{60 \text{ (Excellent)}}{60}$ (Consider Math Major (double major) or minor)

QUESTION 1. (6 points) (1) Given $y' = y^2(4 - y^2)$. Find the critical points (values). Sketch all possible solution curves in the region. Classify each critical point

$y = 0$ $\quad \sqrt{4} = y^2$ $\quad y = \pm 2$



(2) If the point $(1, 1.5)$ lies on the curve, then sketch the solution curve.



QUESTION 2. (6 points) Solve the diff. equation $y' = \frac{e^{2x-y}}{y}$

$$\frac{dy}{dx} = \frac{e^{2x-y}}{y} \qquad e^{2x} \cdot e^{-y}$$

$$\int ye^y = ye^y - e^y$$

$$\begin{array}{c|c} y & \int \\ & (+)\, e^y \\ 1 & (-)\, e^y \\ 0 & e^y \end{array}$$

$$\frac{dy}{dx} = \frac{e^{2x}}{ye^y}$$

$$\int ye^y \, dy = \int e^{2x} \, dx$$

$$\boxed{ye^y - e^y = \frac{e^{2x}}{2} + C}$$

Nadeen

QUESTION 3. (6 points) Solve the diff. equation $y' = \frac{-2xy}{1-x^2}$, where $x \geq 4$

$$y' = \frac{-2xy}{1-x^2}$$

$$\frac{dy}{dx} = \frac{-2xy}{1-x^2}$$

$$\frac{dy}{dx} = \frac{-2x}{1-x^2} \cdot y$$

$$\int \frac{1}{y}\,dy = \int \frac{-2x}{1-x^2}\,dx$$

$$\boxed{\ln|y| = \ln|1-x^2| + C}$$

✓

QUESTION 4. (6 points) Solve the diff. equation $y' = \frac{y\cos(xy)-e^{2y}}{2xe^{2y}-x\cos(xy)+2y}$ [Hint: assume that it is exact, no need to check $F_{xy} = F_{yx}$]

$$y' = \frac{\overbrace{y\cos(xy) - e^{2y}}^{-F_x}}{\underbrace{2xe^{2y} - x\cos(xy) + 2y}_{F_y}}$$

$$[\overbrace{2xe^{2y} - x\cos(xy) + 2y}^{F_y}]\,dy + [\overbrace{-y\cos(xy) + e^{2y}}^{F_x}]\,dx = 0$$

$$\int F_x\,dx = \int -y\cos(xy) + e^{2y}\,dx = \frac{-\cancel{y}}{\cancel{y}}\sin(xy) + e^{2y}x + C(y)$$

$$= -\sin(xy) + x e^{2y} + C(y)$$

$$F_y = -x\cos(xy) + 2xe^{2y} + C'(y) = 2xe^{2y} - x\cos(xy) + 2y$$

$$\int C'(y) = \int 2y\,dy \quad C(y) = y^2 + C$$

$$\boxed{-\sin(xy) + xe^{2y} + y^2 + C = 0}$$

✓

Nadeen

QUESTION 5. (6 points) Imagine a cake is removed from an oven, its temperature is measured 300 F. The cake was placed in a room that has temperature 70 F. Three minutes later its temperature is 200 F. Find the temperature of the cake at any time t. How long will it take for the cake to reach temperature 74 F?

$T(0) = 300 \quad T(3) = 200 \qquad T_m = 70^\circ$

$\frac{dT}{dt} = T' = \alpha(T - T_m)$

$T' = \alpha(T - 70)$

$T' - \alpha T = -70\alpha \qquad I = e^{\int -\alpha\, dt} = e^{-\alpha t}$

$$T = \frac{\int e^{-\alpha t} \cdot -70\alpha\, dt}{e^{-\alpha t}} = \frac{\frac{\cancel{-}70\cancel{\alpha}}{\cancel{-}\cancel{\alpha}} e^{-\alpha t} + C}{e^{-\alpha t}} = \underline{\underline{70 + Ce^{\alpha t}}}$$

$\boxed{T = 70 + Ce^{\alpha t}}$

$300 = 70 + Ce^{0}$

$C = 230$

$T = 70 + 230e^{\alpha t}$

$200 = 70 + 230e^{3\alpha}$

$e^{3\alpha} = \frac{200 - 70}{230}$

$\alpha = \frac{\ln\left(\frac{13}{23}\right)}{3}$

$\alpha = -0.190$

$\boxed{T = 70 + 230e^{-0.190t}}$

$74 = 70 + 230e^{-0.190t}$

$e^{-0.190t} = \frac{74 - 70}{230}$

$t = \frac{\ln\left(\frac{2}{115}\right)}{-0.190} = \underline{\underline{21.3 \text{ min}}}$

✓

QUESTION 6. (6 points) Given $(7x + 2)y'' - 7y' + (-9 - 7x)y = 0$. Given $y_1 = e^{-x}$ is a solution. Find y_2, then find the general solution.

$y_1 = e^{-x}$

$\frac{\cancel{(7x+2)}}{\cancel{7x+2}} y'' - \frac{7}{7x+2}y' + \frac{(-9-7x)}{7x+2}y = 0$

$y'' - \frac{7}{7x+2}y' + \frac{(-9-7x)}{7x+2}y = 0$

$e^{\int \frac{7}{7x+2}dx} = \cancel{e}^{\cancel{\ln}(7x+2)}$

$= 7x + 2$

$$y_2 = y_1 \int \frac{e^{\int -Q_x dx}}{y_1^2} dx = e^{-x} \int \frac{7x+2}{e^{-2x}} dx$$

$\int (7x+2)e^{2x}$

$$\begin{array}{c|c} \text{'} & \int \\ 7x+2 & e^{2x} \\ 7 & \oplus \frac{e^{2x}}{2} \\ 0 & \ominus \frac{e^{2x}}{4} \end{array}$$

$\frac{1}{2}(7x+2)e^{2x} - \frac{7}{4}e^{2x}$

$y_2 = e^{-x}\left[\frac{1}{2}(7x+2)e^{2x} - \frac{7}{4}e^{2x}\right]$

$y_2 = \frac{1}{2}(7x+2)e^{x} - \frac{7}{4}e^{x}$

$y_g = C_1 e^{-x} + C_2\left[\frac{1}{2}(7x+2)e^{x} - \frac{7}{4}e^{x}\right]$

✓

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$m(m-1)-3m+3$
$m^2-m-3m+3$
m^2-4m+3

QUESTION 7. (10 points) (1) Solve for y, $x^2y'' - 3xy' + 3y = 2x^4e^x$ [Hint : $y = y_h + y_p$]

$x^2y'' - 3xy' + 3y = 2x^4e^x \quad y = x^m \quad y' = mx^{m-1} \quad y'' = m(m-1)x^{m-2}$

$(\cancel{x^2}(m(m-1)x^{\cancel{m-2}})) - (3\cancel{x}(m x^{\cancel{m-1}})) + 3(x^m) = 0$

$x^m[m^2 - m - 3m + 3] = 0$

$m^2 - 4m + 3 = 0$

$m = 3 \quad m = 1$

$\boxed{y_h = c_1 \underbrace{x^3}_{y_1} + c_2 \underbrace{x}_{y_2}}$

$y_p = v_1y_1 + v_2y_2$

$v_1'y_1 + v_2'y_2 = 0$

$v_1'y_1' + v_2'y_2' = \frac{2x^{\cancel{4}}e^x}{\cancel{x^2}}$

$v_1'x^3 + v_2'x = 0$

$v_1'(3x^2) + v_2'(1) = 2x^2e^x$

$W = \begin{vmatrix} x^3 & x \\ 3x^2 & 1 \end{vmatrix} = x^3 - 3x^3 = -2x^3$

$v_1' = \frac{\begin{vmatrix} 0 & x \\ 2x^2e^x & 1 \end{vmatrix}}{-2x^3} = \frac{\cancel{-2x^3}e^x}{\cancel{-2x^3}} = e^x$

$\boxed{v_1 = e^x}$

$v_2' = \frac{\begin{vmatrix} x^3 & 0 \\ 3x^2 & 2x^2e^x \end{vmatrix}}{-2x^3} = \frac{\cancel{2}x^{5}e^x}{-\cancel{2}\cancel{x^3}} = -x^2e^x$

$v_2 = \int -x^2e^x$

$\int$

$-x^2 \quad (+) \quad e^x$
$-2x \quad (-) \quad e^x$
$-2 \quad (+) \quad e^x$
$0 \quad e^x$

$\boxed{v_2 = -x^2e^x + 2xe^x - 2e^x}$

$y_p = x^3e^x + (-x^2e^x + 2xe^x - 2e^x)x$

$y_p = \cancel{x^3e^x} - \cancel{x^3e^x} + 2x^2e^x - 2xe^x$

$\boxed{y_p = 2xe^x(x-1)}$

$\boxed{y = c_1x^3 + c_2x + 2xe^x(x-1)}$

✓

(2) $(\sqrt{2x+6})y' + \frac{1}{x-4}y = \frac{1}{x-6}$, where $y(1) = 7$. Find the largest interval for the values of x so that the solution is unique.

$\sqrt{2x+6}\,y' + \frac{1}{x-4}y = \frac{1}{x-6}$

$2x + 6 = 0$

$x = \frac{-6}{2}$

$\boxed{x = -3}$

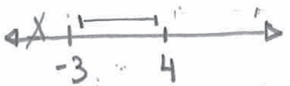
$x \neq -3$

$x \leq -3$

$x \neq 4$

interval
$(-3, 4)$

✓



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QUESTION 8. (6 points) Solve the diff. equation $\frac{dy}{dx} = \frac{1}{-2x+y^2+1}$

$$\frac{dy}{dx} = \frac{1}{-2x+y^2+1}$$

$$\frac{dx}{dy} = -2x+y^2+1$$

$$x' = -2x+y^2+1$$

$$x'+2x = y^2+1 \qquad I = e^{\int 2\,dy} = e^{2y}$$

$$x = \frac{\int e^{2y}\cdot(y^2+1)\,dy}{e^{2y}}$$

$$x = \frac{\int y^2e^{2y}+e^{2y}}{e^{2y}} = \frac{\frac{1}{2}y^2\cancel{e^{2y}} - \frac{1}{2}y\cancel{e^{2y}} + \frac{3}{4}\cancel{e^{2y}}+C}{\cancel{e^{2y}}}$$

$$\boxed{x = \frac{1}{2}y^2 - \frac{1}{2}y + \frac{3}{4} + Ce^{-2y}}$$

$$y^2\underline{\underline{e^{2y}}} + e^{2y}$$

$$\int \underset{\downarrow}{y^2}e^{2y} + \underset{\downarrow}{e^{2y}} \quad \frac{e^{2y}}{2}$$

$$\begin{array}{c|l} \downarrow & \int \\ y^2 & \oplus\, e^{2y} \\ 2y & \ominus\, \frac{1}{2}e^{2y} \\ 2 & \oplus\, \frac{1}{4}e^{2y} \\ 0 & \frac{1}{8}e^{2y} \end{array}$$

$$\frac{1}{2}y^2e^{2y} - \frac{1}{2}ye^{2y} + \frac{1}{4}e^{2y}$$

$$+\frac{1}{2}e^{2y}$$

$$= \frac{1}{2}y^2e^{2y} - \frac{1}{2}ye^{2y} + \frac{3}{4}e^{2y} + C$$

QUESTION 9. (8 points) Imagine a company sells fake-honey. A tank contains 200 liters of fluid in which 30 grams of honey is dissolved (i.e, $A(0) = 30$). Brine containing 3 grams of honey per liter is then pumped into the tank at rate 4L/min. The well-mixed solution is pumped out at 6L/min. Find the number $A(t)$ of grams of honey in the tank at time t. When is the tank empty?

$$A' = In - out$$

$$A' = (3)(4) - \underset{\substack{\downarrow \\ \frac{A(t)}{200+(4-6)t}}}{C(t)}(6)$$

$$A' = 12 - \frac{6A(t)/2}{200-2t\ /2}$$

$$A' + \frac{3A(t)}{100-t} = 12$$

$$I = e^{\int \frac{3}{100-t}dt^{-1}}$$

$$e^{-3\ln(100-t)}$$

$$= (100-t)^{-3}$$

$$A = \frac{\int (100-t)^{-3}\cdot 12\,dt}{(100-t)^{-3}}$$

$$\int 12(100-t)^{-3}\,dt$$

$$12\int(100-t)^{-3}\,dt$$

$$= \cancel{+}\frac{12}{2} \cancel{*-1} (100-t)^{-2}$$

$$= 6(100-t)^{-2} + C$$

$$A = \frac{6(100-t)^{-2}+C}{(100-t)^{-3}}$$

$$\boxed{A = 6(100-t) + C(100-t)}$$

$$30 = 6(100-0) + C(100-0)^3$$

$$30 = 600 + (100)^3C$$

$$C = \frac{-57}{1*10^5}$$

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$$200 + (4-6)t = 0$$

$$-2t = -200$$

$$t = \frac{\cancel{+}200}{\cancel{+}2}$$

$$\underline{\underline{t = 100 \text{ min}}}$$

the tank is empty

$$\boxed{A = 6(100-t) - \frac{57}{10^5}(100-t)^3}$$

QUESTION 1. (i) **(3 points)** Find the values of the constants $a, \overset{k}{\text{[illegible]}}, c$ which makes the differential equation $(12x^2y - aye^{cx})dx + (kx^3 - e^{3x})dy$ exact (**DO NOT SOLVE IT**)

$F_{xy} = F_{yx}$

$F_{xy} = 12x^2 - ae^{cx}$

$F_{yx} = 3kx^2 - 3e^{3x}$

$12x^2 - ae^{cx} = 3kx^2 - 3e^{3x}$

$12 = 3k \qquad \cancel{-}ae^{cx} = \cancel{-}3e^{3x}$

$\boxed{k = 4} \qquad \boxed{a = 3}$

$\boxed{c = 3}$

(ii) **(6 points)** Stare really good at the following diff. equation $\frac{dy}{dx} = \frac{y^3}{x^3 - xy^2}$, change it to Bernoulli and solve it.

$$\frac{dx}{dy} = \frac{x^3 - xy^2}{y^3}$$

$$x' = \frac{1}{y^3}x^3 - \frac{1}{y}x$$

$$x' + \frac{1}{y}x = \frac{1}{y^3}x^3$$

$$v = x^{\cancel{1}-3} = x^{-2}$$

$$v' + (-2) \times \frac{1}{y}v = (-2)\frac{1}{y^3}$$

$$v' - \frac{2}{y}v = -\frac{2}{y^3}$$

$$I = e^{\int -\frac{2}{y}dy} = e^{-2\ln y}$$
$$= \frac{1}{y^2}$$

$$v = \frac{\int \frac{1}{y^2} \times -\frac{2}{y^3}\,dy}{\frac{1}{y^2}}$$

$$v = \frac{\int \frac{-2}{y^5}\,dy}{\frac{1}{y^2}}$$

$$v = \frac{\frac{1}{2}y^{-4} + C}{\frac{1}{y^2}}$$

$$v = \frac{1}{2}y^{-2} + y^2C$$

$$x = \left(\frac{1}{2}y^{-2} + y^2C\right)^{-\frac{1}{2}}$$

QUESTION 3. (10 points) Imagine a company is making fake-sweet-drink(only water and sugar). The Tank has a capacity of 700 Liters. Initially, it contains 250 Liters of brine (water and sugar) that contains 25grams of sugar (i.e., assume $A(0) = 25$). A solution containing 4 grams of sugar per liter is pumped into the tank at rate 4

liter per min. The solution is pumped out at rate 3 liter per min. Find A(t)/ amount of sugar in the tank.

$$\frac{dA}{dt} = In - out$$

$$A' = (4)(4) - C(t)3$$

$$A(t) = \frac{4(250+t)^4 - 1.52\times10^{10}}{(250+t)^3}$$

$$C(t) = \frac{A}{250+(4-3)t}$$

$$A' = 16 - \frac{3A}{250+t}$$ ✓

$$A' + \frac{3}{250+t}A = 16$$

$$I = e^{\int\frac{3}{250+t}dt}$$

$$I = e^{+3\ln|250+t|}$$

$$I = (250+t)^3$$ ✓

$$A = \frac{\int(250+t)^3\times16}{(250+t)^3}$$

$$A = \frac{4(250+t)^4+C}{(250+t)^3}$$ ✓

~~20~~/20

$$A(0) = \frac{4(250)^4+C}{(250)^3} = 25$$

$$C = -1.52\times10^{10}$$ ✓

~~A(10)~~ =

(ii) Find the amount of sugar in the tank after 10 min. ✓

$$A(10) = \frac{4(250+10)^4-1.52\times10^{10}}{(250)^3} = 195\ Kg$$

$(250+10)^3$

(iii) When an overflow will occur?

$$250+(4-3)t = 700$$

$$t = 450\ mins$$ ✓

QUESTION 8. (6 points) Solve for $y(t)$: $(cos(t) - t)y'' + (1 + sin(t))y' = 0$

$V_1 = y'$

$v' = y''$

$(\cos(t) - t)\, v' + (1 + \sin(t))\, v = 0$

$v' + \frac{(1 + \sin(t))}{(\cos(t) - t)} v = 0$

$I = e^{\int \frac{1 + \sin(t)}{\cos(t) - t}}$

$u = -(\cos(t) - t)$

$du = +\sin(t) + 1\, dt$

$I = e^{\int \frac{1}{-u} du}$

$I = e^{-\ln|u|} = \frac{1}{-\cos(t) + t}$

$V = \frac{\int \frac{1}{t - \cos(t)} \times 0 \, dt}{\frac{1}{t - \cos(t)}}$

$V = \frac{0 + C}{\cancel{\frac{1}{t - \cos t)}}} \rightarrow C[t - \cos(t)]$

$y = \int Ct - C\cos(t)\, dt$

$y = \frac{1}{2} Ct^2 - C\sin t + C_1$

$y = Ct^2 - C\sin t + C_1$

QUESTION 9. (10 points)

(i) Solve for $y(t)$, $t^2y'' - 2ty' + 2y = 0$

$$y = t^m$$
$$y' = m\,t^{m-1}$$
$$y'' = (m^2 - m)\,t^{m-2}$$
$$t^m(m^2 - m - 2m + 2) = 0$$
$$m^2 - 3m + 2 = 0$$
$$m = 2 \text{ or } m = 1$$
$$y = C_1 t^2 + C_2 t$$

(ii) Use (1) and solve for $y(t)$: $t^2y'' - 2ty' + 2y = 2t^3e^t$

$$y = y_h + y_p$$
$$y_h = C_1 \underbrace{t^2}_{y_1} + C_2 \underbrace{t}_{y_2}$$
$$y_p = v_1 y_1 + v_2 y_2$$
$$v_1' y_1 + v_2' y_2 = 0$$
$$v_1' y_1' + v_2' y_2' = \frac{2t^3e^t}{t^2}$$

$$v_1' t^2 + v_2' t = 0$$
$$v_1' 2t + v_2' = 2te^t$$

$$w_1 = \begin{vmatrix} t^2 & t \\ 2t & 1 \end{vmatrix} = t^2 - 2t^2$$
$$= -t^2$$

$$v_1' = \frac{\begin{vmatrix} 0 & t \\ 2te^t & 1 \end{vmatrix}}{-t^2} = \frac{-2\cancel{t^2}e^t}{-\cancel{t^2}}$$
$$v_1' = 2e^t$$
$$v_1 \int 2e^t dt = 2e^t$$

$$v_2' = \frac{\begin{vmatrix} t^2 & 0 \\ 2t & 2te^t \end{vmatrix}}{-t^2} = \frac{2t^{3\,1}e^t}{-\cancel{t^2}}$$
$$v_2' = -2te^t$$
$$v_2 \int -2te^t\,dt$$

$$\begin{array}{c|c} ()' & \int \\ \hline -2t & e^t \\ -2 & e^t \\ 0 & e^t \end{array} = -2te^t + 2e^t$$

$$y_p = (2e^t)(t^2) + (-2te^t + 2e^t)(t)$$
$$y_p = 2\cancel{t^2}e^t - \cancel{2t^2}e^t + 2te^t$$
$$y_p = 2te^t$$

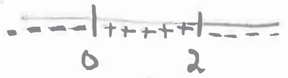
$$\Longrightarrow y = C_1t^2 + C_2t + 2te^t$$

SHORT ANSWERS, JUST STARE well and Think

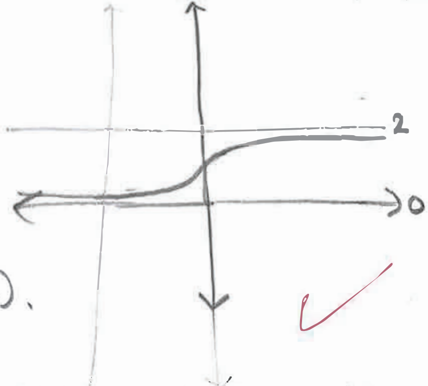
QUESTION 10. (i) **(3 points)** Draw the solution curve for $y(x)$ that contains the point (0, 1.5) and find $lim_{x\to\infty} y(x)$, where $y' = -y^2 + 2y$.

$-y^2 + 2y = 0$

$y = 2$ or $y = 0$



(0, 1.5) lies in (0, 2).



as x goes ∞ → y(x) will approach 2

$\lim_{x\to\infty} = 2.$

(ii) **(3 points)** Given y_1 and y_2 are two distinct solutions for the diff. equation $e^{x^3} y'' + cos(x)y = \frac{ln(x)}{1+x^3}$. Then one can quickly form a third solution $y_3 = \pi^2 y_1 + a y_2$ and a forth solution $y_4 = b y_1 + (e^2 + 1) y_2$. Find the values of the constants a, b.

$e^{x^2} y'' + \cos(x) y = \frac{\ln(x)}{1+x^3}$

$b = \pi^2$

$a = (e^2 + 1)$

0/3

(iii) **(4 points)** Solve the diff. equation $\frac{dy}{dx} = (\sqrt{y} + y)e^x(x^2 + 2x)$

$\int \frac{dy}{\sqrt{y} + y} = \int e^x(x^2 + 2x)\, dx$

$\int \frac{1}{(\sqrt{y})(1+\sqrt{y})} dy = x^2 e^x + C$

$u = 1 + \sqrt{y}$

$du = \frac{1}{2\sqrt{y}} dy$

$2\int \frac{1}{u} du$

$2\ln|1+\sqrt{y}|$

$2\ln|1+\sqrt{y}| = x^2 e^x + C$

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3.23 Questions with Solutions, Review Final Exam

EXERCISES 5.1

Answers to selected odd-numbered problems begin on page ANS-7.

Questions on Spring

3. A mass weighing 24 pounds, attached to the end of a spring, stretches it 4 inches. Initially, the mass is released from rest from a point 3 inches above the equilibrium position. Find the equation of motion.

4. Determine the equation of motion if the mass in Problem 3 is initially released from the equilibrium position with a downward velocity of 2 ft/s.

5. A mass weighing 20 pounds stretches a spring 6 inches. The mass is initially released from rest from a point 6 inches below the equilibrium position.

(a) Find the position of the mass at the times $t = \pi/12$, $\pi/8$, $\pi/6$, $\pi/4$, and $9\pi/32$ s.

(b) What is the velocity of the mass when $t = 3\pi/16$ s? In which direction is the mass heading at this instant?

(c) At what times does the mass pass through the equilibrium position?

This mass is removed and replaced with a mass of

11.

5 Modeling with Higher-Order Differential Equations

Exercises 5.1

Answers to Questions on SPRING

0 then

3. From $\frac{3}{4}x'' + 72x = 0$, $x(0) = -1/4$, and $x'(0) = 0$ we obtain $x = -\frac{1}{4}\cos 4\sqrt{6}\,t$.

4. From $\frac{3}{4}x'' + 72x = 0$, $x(0) = 0$, and $x'(0) = 2$ we obtain $x = \frac{\sqrt{6}}{12}\sin 4\sqrt{6}\,t$.

5. From $\frac{5}{8}x'' + 40x = 0$, $x(0) = 1/2$, and $x'(0) = 0$ we obtain $x = \frac{1}{2}\cos 8t$.

   (a) $x(\pi/12) = -1/4$, $x(\pi/8) = -1/2$, $x(\pi/6) = -1/4$, $x(\pi/4) = 1/2$, $x(9\pi/32) = \sqrt{2}/4$.

   (b) $x' = -4\sin 8t$ so that $x'(3\pi/16) = 4$ ft/s directed downward.

   (c) If $x = \frac{1}{2}\cos 8t = 0$ then $t = (2n+1)\pi/16$ for $n = 0, 1, 2, \ldots$.

6. From $50x'' + 200x = 0$, $x(0) = 0$, and $x'(0) = -10$ we obtain $x = -5\sin 2t$ and $x' = -10\cos 2t$.

Circuit Questions

5.1.4 SERIES CIRCUIT ANALOGUE

45. Find the charge on the capacitor in an *LRC* series circuit at $t = 0.01$ s when $L = 0.05$ h, $R = 2\ \Omega$, $C = 0.01$ f, $E(t) = 0$ V, $q(0) = 5$ C, and $i(0) = 0$ A. Determine the first time at which the charge on the capacitor is equal to zero.

46. Find the charge on the capacitor in an *LRC* series circuit when $L = \frac{1}{4}$ h, $R = 20\ \Omega$, $C = \frac{1}{300}$ f, $E(t) = 0$ V, $q(0) = 4$ C, and $i(0) = 0$ A. Is the charge on the capacitor ever equal to zero?

In Problems 47 and 48 find the charge on the capacitor and the current in the given *LRC* series circuit. Find the maximum charge on the capacitor.

47. $L = \frac{5}{3}$ h, $R = 10\ \Omega$, $C = \frac{1}{30}$ f, $E(t) = 300$ V, $q(0) = 0$ C, $i(0) = 0$ A

48. $L = 1$ h, $R = 100\ \Omega$, $C = 0.0004$ f, $E(t) = 30$ V, $q(0) = 0$ C, $i(0) = 2$ A

Answers to Questions on Circuit

45. Solving $\frac{1}{20}q'' + 2q' + 100q = 0$ we obtain $q(t) = e^{-20t}(c_1 \cos 40t + c_2 \sin 40t)$. The initial conditions $q(0) = 5$ and $q'(0) = 0$ imply $c_1 = 5$ and $c_2 = 5/2$. Thus

$$q(t) = e^{-20t}\left(5\cos 40t + \frac{5}{2}\sin 40t\right) = \sqrt{25 + 25/4}\,e^{-20t}\sin(40t + 1.1071)$$

and $q(0.01) \approx 4.5676$ coulombs. The charge is zero for the first time when $40t + 1.1071 = \pi$ or $t \approx 0.0509$ second.

46. Solving $\frac{1}{4}q'' + 20q' + 300q = 0$ we obtain $q(t) = c_1 e^{-20t} + c_2 e^{-60t}$. The initial conditions $q(0) = 4$ and $q'(0) = 0$ imply $c_1 = 6$ and $c_2 = -2$. Thus

$$q(t) = 6e^{-20t} - 2e^{-60t}.$$

Setting $q = 0$ we find $e^{40t} = 1/3$ which implies $t < 0$. Therefore the charge is not 0 for $t \geq 0$.

47. Solving $\frac{5}{3}q'' + 10q' + 30q = 300$ we obtain $q(t) = e^{-3t}(c_1 \cos 3t + c_2 \sin 3t) + 10$. The initial conditions $q(0) = q'(0) = 0$ imply $c_1 = c_2 = -10$. Thus

$$q(t) = 10 - 10e^{-3t}(\cos 3t + \sin 3t) \quad \text{and} \quad i(t) = 60e^{-3t}\sin 3t.$$

Solving $i(t) = 0$ we see that the maximum charge occurs when $t = \pi/3$ and $q(\pi/3) \approx 10.432$.

48. Solving $q'' + 100q' + 2500q = 30$ we obtain $q(t) = c_1 e^{-50t} + c_2 te^{-50t} + 0.012$. The initial conditions $q(0) = 0$ and $q'(0) = 2$ imply $c_1 = -0.012$ and $c_2 = 1.4$. Thus, using $i(t) = q'(t)$ we get

$$q(t) = -0.012e^{-50t} + 1.4te^{-50t} + 0.012 \quad \text{and} \quad i(t) = 2e^{-50t} - 70te^{-50t}.$$

Solving $i(t) = 0$ we see that the maximum charge occurs when $t = 1/35$ second and $q(1/35) \approx$ .01871 coulomb.

Questions on Substitution, y = ux or u = ax + by

Each DE in Problems 1–14 is homogeneous.

In Problems 1–10 solve the given differential equation by using an appropriate substitution.

[redacted]

2. $(x + y)\,dx + x\,dy = 0$

3. $x\,dx + (y - 2x)\,dy = 0$

4. $y\,dx = 2(x + y)\,dy$

5. $(y^2 + yx)\,dx - x^2\,dy = 0$

6. $(y^2 + yx)\,dx + x^2\,dy = 0$

7. $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$

8. $\dfrac{dy}{dx} = \dfrac{x + 3y}{3x + y}$

9. $-y\,dx + \left(x + \sqrt{xy}\right)dy = 0$

[redacted]

Each DE in Problems 23–30 is of the form given in (5).

In Problems 23–28 solve the given differential equation by using an appropriate substitution.

23. $\dfrac{dy}{dx} = (x + y + 1)^2$

24. $\dfrac{dy}{dx} = \dfrac{1 - x - y}{x + y}$

25. $\dfrac{dy}{dx} = \tan^2(x + y)$

26. $\dfrac{dy}{dx} = \sin(x + y)$

27. $\dfrac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$

28. $\dfrac{dy}{dx} = 1 + e^{y - x + 5}$

In Problems 29 and 30 solve the given initial-value problem.

29. $\dfrac{dy}{dx} = \cos(x + y), \quad y(0) = \pi/4$

30. $\dfrac{dy}{dx} = \dfrac{3x + 2y}{3x + 2y + 2}, \quad y(-1) = -1$

[redacted]

2. Letting $y = ux$ we have

$$\begin{aligned}(x + ux)\,dx + x(u\,dx + x\,du) &= 0\\ (1 + 2u)\,dx + x\,du &= 0\\ \frac{dx}{x} + \frac{du}{1 + 2u} &= 0\\ \ln|x| + \frac{1}{2}\ln|1 + 2u| &= c\\ x^2\left(1 + 2\frac{y}{x}\right) &= c_1\\ x^2 + 2xy &= c_1.\end{aligned}$$

3. Letting $x = vy$ we have

$$\begin{aligned}vy(v\,dy + y\,dv) + (y - 2vy)\,dy &= 0\\ vy^2\,dv + y\left(v^2 - 2v + 1\right)dy &= 0\\ \frac{v\,dv}{(v-1)^2} + \frac{dy}{y} &= 0\\ \ln|v - 1| - \frac{1}{v - 1} + \ln|y| &= c\\ \ln\left|\frac{x}{y} - 1\right| - \frac{1}{x/y - 1} + \ln y &= c\\ (x - y)\ln|x - y| - y &= c(x - y).\end{aligned}$$

4. Letting $x = vy$ we have

$$\begin{aligned}y(v\,dy + y\,dv) - 2(vy + y)\,dy &= 0\\ y\,dv - (v + 2)\,dy &= 0\\ \frac{dv}{v + 2} - \frac{dy}{y} &= 0\\ \ln|v + 2| - \ln|y| &= c\\ \ln\left|\frac{x}{y} + 2\right| - \ln|y| &= c\\ x + 2y &= c_1y^2.\end{aligned}$$

5. Letting $y = ux$ we have

$$\begin{aligned}\left(u^2x^2 + ux^2\right) dx - x^2(u\,dx + x\,du) &= 0\\ u^2\,dx - x\,du &= 0\\ \frac{dx}{x} - \frac{du}{u^2} &= 0\\ \ln|x| + \frac{1}{u} &= c\\ \ln|x| + \frac{x}{y} &= c\\ y\ln|x| + x &= cy.\end{aligned}$$

6. Letting $y = ux$ and using partial fractions, we have

$$\begin{aligned}\left(u^2x^2 + ux^2\right) dx + x^2(u\,dx + x\,du) &= 0\\ x^2\left(u^2 + 2u\right) dx + x^3\,du &= 0\\ \frac{dx}{x} + \frac{du}{u(u+2)} &= 0\\ \ln|x| + \frac{1}{2}\ln|u| - \frac{1}{2}\ln|u+2| &= c\\ \frac{x^2u}{u+2} &= c_1\\ x^2\frac{y}{x} &= c_1\left(\frac{y}{x} + 2\right)\\ x^2y &= c_1(y + 2x).\end{aligned}$$

7. Letting $y = ux$ we have

$$\begin{aligned}(ux - x)\,dx - (ux + x)(u\,dx + x\,du) &= 0\\ \left(u^2 + 1\right) dx + x(u+1)\,du &= 0\\ \frac{dx}{x} + \frac{u+1}{u^2+1}\,du &= 0\\ \ln|x| + \frac{1}{2}\ln\left(u^2 + 1\right) + \tan^{-1}u &= c\\ \ln x^2\left(\frac{y^2}{x^2} + 1\right) + 2\tan^{-1}\frac{y}{x} &= c_1\\ \ln\left(x^2 + y^2\right) + 2\tan^{-1}\frac{y}{x} &= c_1.\end{aligned}$$

8. Letting $y = ux$ we have

$$\begin{aligned}
(x + 3ux)\,dx - (3x + ux)(u\,dx + x\,du) &= 0 \\
\left(u^2 - 1\right) dx + x(u + 3)\,du &= 0 \\
\frac{dx}{x} + \frac{u + 3}{(u - 1)(u + 1)}\,du &= 0 \\
\ln|x| + 2\ln|u - 1| - \ln|u + 1| &= c \\
\frac{x(u - 1)^2}{u + 1} &= c_1 \\
x\left(\frac{y}{x} - 1\right)^2 &= c_1\left(\frac{y}{x} + 1\right) \\
(y - x)^2 &= c_1(y + x).
\end{aligned}$$

9. Letting $y = ux$ we have

$$\begin{aligned}
-ux\,dx + (x + \sqrt{u}\,x)(u\,dx + x\,du) &= 0 \\
(x^2 + x^2\sqrt{u}\,)\,du + xu^{3/2}\,dx &= 0 \\
\left(u^{-3/2} + \frac{1}{u}\right) du + \frac{dx}{x} &= 0 \\
-2u^{-1/2} + \ln|u| + \ln|x| &= c \\
\ln|y/x| + \ln|x| &= 2\sqrt{x/y} + c
\end{aligned}$$

[redacted]

[redacted]

23. Let $u = x + y + 1$ so that $du/dx = 1 + dy/dx$. Then $\dfrac{du}{dx} - 1 = u^2$ or $\dfrac{1}{1+u^2}\,du = dx$. Thu
$\tan^{-1} u = x + c$ or $u = \tan(x + c)$, and $x + y + 1 = \tan(x + c)$ or $y = \tan(x + c) - x - 1$.

24. Let $u = x + y$ so that $du/dx = 1 + dy/dx$. Then $\dfrac{du}{dx} - 1 = \dfrac{1-u}{u}$ or $u\,du = dx$. Thus $\frac{1}{2}u^2 = x +$
or $u^2 = 2x + c_1$, and $(x + y)^2 = 2x + c_1$.

25. Let $u = x + y$ so that $du/dx = 1 + dy/dx$. Then $\dfrac{du}{dx} - 1 = \tan^2 u$ or $\cos^2 u\,du = dx$. Thu
$\frac{1}{2}u + \frac{1}{4}\sin 2u = x + c$ or $2u + \sin 2u = 4x + c_1$, and $2(x+y) + \sin 2(x+y) = 4x + c_1$ or $2y + \sin 2(x+y) =$
$2x + c_1$.

26. Let $u = x + y$ so that $du/dx = 1 + dy/dx$. Then $\dfrac{du}{dx} - 1 = \sin u$ or $\dfrac{1}{1+\sin u}\,du = dx$. Multiplyin
by $(1 - \sin u)/(1 - \sin u)$ we have $\dfrac{1-\sin u}{\cos^2 u}\,du = dx$ or $(\sec^2 u - \sec u \tan u)du = dx$. Thu
$\tan u - \sec u = x + c$ or $\tan(x + y) - \sec(x + y) = x + c$.

27. Let $u = y - 2x + 3$ so that $du/dx = dy/dx - 2$. Then $\dfrac{du}{dx} + 2 = 2 + \sqrt{u}$ or $\dfrac{1}{\sqrt{u}}\,du = dx$. Thu
$2\sqrt{u} = x + c$ and $2\sqrt{y - 2x + 3} = x + c$.

28. Let $u = y - x + 5$ so that $du/dx = dy/dx - 1$. Then $\dfrac{du}{dx} + 1 = 1 + e^u$ or $e^{-u}du = dx$. Thu
$-e^{-u} = x + c$ and $-e^{y-x+5} = x + c$.

29. Let $u = x + y$ so that $du/dx = 1 + dy/dx$. Then $\dfrac{du}{dx} - 1 = \cos u$ and $\dfrac{1}{1+\cos u}\,du = dx$. Now

$$\frac{1}{1+\cos u} = \frac{1-\cos u}{1-\cos^2 u} = \frac{1-\cos u}{\sin^2 u} = \csc^2 u - \csc u \cot u$$

so we have $\int(\csc^2 u - \csc u \cot u)du = \int dx$ and $-\cot u + \csc u = x + c$. Thus $-\cot(x+y) + \csc(x+y) =$
$x + c$. Setting $x = 0$ and $y = \pi/4$ we obtain $c = \sqrt{2} - 1$. The solution is

$$\csc(x + y) - \cot(x + y) = x + \sqrt{2} - 1.$$

30. Let $u = 3x + 2y$ so that $du/dx = 3 + 2\,dy/dx$. Then $\dfrac{du}{dx} = 3 + \dfrac{2u}{u+2} = \dfrac{5u+6}{u+2}$ and $\dfrac{u+2}{5u+6}\,du = d$
Now by long division

$$\frac{u+2}{5u+6} = \frac{1}{5} + \frac{4}{25u+30}$$

77/80

Department of Mathematics and Statistics
American University of Sharjah
Final Exam – Fall 2019
MTH 205-Differential Equations

Date: Sunday, December 15, 2019 **Time: 2pm to 4pm**

Student Name	**Student ID Number**
Aya Tarek	78806

Instructor Name	**Class Time**
Ayman Badawi	M, W : 11-12:15

1. ***Do not open*** *this exam until you are told to begin.*
2. ***No questions are allowed during the examination.***
3. *This exam has 8 pages + this cover exam page + Laplace Formula Sheet.*
4. *Do not separate the pages of the exam.*
5. *Scientific calculators are allowed.*
6. ***Turn off all cell phones and remove all headphones.***
7. *Take off your cap.*
8. ***No communication of any kind is allowed during the examination***
9. ***If you are found wearing a smart watch or holding a mobile phone at any point during the exam then it will be considered an academic violation and will be reported to the dean's office.***

Student signature: ______________________

Final Exam , MTH 205 , Fall 2019

Ayman Badawi

QUESTION 1. (i) **(3 points)** Find the values of the constants $a, \overset{k}{b}, c$ which makes the differential equation $(12x^2y - aye^{cx})dx + (kx^3 - e^{3x})dy$ exact (**DO NOT SOLVE IT**)

$F_{xy} = F_{yx}$

$F_{xy} = 12x^2 - ae^{cx}$

$F_{yx} = 3kx^2 - 3e^{3x}$

$12x^2 - ae^{cx} = 3kx^2 - 3e^{3x}$

$12 = 3k \qquad \cancel{-}ae^{cx} = \cancel{-}3e^{3x}$

$\boxed{k = 4} \quad \boxed{a = 3}$

$\boxed{c = 3}$

✓

(ii) **(6 points)** Stare really good at the following diff. equation $\frac{dy}{dx} = \frac{y^3}{x^3 - xy^2}$, change it to Bernoulli and solve it.

$\frac{dx}{dy} = \frac{x^3 - xy^2}{y^3}$

$x' = \frac{1}{y^3}x^3 - \frac{1}{y}x$

$x' + \frac{1}{y}x = \frac{1}{y^3}x^3$

$v = x^{\cancel{1}-3} = x^{-2}$

$v' + (-2)x\frac{1}{y}v = (-2)\frac{1}{y^3}$

$v' - \frac{2}{y}v = -\frac{2}{y^3}$ ✓

$I = e^{\int -\frac{2}{y}dy} = e^{-2\ln y}$

$= \frac{1}{y^2}$

$v = \dfrac{\int \frac{1}{y^2} \times \frac{-2}{y^3}\,dy}{\frac{1}{y^2}}$

$v = \dfrac{\int \frac{-2}{y^5}\,dy}{\frac{1}{y^2}}$

$v = \dfrac{\frac{1}{2}y^{-4} + c}{\frac{1}{y^2}}$

$v = \frac{1}{2}y^{-2} + y^2c$

$x = \left(\frac{1}{2}y^{-2} + y^2c\right)^{-\frac{1}{2}}$

✓

QUESTION 2. (8 points) Use Laplace to solve the differential equation :

$$y'(t) = e^{3t} + \int_0^t 4y(u)\,du,\ y(0) = 0$$

$\int 4y(u)\,du$

$4 * y(t)$

$y'(t) = e^{3t} + 4 * y(t)$

$\mathcal{L}(y'(t)) = \mathcal{L}(e^{3t}) + \mathcal{L}(4 * y(t))$

$sY(s) - y(0) = \frac{1}{s-3} + \frac{4Y(s)}{s}$

$sY(s) - \frac{4}{s}Y(s) = \frac{1}{s-3}$

$Y(s)\left[s - \frac{4}{s}\right] = \frac{1}{s-3}$

$Y(s) = \frac{1}{(s-3)} \times \frac{s}{(s^2-4)}$

$Y(s) = \frac{s}{(s-3)(s-2)(s+2)}$

$\frac{s}{(s-3)(s-2)(s+2)} = \frac{A}{s-3} + \frac{B}{s-2} + \frac{C}{s+2}$

$s = 3$ $s = 2$ $s = -2$

$A = \frac{3}{5}$ $B = \frac{-1}{2}$ $C = -\frac{1}{10}$

$y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left\{\frac{3/5}{s-3} - \frac{1/2}{s-2} - \frac{1/10}{s+2}\right\}$

$y(t) = \frac{3}{5}e^{3t} - \frac{1}{2}e^{2t} - \frac{1}{10}e^{-2t}$

QUESTION 3. (10 points) Imagine a company is making fake-sweet-drink(only water and sugar). The Tank has a capacity of 700 Liters. Initially, it contains 250 Liters of brine (water and sugar) that contains 25 kg of sugar (i.e., assume $A(0) = 25$). A solution containing ~~4kg/L~~ 4L/min of sugar is pumped into the tank and solution is pumped out at 3~~kg/L~~ L/min.

(i) Find $A(t)$, the amount of sugar in the tank at time t.

$$\frac{dA}{dt} = In - out$$

$$A' = (4)(4) - C(t)(3)$$

$$A(t) = \frac{4(250+t)^4 - 1.52\times10^{10}}{(250+t)^3}$$

$$C(t) = \frac{A}{250+(4-3)t}$$

$$A' = 16 - \frac{3A}{250+t} \checkmark$$

$$A' + \frac{3}{250+t}A = 16$$

$$I = e^{\int\frac{3}{250+t}dt}$$

$$I = e^{+3\ln|250+t|}$$

$$I = (250+t)^3 \checkmark$$

$$A = \frac{\int(250+t)^3\times16}{(250+t)^3}$$

$$A = \frac{4(250+t)^4 + C}{(250+t)^3} \checkmark$$

~~9~~0/10

$$A(0) = \frac{4(250)^4 + C}{(250)^3} = 25$$

$$C = -1.52\times10^{10} \checkmark$$

(ii) Find the amount of sugar in the tank after 10 min. ✓

~~A(10)~~) =

$$A(10) = \frac{4(250+10)^4 - 1.52\times10^{10}}{(250)^3} = 195\ kg \checkmark$$

$(250+10)^3$

(iii) When an overflow will occur?

$$250+(4-3)t = 700$$

$t = 450$ mins ✓

QUESTION 4. (4 points) Consider the diff. equation $y' - 2xy = 0$, $y(0) = 1$. Now use power series to solve it (as explained in class), i.e., do the following:

(i) Find the recurrence formula. Calculate the coefficients of the first 5 terms (i.e., a_0, a_1, a_2, a_3, a_4)

$y = \sum_{n=0} a_n t^n \quad t = x$

$y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 \ldots a_{n-1} t^{n-1} + a_n t^n + a_n t^{n+1}$

$y' = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 \ldots n a_n t^{n-1} + (n+1) a_{n+1} t^n$

$(a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 \ldots n a_n t^{n-1} + (n+1) a_{n+1} t^{n+1}) - 2t[a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 \ldots a_n t^n + a_{n+1} t^n$

$a_1 + (2a_2 - 2a_0)t + (3a_3 - 2a_1)t^2 + (4a_4 - 2a_2)t^3 \ldots (a_{n+1}(n+1) - 2a_{n-1})t^n = 0$

$a_1 = 0 \quad a_0 = 1 \quad 2a_2 - 2a_0 = 0 \quad a_2 =$

$a_2 = 1$

$n = 3 \to a_4 = \frac{2a_2}{4} = \frac{2 \times 1}{4} = \frac{1}{2}$

$(n+1)(a_{n+1}) - 2a_{n-1} = 0$

$a_{n+1} = \frac{2a_{n-1}}{n+1} \quad n \geq 1$

$\to y = 1 + t^2 + \frac{1}{2} t^4 \ldots$

$n = 1 \quad a_2 = \frac{2a_0}{2} = \frac{2}{2} = 1$

$n = 2 \to a_3 = \frac{2a_1}{3} = \frac{0}{3} = 0$

$(a_0 = 1, \ a_1 = 0, \ a_2 = 1, \ a_3 = 0, \ a_4 = \frac{1}{2})$.

(ii) The power series in (1) converges to a well-known function, what is this function?(i.e., solve the diff. equation without using power series)

$I = e^{\int -2x} = e^{-x^2}$

$y = \frac{\int e^{-x^2} \times 0 \, du}{e^{-x^2}} = \frac{0 + c}{e^{-x^2}}$

$y = c e^{x^2}$

$y(0) = 1 \longrightarrow c = 1$

$y = e^{x^2}$

QUESTION 5. (7 points) Imagine that a 10-kg mass is attached to a spring, stretching it 0.7 m from its natural length. The mass is started in motion from the equilibrium position (i.e., $M(0) = 0$, note $M(t)$ is the motion of the spring, where small m is the mass) with an initial velocity of 1 m/sec in the upward direction (i.e., $M'(0) = -1$). Find the motion, $M(t)$, if the force due to air resistance is -90N.($g(gravity) = 9.8m/sec^2$)

mass = 10 kg

$L = 0.7$

$F = 10 \times 9.8 = 98 \text{ N}$

$k = \frac{F}{L} = \frac{98}{0.7} = 140$

$F_{air} = -90$

-90

opposite direction

$M'' + \frac{90}{10} M' + \frac{140}{10} M = 0$

$M'' + 9M' + 14M = 0$

$M = e^{mt}$

$m^2 + 9m + 14 = 0$

$m = -7 \quad m = -2$

$M = C_1 e^{-7t} + C_2 e^{-2t}$

$M(0) = C_1 + C_2 = 0$

$C_1 = -C_2$

$M' = -7C_1 e^{-7t} - 2C_2 e^{-2t}$

$M'(0) = -7C_1 - 2C_2 = -1$

$7(-C_2) + 2C_2 = -1$

$C_2 = \frac{1}{5}$

$C_1 = -\frac{1}{5}$

$\to M = -\frac{1}{5} e^{7t} + \frac{1}{5} e^{2t}$

QUESTION 6. (10 points) Use Laplace and solve the following system of Linear Diff. Equations:

$$x'(t) - y(t) = 0, x(0) = 2$$

$$y'(t) - x(t) = -t, y(0) = 1$$

$sX(s) - x(0) - Y(s) = 0$

$sX(s) - Y(s) = 2 \cdots$ ①

$sY(s) - y(0) - X(s) = -\frac{1}{s^2}$

$-X(s) + sY(s) = -\frac{1}{s^2} + 1 \to \frac{s^2-1}{s^2}$ --- ②

$$X(s) = \frac{\begin{vmatrix} 2 & -1 \\ \frac{s^2-1}{s^2} & s \end{vmatrix}}{\begin{vmatrix} s & -1 \\ -1 & s \end{vmatrix}} = \frac{2s + \frac{s^2-1}{s^2}}{s^2-1}$$

$$X(s) = \frac{2s^3+s^2-1}{s^2(s^2-1)} = \frac{2s^{\cancel{3}}}{\cancel{s^2}(s^2-1)} + \frac{\cancel{s^2-1}}{s^2\cancel{(s^2-1)}}$$

$$X(s) = \frac{2s}{s^2-1} + \frac{1}{s^2}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{2s}{s^2-1} + \frac{1}{s^2}\right\}$$

$$x(t) = 2\cosh(t) + t$$ ✓

$$Y(s) = \frac{\begin{vmatrix} s & 2 \\ -1 & \frac{s^2-1}{s^2} \end{vmatrix}}{\begin{vmatrix} s & -1 \\ -1 & s \end{vmatrix}} = \frac{\frac{s(s^2-1)}{s^2} + 2}{s^2-1} = \frac{s(s^2-1)+2s^2}{s^2(s^2-1)}$$

$$Y(s) = \frac{\cancel{s}\cancel{(s^2-1)}}{s^2\cancel{(s^2-1)}} + \frac{2\cancel{s^2}}{\cancel{s^2}(s^2-1)}$$

$$Y(s) = \frac{1}{s} + \frac{2}{s^2-1}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{2}{s^2-1}\right\} \Rightarrow y(t) = 1 + 2\sinh(t)$$ ✓

QUESTION 7. (6 points)

(i) $\ell\{\int_0^t e^{(t-u)} cos(t-u) sin(u)\, du\}$

$e^t Cos(t) * Sin(t)$

$\mathcal{L}\{e^t Cos(t) * Sin(t)\}$

$= \frac{S-1}{(S-1)^2+1} \cdot \frac{1}{S^2+1}$ ✓

✓

(ii) Find $\ell^{-1}\{\frac{s(e^{-2s})}{(s+1)^2+4}\}$

$u(t-2)\mathcal{L}^{-1}\left\{\frac{S}{(S+1)^2+4}\right\}$

$\mathcal{L}^{-1}\left\{\frac{S+1-1}{(S+1)^2+4}\right\} = \frac{S+1}{(S+1)^2+4} - \frac{1}{(S+1)^2+4}$

$= e^{-t} Cos(2t) - \frac{1}{2} e^{-t} Sin(2t)$

$u(t-2)\left[e^{-(t-2)}\left(Cos(2t-4) - \frac{1}{2} Sin(2t-4)\right)\right]$

✓

QUESTION 8. (6 points) Solve for $y(t)$: $(cos(t) - t)y'' + (1 + sin(t))y' = 0$

$V = y'$

$V' = y''$

$(Cos(t) - t)V' + (1 + Sin(t))V = 0$

$V' + \frac{(1 + Sin(t))}{(Cos(t) - t)} V = 0$

$I = e^{\int \frac{1 + Sin(t)}{Cos(t) - t}}$

$u = -(Cos(t) - t)$

$du = + Sin(t) + 1 dt$

$I = e^{\int \frac{1}{-u} du}$

$I = e^{-\ln|u|} = \frac{1}{-Cos(t) + t}$

✓

$$V = \frac{\int \frac{1}{t - Cos(t)} \times 0 \, dt}{\frac{1}{t - Cos(t)}}$$

$$V = \frac{0 + C}{\cancel{\frac{1}{t - Cos t}}} \rightarrow C[t - Cos(t)]$$

$y = \int Ct - C Cos(t)\, dt$

$y = \frac{1}{2} Ct^2 - C Sin t + C_1$

$y = Ct^2 - C Sin t + C_1$

QUESTION 9. (10 points)

(i) Solve for $y(t)$, $t^2y'' - 2ty' + 2y = 0$

$y = t^m$

$y' = m\,t^{m-1}$

$y'' = (m^2 - m)\,t^{m-2}$

$t^m(m^2 - m - 2m + 2) = 0$

$m^2 - 3m + 2 = 0$

$m = 2$ or $m = 1$

$y = C_1 t^2 + C_2 t$

(ii) Use (1) and solve for $y(t)$: $t^2y'' - 2ty' + 2y = 2t^3e^t$

$y = y_h + y_p$

$y_h = C_1 \underbrace{t^2}_{y_1} + C_2 \underbrace{t}_{y_2}$

$y_p = v_1 y_1 + v_2 y_2$

$v_1' y_1 + v_2' y_2 = 0$

$v_1' y_1' + v_2' y_2' = \dfrac{2t^3e^t}{t^2}$

$v_1' t^2 + v_2' t = 0$

$v_1' 2t + v_2' = 2te^t$

$w = \begin{vmatrix} t^2 & t \\ 2t & 1 \end{vmatrix} = t^2 - 2t^2$
$= -t^2$

$v_1' = \dfrac{\begin{vmatrix} 0 & t \\ 2te^t & 1 \end{vmatrix}}{-t^2} = \dfrac{-2\cancel{t^2}e^t}{-\cancel{t^2}}$

$v_1' = 2e^t$

$v_1 \int 2e^t dt = 2e^t$

$v_2' = \dfrac{\begin{vmatrix} t^2 & 0 \\ 2t & 2te^t \end{vmatrix}}{-t^2} = \dfrac{2t^{3\,1}e^t}{-\cancel{t^2}}$

$v_2' = -2te^t$

$v_2 \int -2te^t dt$

$\begin{array}{c|c} ( )' & \int \\ \hline -2t & e^t \\ -2 & e^t \\ 0 & e^t \end{array} \quad = -2te^t + 2e^t$

$y_p = (2e^t)(t^2) + (-2te^t + 2e^t)(t)$

$y_p = \cancel{2t^2e^t} - \cancel{2t^2e^t} + 2te^t$

$y_p = 2te^t$

$\Rightarrow y = C_1t^2 + C_2t + 2te^t$.

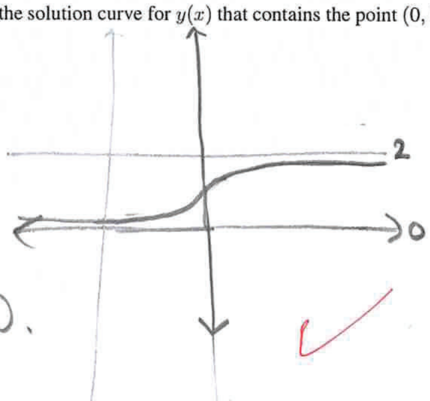
SHORT ANSWERS, JUST STARE well and Think

QUESTION 10. (i) **(3 points)** Draw the solution curve for $y(x)$ that contains the point $(0, 1.5)$ and find $lim_{x\to\infty} y(x)$, where $y' = -y^2 + 2y$.

$-y^2 + 2y = 0$

$y = 2$ or $y = 0$

$(0, 1.5)$ lies in $(0, 2)$.



as x goes ∞ → $y(x)$ will approach 2

$\lim_{x\to\infty} = 2$.

✓

(ii) **(3 points)** Given y_1 and y_2 are two distinct solutions for the diff. equation $e^{x^2} y'' + cos(x)y = \frac{ln(x)}{1+x^3}$. Then one can quickly form a third solution $y_3 = \pi^2 y_1 + a y_2$ and a forth solution $y_4 = b y_1 + (e^2 + 1) y_2$. Find the values of the constants a, b.

$e^{x^2} y'' + Cos(x) y = \frac{ln(x)}{1+x^3}$

$b = \pi^2$

$a = (e^2+1)$

0/3

(iii) **(4 points)** Solve the diff. equation $\frac{dy}{dx} = (\sqrt{y} + y)e^x(x^2 + 2x)$

$\int \frac{dy}{\sqrt{y} + y} = \int e^x(x^2 + 2x)\, dx$

$\int \frac{1}{(\sqrt{y})(1+\sqrt{y})} dy = x^2 e^x + C$

$u = 1 + \sqrt{y}$

$du = \frac{1}{2\sqrt{y}} dy$

$2\int \frac{1}{u} du$

$2 \ln|1 + \sqrt{y}|$

$2 \ln|1 + \sqrt{y}| = x^2 e^x + C$

✓

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Name Abdull Kayyani ID b00059758

Exam II, MTH 205, Fall 2015

Ayman Badawi

Excellent 50/50

QUESTION 1. (i) Find the general solution to the Diff. Equation $\underbrace{(2x+1)y' - y = y^3(2x+1)e^{(-2x^2-2x+7)}}_{\text{bernouli}}$

$$\Rightarrow \underbrace{y' - \frac{1}{(2x+1)}y = e^{-2x^2-2x+7}y^3}_{n=3}$$

$$w = y^{1-n} = y^{-2}$$

$$\Rightarrow w' - \frac{(1-3)}{(2x+1)}w = (1-3)e^{-2x^2-2x+7}$$

$$\Rightarrow w' + \underbrace{\frac{2}{2x+1}}_{Q(x)}w = \underbrace{-2e^{-2x^2-2x+7}}_{K(x)}$$

$$\Rightarrow w = \frac{\int K(x)e^{\int Q(x)dx}}{e^{\int Q(x)dx}} = \frac{\int -2e^{-2x^2-2x+7}e^{\int\frac{2}{2x+1}}}{e^{\int\frac{2}{2x+1}dx}}$$

$$= \frac{\int -2(2x+1)e^{-2x^2-2x+7}dx}{2x+1}$$

let $u = -2x^2-2x+7$

$du = (-4x-2$

$= -2(2x+1)$

$$= \frac{e^{-2x^2-2x+7}+C}{2x+1}$$

$$\Rightarrow w = y^{-2} = \frac{e^{-2x^2-2x+7}+C}{2x+1}$$

10/10

$$\Rightarrow \boxed{y = \sqrt{\left(\frac{2x+1}{e^{-2x^2-2x+7}+C}\right)}}$$

(ii) Find the general solution to the Diff. Equation $\underbrace{x^2y^{(2)} + xy' + y}_{\text{Cauchy}} = ln(x)$ $\rightarrow$ $y'' + \frac{y'}{x} + \frac{y}{x^2} = \frac{\ln x}{x^2}$

$\Rightarrow$ for $\underline{\underline{y_h}}$ $\Rightarrow$ let $y = x^n$, $y' = nx^{n-1}$, $y'' = n(n-1)x^{n-2}$

$\Rightarrow$ so $[n(n-1) + n + 1]x^n = 0$

$\Rightarrow$ so $n(n-1) + n + 1 = 0$

$\Rightarrow n^2 - n + n + 1 = 0$

$\Rightarrow n^2 + 1 = 0$

$\Rightarrow n = \pm i$

$\Rightarrow$ so $y_h = c_1\cos(\ln x) + c_2\sin(\ln x)$

$-u\cos u + \sin u$

Flip page! $\longrightarrow$

$\Rightarrow$ for y_p $\Rightarrow$ let $y_1 = \cos(\ln x)$, $y_2 = \sin(\ln x)$, $k(x) = \frac{\ln x}{x^2}$

$$\Rightarrow W(y_1, y_2) = \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ \frac{-\sin(\ln x)}{x} & \frac{\cos(\ln x)}{x} \end{vmatrix} = \frac{1}{x}$$

$$\Rightarrow y_p = y_2\int\frac{y_1 k(x)}{W(y_1, y_2)}dx - y_1\int\frac{y_2 k(x)}{W(y_1,y_2)}dx$$

$$= \sin(\ln x)\cdot\int\frac{\cos(\ln x)\ln x/\cancel{x^2}}{1/x}dx - \cos(\ln x)\int\frac{\sin(\ln x)\ln x/\cancel{x^2}}{1/x}dx$$

$$= \sin(\ln x)\cdot\int\frac{\ln x\cos(\ln x)}{x}dx - \cos(\ln x)\int\frac{\ln x\sin(\ln x)}{x}dx$$

Flip page $\downarrow$

$$= \sin(\ln x)\cdot\left(\ln(x)\sin(\ln x) + \cos(\ln x)\right) - \cos(\ln x)\cdot\left(-\ln(x)\cos(\ln x) + \sin(\ln x)\right)$$

$$= \ln(x)\left(\sin^2(\ln x) + \cos^2(\ln x)\right) + \cancel{\cos(\ln x)\sin(\ln x)} - \cancel{\cos(\ln x)\sin(\ln x)}$$

$$= \ln(x)$$

$$\Rightarrow \quad y_p = \ln x$$

$$\Rightarrow \quad y = y_h + y_p = \boxed{\ln x + c_1 \cos(\ln x) + c_2 \sin(\ln x)}$$

$$\frac{10}{10}$$

(iii) Find the solution to the Diff. equation $y' - \frac{1}{x}y = (1 + x\ln(x))e^x$, $y(1) = 4$

$$\Rightarrow \quad y' \underbrace{- \frac{1}{x}}_{G(x)} y = \underbrace{(1 + x\ln x)e^x}_{K(x)}$$

$$\Rightarrow \quad y = \frac{\int K(x)\, e^{\int G(x)dx}}{e^{\int G(x)\,dx}} = \frac{\int (1+x\ln x)e^x \cdot e^{-\int \frac{1}{x}dx}\,dx}{e^{-\int \frac{1}{x}dx}}$$

$$= \frac{\int \frac{(1+x\ln x)e^x}{x}\,dx}{1/x}$$

$$\Rightarrow \int (f(x) + f'(x))e^x dx$$
$$= f(x)e^x + c$$

$$= \frac{\int \left(\frac{1}{x} + \ln x\right)e^x dx}{1/x}$$

$$= \frac{(\ln x)e^x + c}{1/x}$$

$$y = (x\ln x)e^x + cx$$

$$\Rightarrow \quad 4 = y(1) = (1\cdot \overset{0}{\cancel{\ln 1}})e^1 + c$$

$$\Rightarrow \quad \underline{c = 4}$$

$$\Rightarrow \quad \boxed{y = (x\ln x)e^x + 4x}$$

10/10

$$y_p = y_2 \int \frac{y_1 k(x)}{w(y_1, y_2)}\,dx - y_1 \int \frac{y_2 k(x)}{w(y_1, y_2)}\,dx$$

$$= e^x \cdot \int \frac{(x+1)\cdot\cancel{(xe^{-x})}}{\cancel{(xe^{-x})}}\,dx - (x+1) \int \frac{e^x \cdot \cancel{(xe^{-x})}}{\cancel{(xe^{-x})}}\,dx$$

$$= e^x \cdot \int (x+1)\,dx - (x+1) \int e^x\,dx$$

$$= e^x \cdot \left(\frac{x^2}{2} + x\right) - (x+1)e^x = e^x \cdot \left(\frac{x^2}{2} \text{[illegible]} 1\right)$$

$$\Rightarrow y = y_h + y_p = \boxed{c_1(x+1) + c_2 e^x + e^x\left(\frac{x^2}{2} - 1\right)}$$

~0/0

[illegible]

(iv) Find the general solution to the Diff. Equation $xy^{(2)} - (x+1)y' + y = x^2e^x$, given $y = -e^x$ is a solution to the homogeneous part.

[redacted]

$$\Rightarrow \quad y'' - \underbrace{(1+\tfrac{1}{x})}_{Q(x)} y' + \frac{y}{x} = xe^x$$

$\Rightarrow$ for y_h, let $y_1 = -e^x$

$$\Rightarrow \text{ so } \quad y_2 = y_1 \int \frac{e^{-\int Q(x)\,dx}}{y_1^2}\,dx$$

$$= -e^x \cdot \int \frac{e^{\int (1+\frac{1}{x})dx}}{e^{2x}}\,dx$$

$$= -e^x \cdot \int \frac{e^{x+\ln x}}{e^{2x}}\,dx$$

$$= e^x \cdot \int \frac{xe^x}{e^{2x}}\,dx$$

$$= -e^x \cdot \int xe^{-x}\,dx$$

$$= -e^x \cdot \left(-(x+1)e^{-x}\right) = (x+1)$$

$$\Rightarrow \text{ so } \quad y_h = c_1(x+1) + c_2 e^x$$

$\Rightarrow$ for y_p, let $y_1 = (x+1)$, $y_2 = e^x$, $k(x) = xe^x$

$$\Rightarrow w(y_1, y_2) = \begin{vmatrix} x+1 & e^x \\ 1 & e^x \end{vmatrix} = xe^x + e^x - e^x = xe^x$$

flip page!
↓

(v) A 39.2 [illegible] is attached to a spring having a spring constant 4N/m. At t = 0, the object is released from a point 1.5 meter below the equilibrium position with an upward velocity 1m/s and with constant external force F(t) = 14.

a) Find the equation of the motion, $x(t)$.

$$\Rightarrow \quad x'' + \frac{a}{m}x' + \frac{k}{m}x = \frac{F(t)}{m}$$

$\Rightarrow m = \frac{39.2}{9.8} = 4kg$

$\Rightarrow a = 0$

$\Rightarrow k = 4$

$\Rightarrow x(0) = 1.5m$

$\Rightarrow x'(0) = -1m/s$

$$\Rightarrow \quad x'' + x = \frac{14}{4}$$

$$\Rightarrow \quad x'' + x = \frac{7}{2}$$

$\Rightarrow$ for x_h, let $x = e^{mt}$ so $m^2+1=0$, $m = \pm i$

$\Rightarrow \quad x_h = c_1 \cos t + c_2 \sin t$

$\Rightarrow$ for x_p, let $x = A$ so $A = \frac{7}{2} = x_p$

[redacted]

$\Rightarrow x(t) = \frac{7}{2} + c_1 \cos t + c_2 \sin t$

$\Rightarrow 1.5 = x(0) = 3.5 + c_1$

$\Rightarrow c_1 = -2$

65

$\Rightarrow x'(t) = 2\sin t + c_2 \cos t$

y time t. Justify my claim or prove me wrong.

$\Rightarrow -1 = x'(0) = c_2$

$\Rightarrow c_2 = -1$

$$\Rightarrow \boxed{x(t) = 3.5 - (2\cos t + \sin t)}$$

onstant external force be ~~so that the~~ object will

jah, P.O. Box 26666, Sharjah, United Arab Emirates.

p page

4 Worked out Solutions for all Assessment Tools

4.1 Solution for Quiz I

Name Farah Ossama, ID 82666

15/15

Quiz One, MTH 205 , Fall 2020

Ayman Badawi

QUESTION 1. (i) $\ell^{-1}\{\frac{3}{2s+5}\}$ $= \frac{3}{2}\ell^{-1}\left\{\frac{1}{s+5/2}\right\}$

$= \frac{3}{2}e^{-5/2t} = \frac{3}{2}e^{-\frac{5t}{2}}$ → power

3/3

(ii) $\ell^{-1}\{\frac{3}{s^2+4}+\frac{7}{s^9}\}$ $= \ell^{-1}\left\{\frac{3}{s^2+4}\right\} + \ell^{-1}\left\{\frac{7}{s^9}\right\}$

$\rightarrow \ell^{-1}\left\{\frac{3}{s^2+4}\right\} = \ell^{-1}\left\{\frac{3}{s^2+2^2}\right\} = \frac{3}{2}\sin(2t)$

$\rightarrow \ell^{-1}\left\{\frac{7}{s^9}\right\} = n=8 = \frac{1}{5760}\ell^{-1}\left\{\frac{8!}{s^9}\right\} = \frac{t^8}{5760}$

$= \frac{3}{2}\sin(2t) + \frac{t^8}{5760}$

3/3

(iii) $\ell\{(t+2)^2\}$

$\ell\{(t+2)(t+2)\} = \ell\{t^2+4t+4\} = \frac{2!}{s^3} + \frac{4}{s^2} + \frac{4}{s} = \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}$

3/3

QUESTION 2. find $y(t)$, where $y^{(2)} - 5y' + 6y = 1, y(0) = y'(0) = 0$

$\ell\{y^{(2)}\} - 5\ell\{y'\} + 6\ell\{y\} = \ell\{1\}$

$\rightarrow s^2Y(s) - s(0) - 0 - 5sY(s) - 0 + 6Y(s) = \frac{1}{s}$

$Y(s)[s^2 - 5s + 6] = \frac{1}{s}$

$Y(s) = \frac{1}{s(s^2-5s+6)} = \frac{1}{s(s-3)(s-2)} = \frac{a}{s} + \frac{b}{s-3} + \frac{c}{s-2}$

$=0 \quad =3 \quad =2$

3/3

$a = \frac{1}{6} \quad b = \frac{1}{3} \quad c = -\frac{1}{2}$

$y(t) = \frac{1}{6}\ell^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{3}\ell^{-1}\left\{\frac{\cancel{5}1}{s-3}\right\} - \frac{1}{2}\ell^{-1}\left\{\frac{1}{s-2}\right\}$

$\boxed{y(t) = \frac{1}{6} + \frac{1}{3}e^{3t} - \frac{1}{2}e^{2t}}$

3/3

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4.2 Solution for Quiz II

15/15

DIFFERENTIAL EQUATIONS QUIZ 2 ROHAN MITRA 85023

Q1) i) $L^{-1}\left\{ e^{-4s} \cdot \frac{1}{s^2+3^2} \right\}$ ~~$= U_4 \frac{1}{3} \sin(3(t+4))$~~ $= \frac{1}{3} U_4(t) \sin(3(t-4))$ ✓ 4/4

~~$= U_4 \cdot \frac{1}{3} \cdot \sin(3(t-4))$~~

ii) $L^{-1}\left\{ \frac{1}{(s-3)^2+2^2} + 6e^{-3s} \frac{1}{s^4} \right\} = \frac{1}{2} e^{3t} \sin(2t) + \cancel{\frac{1}{4}}$ ~~$U_3 (t-3)^3$~~

$+ U_3(t) \cdot (t-3)^3$ ✓ 4/4

iii) $L\left\{ \underbrace{U_5(t) e^{t-5} \cosh(t-5)}_{f(t)} \right\} = e^{-5s} \cdot L\{ e^{t+5-5} \cosh(t+5-5) \} = e^{-5s} \cdot L\{ e^t \cosh(t) \}$

$= e^{-5s} \cdot \frac{(s-1)}{(s-1)^2-1}$ ✓ 4/4

Q2) $y' - 2y = U_3 e^{t-3}$, $y(0)=0$

$L\{y'\} - 2L\{y\} = L\{U_3 e^{t-3}\}$.

$sY(s) - 2Y(s) - 0 = e^{-3s} \cdot L\{e^t\} = e^{-3s} \cdot \frac{1}{s-1}$

$Y(s)[s-2] = \frac{e^{-3s}}{s-1}$

$Y(s) = \frac{e^{-3s}}{(s-1)(s-2)} = e^{-3s}\left(\frac{1}{(s-1)(s-2)} \right)$

$Y(s) = e^{-3s}\left(\frac{1}{s-2} - \frac{1}{s-1} \right)$

3/3

Partial Fractions:

$\frac{1}{(s-1)(s-2)} \equiv \frac{A}{s-1} + \frac{B}{s-2}$ ✓

$1 = A(s-2) + B(s-1)$

$s=1 \rightarrow \quad 1 = -A \qquad A=-1$

$s=2 \rightarrow \quad 1 = B \qquad B=1$

$\therefore \frac{1}{(s-1)(s-2)} = \frac{1}{s-2} - \frac{1}{s-1}$ ✓

$L^{-1}\{Y(s)\} = L^{-1}\left\{ e^{-3s}\left(\frac{1}{s-2} - \frac{1}{s-1} \right) \right\}$.

$y(t) = L^{-1}\left\{ e^{-3s} \cdot \frac{1}{s-2} \right\} - L^{-1}\left\{ e^{-3s} \cdot \frac{1}{s-1} \right\}$. ✓

$y(t) = U_3(t) \cdot e^{2(t-3)} - U_3(t) e^{t-3}$ ✓ 4/4

4.3 Solution for Quiz III (3 different versions)

15/15

DIFFERENTIAL EQUATIONS QUIZ 3 ROHAN MITRA 85023

Q1) $f(t) = \begin{cases} 1 & \text{if } 0 \leq t < 3 \\ 0 & \text{if } 3 \leq t < \infty \end{cases}$

i) $f(t) = 1[\underbrace{U_0(t)}_{=1} - U_3(t)] + \underbrace{0 \cdot [U_3(t) - U_\infty(t)]}_{=0.}$

$f(t) = 1 - U_3(t)$

✓ 2/2

ii) $y' - 4y = f(t)$ ~~Definition of f(t) implies f(0) = 1~~. $y(0) = 0$

$\mathcal{L}\{y'\} - 4\mathcal{L}\{y\} = \cancel{\mathcal{L}\{U_3(t)\}}\ \mathcal{L}\{1 - U_3(t)\}$

$sY(s) - \underbrace{y(0)}_{=0} - 4Y(s) = \frac{1}{s} - \frac{e^{-3s}}{s}$

$Y(s)[s-4] = \frac{1}{s} - \frac{e^{-3s}}{s}$

2/2

Partial Fractions:

$\frac{1}{s(s-4)} \equiv \frac{A}{s} + \frac{B}{s-4}$

$1 = A(s-4) + Bs$

$s=0 \rightarrow 1 = -4A, A = -1/4$

$s=4 \rightarrow 1 = 4B \quad B = 1/4.$

$\therefore \frac{1}{s(s-4)} = \frac{1}{4(s-4)} - \frac{1}{4s}.$ ✓

$Y(s) = \frac{1}{s(s-4)} - \frac{e^{-3s}}{s(s-4)}$

$Y(s) = \frac{1}{4(s-4)} - \frac{1}{4s} - e^{-3s}\left[\frac{1}{4(s-4)} - \frac{1}{4s}\right].$ ✓

~~$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{4(s-4)} - \frac{1}{4s}\right\} - U_3(t) \cdot \mathcal{L}^{-1}\left\{\frac{1}{4(s-4)} - \frac{1}{4s}\right\}\Big|_{s \to t}$~~

$y(t) = \frac{1}{4}e^{4t} - \frac{1}{4} - U_3(t) \cdot \left[\frac{1}{4}e^{4(t-3)} - \frac{1}{4}\right]$ // ✓ 4/4

Q2) $y'' - 6y' - 5y = 0$, $y(0) = 0$, $y'(0) = 2$.

$s^2Y(s) - \underbrace{sy(0)}_{=0} - \underbrace{y'(0)}_{=2} - 6sY(s) + \underbrace{6y(0)}_{=0} - 5Y(s) = 0$

$Y(s)[s^2 - 6s - 5] = 2.$

$Y(s) = \frac{2}{s^2 - 6s - 5} = \frac{2}{(s-3)^2 - 14} =$ [illegible]

2/2

$Y(s) = \frac{2}{\sqrt{14}} \cdot \frac{\sqrt{14}}{(s-3)^2 - 14}$

$y(t) = \frac{2}{\sqrt{14}} e^{3t} \cdot \sinh(\sqrt{14}t)$ // ✓ 5/5

15/15

① i) $f(t) = \left\{ \begin{matrix} 1 & \text{if} & 0 \le t < 3 \\ 0 & \text{if} & t \geq 3 \end{matrix} \right\}.$

$f(t) = 1\ (U_0(t) - U_3(t)) + 0\ (\overset{0}{\cancel{U_3}}(t))$

$f(t) = U_0(t) - U_3(t).$ ✓ 2/2

ii) $y' - 4y = f(t)$

$\mathcal{l}\ \{y' - 4y = 1U_0(t) - U_3(t)\}$

$sY(s) - 4Y(s) = \frac{e^{-0}}{s} - \frac{e^{-3s}}{s}$ ✓ 2/2

$Y(s)\,[s-4] = \frac{1-e^{-3s}}{s}$

$Y(s) = \frac{1-e^{-3s}}{s(s-4)} = \frac{a}{s} + \frac{b}{s-4}$

$A = -\frac{1}{4},\ B = \frac{1}{4} \rightarrow \frac{1}{s(s-4)} = \frac{-\frac{1}{4}}{s} + \frac{\frac{1}{4}}{s-4}.$

$y(t)\ \mathcal{l}^{-1}\left\{ \frac{-\frac{1}{4}}{s} + \frac{\frac{1}{4}}{s-4} \right\} - \mathcal{l}^{-1}\left\{ e^{-3s} \ast \left(\frac{-1}{4s} + \frac{\frac{1}{4}}{s-4} \right) \right\}$

$y(t) = -\frac{1}{4} + \frac{1}{4}e^{4t} - \left(U_3(t)\left(-\frac{1}{4} + \frac{1}{4}\ e^{4(t-3)}\right)\right)$ 4/4

✓ ✓ ✓ ✓ ✓

② $y'' - 6y' - 5y = 0 \quad y(0) = 0\ y'(0) = 2$

$s^2Y(s) - 0 - 2 - 6sY(s) - 5Y(s) = 0$

$Y(s)\left[s^2 - 6s - 5\right] = 2.$ ✓ 2/2

$Y(s) = \frac{2}{\cancel{(s-3)^2+4}\ (s^2 - 6s - 5)} = \frac{2}{(s^2 - 6s + 9) - 9 - 5.}$

$= \frac{2}{(s-3)^2 - 14.}$ ✓

$y(t) = \mathcal{l}^{-1}\left\{ \frac{2}{(s-3)^2 - 14} \right\}$

$y(t) = \frac{2e^{3t}}{\sqrt{14}} \sinh(\sqrt{14}\,t)$

✓ 5/5

15/15

1) i) $f(t) = 1\left[u_0(t) - u_3(t)\right] + 0$

$= 1 - u_3(t)$ ✓ 2/2

ii) $y' - 4y = f(t)$

$y' - 4y = 1 - u_3(t)$

$L\{y' - 4y\} = L\{1 - u_3(t)\}$

$sY(s) - y(0) - 4Y(s) = \frac{1}{s} - \frac{e^{-3s}}{s}$ ✓ $y(0) = 0$

$Y(s)[s-4] = \frac{1}{s} - \frac{e^{-3s}}{s}$

$Y(s) = \frac{1}{s(s-4)} - \frac{e^{-3s}}{s(s-4)}$ ✓ 2/2

$\frac{1}{s(s-4)} = \frac{A}{s} + \frac{B}{s-4}$ $A = -\frac{1}{4}$ $B = \frac{1}{4}$ ✓

$= -\frac{1}{4}\frac{1}{s} + \frac{1}{4}\frac{1}{s-4}$

$y(t) = L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{1}{s(s-4)}\right\} - L^{-1}\left\{e^{-3s} \cdot \frac{1}{s(s-4)}\right\}$

- $L^{-1}\left\{\frac{1}{s(s-4)}\right\} = L^{-1}\left\{-\frac{1}{4}\frac{1}{s} + \frac{1}{4}\frac{1}{s-4}\right\} = -\frac{1}{4} + \frac{1}{4}e^{4t}$ ✓

- $L^{-1}\left\{e^{-3s} \cdot \frac{1}{s(s-4)}\right\} = f(t-3)u_3(t)$

$f(t) = -\frac{1}{4} + \frac{1}{4}e^{4t}$

$f(t-3) = -\frac{1}{4} + \frac{1}{4}e^{4(t-3)}$

$$L^{-1}\left\{\frac{1}{s(s-4)}\right\} \cancel{+} - L^{-1}\left\{e^{-3s}\cdot\frac{1}{s(s-4)}\right\}$$

$$= -\frac{1}{4} + \frac{1}{4}e^{4t} - \left[u_3(t)\left[-\frac{1}{4} + \frac{1}{4}e^{4(t-3)}\right]\right]$$

$$= -\frac{1}{4} + \frac{1}{4}e^{4t} + u_3(t)\left[\frac{1}{4} - \frac{1}{4}e^{4(t-3)}\right]$$

✓ ✓ ✓

4/4

2) $y'' - 6y' - 5y = 0 \qquad y(0) = 0 \;,\; y'(0) = 2$

$$s^2Y(s) - sy(0) - y'(0) - 6[sY(s) - y(0)] - 5Y(s) = 0$$

$$s^2Y(s) - 2 - 6sY(s) - 5Y(s) = 0$$

$$Y(s)\left[s^2 - 6s - 5\right] = 2$$

$$Y(s) = \frac{2}{s^2 - 6s - 5} = \frac{2}{(s-3)^2 - 14}$$ ✓ 2/2

$$y(t) = L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{2}{(s-3)^2 - 14}\right\}$$

$$= \frac{2}{\sqrt{14}} L^{-1}\left\{\frac{\sqrt{14}}{(s-3)^2 - 14}\right\}$$

$$= \frac{2}{\sqrt{14}} e^{3t} \sinh(\sqrt{14}\, t)$$ 5/5

✓

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Quiz Three

$\frac{15}{15}$

Q₁) let $f(t) = \begin{cases} 1 & \text{if } 0 \leq t < 3 \\ 0 & \text{if } t \geq 3 \end{cases}$

write $f(t)$ in terms of unit-step functions:

i) $U_0(t) - U_3(t) + \underset{0}{\cancel{0(U_3(t))}}$ ✓ 2/2

ii) $\mathcal{L} \times \{$Find $y(t)$, where $y' - 4y = f(t)\}$, $y(0)=0$

$$sY(s) - \cancel{y(0)} - 4Y(s) = \mathcal{L}\{U_0(t) - U_3(t)\}$$

$$sY(s) - 4Y(s) = \frac{e^0}{s} - \frac{e^{-3s}}{s}$$

$$Y(s) = \frac{1 - e^{-3s}}{s} \times \frac{1}{s-4}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1 - e^{-3s}}{s(s-4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s-4)} - \frac{e^{-3s}}{s(s-4)}\right\}$$ ✓ 2/2

$$\frac{1}{s(s-4)} = \frac{A}{s} + \frac{B}{s-4}$$

$A = -\frac{1}{4}$, $B = \frac{1}{4}$ $\rightarrow$ $\frac{1}{s(s-4)} = \frac{-\frac{1}{4}}{s} + \frac{\frac{1}{4}}{s-4}$. ✓

$$y(t) = \mathcal{L}^{-1}\left\{\frac{-\frac{1}{4}}{s} + \frac{\frac{1}{4}}{s-4}\right\} - \mathcal{L}^{-1}\left\{e^{-3s} \times \left(-\frac{1}{4s} + \frac{\frac{1}{4}}{s-4}\right)\right\}$$

$$y(t) = -\frac{1}{4} + \frac{1}{4}e^{4t} - \left(U_3(t)\left(-\frac{1}{4} + \frac{1}{4}e^{4(t-3)}\right)\right)$$

$$y(t) = -\frac{1}{4} + \frac{1}{4}e^{4t} - U_3(t)\left(-\frac{1}{4} + \frac{1}{4}e^{4(t-3)}\right)$$

✓ ✓ ✓ ✓ ✓

4/4

Find $y(t)$

$y^{(2)} - 6y' - 5y = 0 \quad , y(0) = 0 \ , \quad y'(0) = 2$

$s^2 Y(s) - s\overset{0}{\cancel{y(0)}} - y'(0) - 6[sY(s) - \overset{0}{\cancel{y(0)}}] - 5Y(s) = 0$ ✓

$s^2 Y(s) - 2 - 6sY(s) - 5Y(s) = 0$

$Y(s) = \dfrac{2}{(s^2 - 6s - 5)} \longrightarrow$ Complete the square ✓ 2/2

$Y(s) = \dfrac{2}{(s^2 - 6s + 9) - 5 - 9} = \dfrac{2}{(s-3)^2 - 14}$

$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\dfrac{2}{(s-3)^2 - 14}\right\}$

$y(t) = \dfrac{2e^{3t}}{\sqrt{14}} \sinh(\sqrt{14}\, t)$ ✓ 5/5

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Quiz- Three
MTH-205

15/15

Q.1 i) $f(t) = \begin{cases} 1, & 0 \le t < 3 \\ 0, & t \geqslant 3 \end{cases}$

$$f(t) = 1\{U_0(t) - U_3(t)\} + 0\{U_3(t) - U_\infty(t)\}$$

$$\underline{f(t) = \{U_0(t) - U_3(t)\}}$$ ✓ 2/2

ii) $y' - 4y = f(t)$, $y(0) = 0$

$$\mathcal{L}\{y'\} - 4\mathcal{L}\{y\} = \mathcal{L}\{U_0(t) - U_3(t)\}$$

$$[sY(s) - y(0)] - 4Y(s) = \frac{e^{-0s}}{s} - \frac{e^{-3s}}{s}$$

$$sY(s) - 4Y(s) = \frac{1}{s} - \frac{e^{-3s}}{s}$$ ✓

2/2

$$Y(s)[s-4] = \frac{1 - e^{-3s}}{s\cancel{(s+4)}}$$

$$Y(s) = \frac{1 - e^{-3s}}{s(s-4)}$$ ✓

$$\frac{1}{s(s-4)} = \frac{A}{s} + \frac{B}{s-4}.$$

$$\underline{A = -1/4, \ B = 1/4}$$ ✓

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s-4)}\right\} - \mathcal{L}^{-1}\left\{e^{-3s} \cdot \frac{1}{s(s-4)}\right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{-1}{4s} + \frac{1}{4(s-4)}\right\} - U_3(t)\left\{\frac{-1}{4} + \frac{e^{4(t-3)}}{4}\right\}$$

$$= \frac{-1}{4} + \frac{e^{4t}}{4} - \underline{U_3(t)}\left\{\frac{-1}{4} + \frac{e^{4(t-3)}}{4}\right\}$$ 4/4

✓ ✓ ✓ ✓ ✓

Q.2 $y'' - 6y' - 5y = 0$, $y(0) = 0,\ y'(0) = 2.$

$$\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} - 5\mathcal{L}\{y\} = 0$$

$$[s^2 Y(s) - 2] - 6\{sY(s)\} - 5Y(s) = 0.$$

$$s^2 Y(s) - 6sY(s) - 5Y(s) = 2.$$

$$Y(s)[s^2 - 6s - 5] = 2.$$

$$Y(s) = \frac{2}{s^2 - 6s - 5}$$ ✓ 2/2

$$= \frac{2}{s^2 - 6s + 9 - 9 - 5}$$

$$Y(s) = \frac{2}{(s-3)^2 - 14}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{2}{\sqrt{14}} \mathcal{L}^{-1}\left\{\frac{\sqrt{14}}{(s-3)^2 - 14}\right\}$$

~~$y(t) = \frac{2}{\sqrt{14}} e^{3t} \sinh 4t$~~

$$y(t) = \frac{2}{\sqrt{14}} e^{3t} \sinh \sqrt{14}\, t$$ ✓ 5/5

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4.4 Solution for Quiz IV

15

① $y''' - 6y'' + 9y' = e^{-2t}$

$m^3 - 6m^2 + 9m = 0$

$m(m^2 - 6m + 9) = 0$

$m = 0 \quad m = 3 \quad m = 3$

$y_h = C_1 + C_2e^{3t} + C_3te^{3t}$

5/5

$y_p = Ae^{-2t}$

$y' = -2Ae^{-2t}$

$y'' = 4Ae^{-2t}$

$y''' = -8Ae^{-2t}$

$= -8Ae^{-2t} - 6(4Ae^{-2t}) + 9(-2Ae^{-2t})$

$\boxed{y_g = C_1 + C_2e^{3t} + C_3e^{3t}t + \frac{1}{-50}e^{-2t}}$ ✓

$= -8Ae^{-2t} - 24Ae^{-2t} - 18Ae^{-2t} = e^{-2t}$

$-8A - 24A - 18A = 1$

$\frac{-50A}{-50} = \frac{1}{-50}$

② $y' + 3y = \cos t$

$m + 3 = 0$

$\boxed{m = -3}$

$y_h = C_1e^{-3t}$

$y_p = A\cos t + B\sin t$

$y' = -A\sin t + B\cos t$

$-A\sin t + B\cos t + 3A\cos t + 3B\sin t = \cos t$

~~3A = A~~

$3A + B = 1$

$3B - A = 0$

$3A + \frac{A}{3} = 1$

$\frac{3B}{3} = \frac{A}{3}$

$\frac{10}{3}A = 1$

$\boxed{A = \frac{3}{10}}$

$3\left(\frac{3}{10}\right) + B = 1$

$\frac{9}{10} + B = 1$

$\boxed{B = \frac{1}{10}}$

$\boxed{y_g = C_1e^{-3t} + \frac{3}{10}\cos t + \frac{1}{10}\sin t}$

✓ 5/5

$y^{(3)} - 3y^{(2)} + 6.25y' = 25$

$y_p = At$

$y' = A$

$y'' = 0$

$y''' = 0$

$m^3 - 3m^2 + 6.25m = 0$

$m(m^2 - 3m + 6.25) = 0$

$m = 0 \qquad m = \frac{3}{2} \pm 2i$

$y_h = c_1 + e^{3/2t}[c_2\cos(2t) + c_3\sin(2t)]$

$\frac{\cancel{6.25}A}{\cancel{6.25}} = \frac{25}{6.25}$

$\boxed{A = 4}$

$$\boxed{y_g = c_1 + e^{3/2t}[c_2\cos(2t) + c_3\sin(2t)] + 4t}$$

5/5

①

4.5 Solution for Quiz V

Q1) $ty^{(2)} - 4y' = t^4$

$n(n-1) - 4n = 0$

$n^2 - n - 4n = 0$

$n^2 - 5n = 0$

$n(n-5) = 0$

$\boxed{n=0}$ $\boxed{n=5}$

$\Downarrow$ $\Downarrow$

$y_1 = \boxed{t^0 = 1}$ $y_2 = \boxed{t^5}$

$$\boxed{Y_h = C_1 + C_2 t^5}$$

$Y_p = U_1 Y_1 + U_2 Y_2$

$$\begin{cases} U_1'(1) + U_2'(t^5) = 0 \\ \cancel{U_1'(0)} + U_2'(5t^4) = t^3 \end{cases}$$

$$D = \begin{vmatrix} 1 & t^5 \\ 0 & 5t^4 \end{vmatrix} = \boxed{5t^4}$$

$$U_1' = \frac{\begin{vmatrix} 0 & t^5 \\ t^3 & 5t^4 \end{vmatrix}}{5t^4} = \frac{-t^8}{5t^4} = -\frac{1}{5}t^4$$

$$U_2' = \frac{\begin{vmatrix} 1 & 0 \\ 0 & t^3 \end{vmatrix}}{5t^4} = \frac{t^3}{5t^4} = \frac{1}{5t}$$

$$U_1 = \int -\frac{1}{5}t^4 = \frac{-1}{25}t^5$$

$$U_2 = \int \frac{1}{5t} = \frac{1}{5}\ln|t|$$

$$Y_g = C_1 + C_2 t^5 + \frac{-1}{25}t^5 + \frac{1}{5}t^5 \ln|t|$$

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Q2) $(ty' + y = t\sin(t^2)) \div t$

$$y' + \frac{1}{t}y = \sin(t^2)$$

$$I = e^{\int \frac{1}{t}\,dt} = e^{\ln t} = t$$

$$y = \frac{\int t\sin(t^2)\,dt}{t} \leftarrow$$

$u = t^2$

$\frac{du}{2} = \frac{2t\,dt}{2} \quad = \frac{1}{2}\int \sin u\,du$

$= \frac{-1}{2}\cos t^2 + C$

$$y = -\frac{1}{2t}\cos t^2 + Ct^{-1}$$

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MTH 205

Quiz 5

Question 1:

$ty^{(2)} - 4y' = t^4$. Find y_g → 2nd order L.D.E with non const coeffs.

$y_g = y_h + y_p$

Assume $y = t^n$, $y' = n t^{n-1}$, $y'' = n^2 - n\, t^{n-2}$

↳ All of the terms have a degree of $\boxed{n-1}$ → I can use Cauchy Euler.

$y_h = ty^{(2)} - 4y' = 0.$

Char (2.D.E) $= n^2 - n - 4n = 0.$

$n^2 - 5n = 0. \rightarrow n(n-5) = 0.$

$n = 0$, $n = 5.$

$y_1 = t^0 = 1$, $y_2 = t^5$

\# $y_h = c_1 + c_2 t^5.$

\# $y_p = a_1(t) y_1 + a_2(t) y_2 = a_1(t) \cdot 1 + a_2(t) t^5.$

subject to:

$u_1' + u_2' t^5 = 0.$

$0 + u_2' 5t^4 = \frac{t^4}{t} = t^3.$

$W = \begin{vmatrix} 1 & t^5 \\ 0 & 5t^4 \end{vmatrix} = 5t^4.$

$u_1' = \frac{\begin{vmatrix} 0 & t^5 \\ t^3 & 5t^4 \end{vmatrix}}{W} = \frac{0 - t^8}{5t^4} = \frac{-t^4}{5}$

$u_2' = \frac{\begin{vmatrix} 1 & 0 \\ 0 & t^3 \end{vmatrix}}{W} = \frac{t^3}{5t^4}$

↳ $u_2' = \frac{1}{5t}$

⇒ $u_1 = \int -\frac{t^4}{5}\, dt$

$\boxed{u_1 = \frac{-t^5}{25}}$

⇒ $u_2 = \int \frac{1}{5t}\, dt$

$\boxed{u_2 = \frac{1}{5} \ln(t)}$

\# $y_p = \frac{-t^5}{25} + \frac{1}{5} \ln(t)\, t^5.$

$$\boxed{y_g = c_1 + c_2 t^5 + t^5 \left[-\frac{1}{25} + \frac{1}{5} \ln(t) \right]}$$

Question 2:

Solve for y_g: $ty' + y = t\sin(t^2)$ → 1st order ODE.

1. Divide by t to get the std form

$$y' + \frac{1}{t}y = \sin(t^2)$$

2. IF: $e^{\int Q(t)\,dt} = e^{\int \frac{1}{t}\,dt} = e^{\ln(t)} = t.$

$$\boxed{IF = t}$$

$$\left[y' + \frac{1}{t}y = \sin(t^2)\right] \times t.$$

$$\int (ty)' = \int t\sin(t^2)$$ ← take integral of both side.

$$ty = -\frac{1}{2}\cos(t^2) + C$$

$$y = -\frac{1}{2t}\cos(t^2) + \frac{C}{t}.$$

DIFFERENTIAL EQUATIONS ROHAN MITRA 85023

Q1) $ty'' - 4y' = t^4$ find y_g.

~~Let $y = x^n \Rightarrow y' = nx^{n-1}$ $y'' = n(n-1)x^{n-2}$.~~

Let $y = t^n \Rightarrow y' = nt^{n-1}$ $y'' = n(n-1)\,t^{n-2}$

~~$nt\cdot t^{n-1}$~~ $\Rightarrow n(n-1)\,t\cdot t^{n-2} - 4nt^{n-1} = 0$

$n(n-1)\,t^{n-1} - 4nt^{n-1} = 0$

$\Rightarrow n^2 - n - 4n = 0$

$n^2 - 5n = 0$ $\quad n = 0,5 \quad \therefore y_h = c_1 + c_2 t^5$

$n(n-5) = 0 \nearrow$

Variation: $y_p \rightarrow$ $u_1'(t)\cdot 1 + u_2'(t)\cdot t^5 = 0$

$u_1'(t)\cdot(0) + u_2'(t)\cdot 5t^4 = \frac{t^4}{t} \Rightarrow$ ~~$u_2'(t) = 0$, $u_2(t) = 0$~~

$$W\begin{vmatrix} 1 & t^5 \\ 0 & 5t^4 \end{vmatrix} = 5t^4,$$

$$u_1'(t) = \frac{\begin{vmatrix} 0 & t^5 \\ t^3 & 5t^4 \end{vmatrix}}{5t^4} = \frac{t^8}{5t^4} = \frac{1}{5}t^4 \quad \cancel{u_1(t) = \frac{1}{9}t^9} \qquad u_1(t) = \frac{1}{25}t^5$$

$$u_2'(t) = \frac{\begin{vmatrix} 1 & 0 \\ 0 & t^3 \end{vmatrix}}{5t^4} = \frac{t^3}{5t^4} = \frac{1}{5}\frac{1}{t}. \qquad u_2(t) = \frac{1}{5}\ln(t).$$

$$\therefore y_p = \frac{1}{25}t^5 + \frac{1}{5}t^5\ln(t).$$

$$y_g = c_1 + c_2 t^5 + \frac{1}{5}t^5\ln(t).$$

Q2) $ty' + y = t\sin(t^2).$

$y' + \frac{1}{t}y = \cancel{t}\sin(t^2).$

$I{:}F \doteq e^{\int \frac{1}{t}dt} = e^{\ln|t|} \overset{\downarrow\, t>0 \text{ Assumption.}}{=} t.$

$ty = \int t\sin(t^2)dt.$

$I = \int t\sin(t^2)dt$ let $u = t^2$

$du = 2t\,dt.$

$t\,dt = \frac{1}{2}du$

$I = \frac{1}{2}\int \sin(u)\,du = \frac{-1}{2}\cos(u) + c = \frac{-1}{2}\cos(t^2) + c$

$ty = \frac{-1}{2}\cos(t^2) + c$

$y_g = \frac{-1}{2}t^{-1}\cos(t^2) + ct^{-1}!$

4.6 Solution for Quiz VI

Farah Ossama - 82666 Quiz

15/15

Q1) $\frac{dy}{dx} = \frac{-e^{x}y + 4x - 3y^2 + 2xy}{e^{x} - x^2 + 6yx + \sin(y) - 7}$

$f_x = e^{x}y - 4x + 3y^2 - 2xy$

$f_y = e^{x} - x^2 + 6yx + \sin(y) - 7$

$f_{xy} = e^{x} + \cancel{4}\, 6y - 2x$

$f_{yx} = e^{x} - 2x + 6y$

$\Big\}$ EXACT $f_{xy} = f_{yx}$

2/2

$\int f_x \, dx = \int e^{x}y - 4x + 3y^2 - 2xy \, dx$

$= e^{x}y - 2x^2 + 3y^2x - x^2y + C(y)$

$\int f_y \, dy = \int e^{x} - x^2 + 6yx + \sin(y) - 7 \, dy$

$= e^{x}y - x^2y + 3xy^2 - \cos(y) - 7y + C(x)$

3/3 $\rightarrow$ $\boxed{e^{x}y - x^2y + 3xy^2 - 2x^2 - \cos(y) - 7y}$

$\boxed{e^{x}y - x^2y + 3xy^2 - 2x^2 - \cos(y) - 7y + C = 0}$

Q2) $\frac{dy}{dx} = y^3 - 6y^2 - 7y$

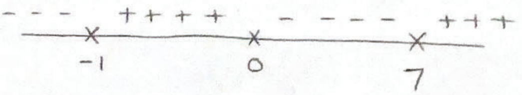
critical points

→ $y^3 - 6y^2 - 7y = 0$

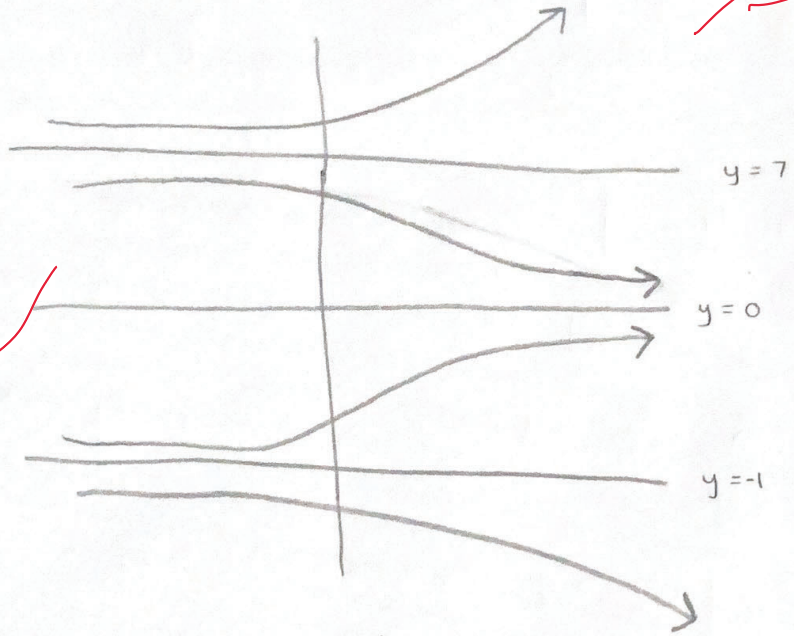
$y(y^2 - 6y - 7) = 0$

$y = 0$ $y = -1$ $y = 7$ → critical points

Find the sign



2/2



@ $y = 7$ → unstable point / non stable

@ $y = 0$ → stable point

@ $y = -1$ → unstable point / nonstable

3/3

Q3) Tank capacity = 1200 L

rate in → 2 grams per L $\quad$ A(0) = 80

6L/min $\quad$ 300L of brine

rate out → 3L/min

Volume of fluid = $300 + (6-3)t$

$$\boxed{300 + 3t}$$ ✓

C(t) concentration @ time t

$$C(t) = \frac{A(t)}{\text{Volume}} = \frac{A}{300+3t}$$ ✓ 2/2

$$\frac{dA}{dt} = \text{rate in} - \text{rate out}$$

$$2 \times 6 - C(t) \times 3$$

$$= 12 - \frac{A}{300+3t} \times 3$$ ✓

$$\frac{dA}{dt} = 12 - \frac{3A}{300+3t}$$ ✓

$$\frac{dA}{dt} + \underbrace{\frac{3A}{300+3t}}_{Q(t)} = 12$$ → 1st order eq.

$$I = e^{\int Q(t)\, dt} = e^{\int \frac{3}{300+3t}\, dt} = e^{\ln|300+3t|} = \boxed{300+3t}$$ ✓

$$A = \frac{\int (300+3t)\,12 \,.\, dt}{300+3t} = \frac{3600t + 18t^2 + C}{300+3t}$$ ✓

i) $c(t) = \dfrac{A}{300 + 3t}$

ii) ~~$A(t) = \dfrac{3600t + 18t^2 + C}{300 + 3t}$~~

$$A(0) \Rightarrow \frac{3600(0) + 18(0)^2 + C}{300 + 3(0)} = 80$$

$$\frac{C}{300} = 80$$

$$C = 24000$$

$$A(t) = \frac{3600t + 18t^2 + 24000}{300 + 3t}$$

✓ 3/3

4.7 Solution for EXAM I

1) $\mathcal{L}\{y'' - 6y' + 5y = u_2(t)(e^{(t-2)})\}$ $y(0)=0,\ y'(0)=0$

$s^2Y(s) - 6sY(s) + 5Y(s) =$

$e^{-2s} \cdot \mathcal{L}\{g(t+2)\} = \frac{1}{s-1} \cdot e^{-2s}$

$\downarrow$
e^{t}

$Y(s)[\cancel{s^2-6s}+5] = \frac{e^{-2s}}{s-1} \cdot \frac{1}{s^2-6s+5}$

$(s-5)(s-1)$

$\mathcal{L}^{-1}\left\{Y(s) = \frac{1e^{-2s}}{(s-1)^2(s-5)}\right\} =$

$\downarrow$

$\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2(s-5)}\right] \rightarrow \frac{a}{s-1} + \frac{b}{(s-1)^2} + \frac{c}{s-5}$

$b = \frac{1}{1-5} = -1/4$

$c = \frac{1}{16}$

$a = \frac{-1}{16}$

$a(s-1)(s-5) + b(s-5) + c(s-1)^2$

$(as-a)(s-5)$ $\qquad c(s^2-2s+1)$

$1 = as^2 - 5as - as + 5a + bs - 5b + cs^2 - 2cs + c$

$a + c = 0 \Rightarrow c = -a$

$-5a - a - 2c = 0 \Rightarrow -6a - 2c = 0$

$5a - 5b + c = 1 \Rightarrow$

$\Rightarrow \left[\frac{-1/16}{s-1} + \frac{-1/4}{(s-1)^2} + \frac{1/16}{s-5}\right]\mathcal{L}^{-1}$

$-1/16\, e^{t} - 1/4\, te^{t} + \frac{1}{16} e^{5t}$

$Y(t) = \left\{-\frac{1}{16} e^{(t-2)} - 1/4\, te^{(t-2)} + \frac{1}{16} e^{5(t-2)}\right] U_2(t)$

2) $\{y'' - 4y' + 13y = 3\,\delta_0(t)\}\ell,\quad y(0)=0,\ y'(0)=0$

$s^2Y(s) - 4sY(s) + 13Y(s) = 3$

$Y(s)[s^2\cancel{-4}s+13) = \dfrac{3}{s^2-4s+13}$

$$Y(s) \overset{\ell^{-1}}{=} \left\{\overset{\ell}{\frac{3}{(s-2)^2+9}}\right\}$$

$$\boxed{Y(t) = e^{2t}\sin(3t)}$$

3) $\{y' - 4y = U_2(t) - 4\int_0^t \underset{1\,*\,y(t)}{y(r)\,dr}\}\ell,\quad y(0)=0$

$sY(s) - 4Y(s) = \dfrac{e^{-2s}}{s} - 4\left(\dfrac{1}{s}\cdot Y(s)\right)$

$Y(s)\left[s-4+\dfrac{4}{s}\right] = \dfrac{e^{-2s}}{s}$

$Y(s)\left[\cancel{\dfrac{s^2-4s+4}{s}}\right] = \dfrac{e^{-2s}}{\cancel{s}}\cdot\dfrac{\cancel{s}}{s^2-4s+4}$

$$Y(s) = \left(\frac{e^{-2s}}{s^2-4s+4}\right) \overset{\ell^{-1}}{=} \left\{\frac{e^{-2s}}{(s-2)^2}\right\}\ell^{-1}$$

$\ell^{-1}\left\{\dfrac{1}{(s-2)^2}\right\} = te^{2t}$

$y(t) = \left[(t-2)e^{2(t-2)}\right]U_2\,t$

4) $y'' - 2y' + y = 2e^{t} \longrightarrow y_p = \underbrace{At^2}_{u}\underbrace{e^{t}}_{v}$

$m^2 - 2m + 1 = 0$

$(m-1)^2$

$m = 1$

$m = 1$

$y' = \underbrace{2At}_{u}\underbrace{e^{t}}_{v} + \underbrace{e^{t}}_{u} \cdot \underbrace{At^2}_{v}$

$y'' = 2Ae^{t} + e^{t} \cdot 2At + e^{t} \cdot At^2 + 2Ate^{t}$

$y_h = c_1 e^{t} + c_2 t e^{t}$

↙

$2Ae^{t} + 2Ate^{t} + At^2e^{t} + 2Ate^{t} - 2(2Ate^{t} + At^2e^{t})$

$+ At^2e^{t}$

$= 2e^{t}$

$2Ae^{t} + 2Ate^{t} + At^2e^{t} + 2Ate^{t} - 4Ate^{t} - 2At^2e^{t} + At^2e^{t} = 2e^{t}$

$\frac{\cancel{2}A}{\cancel{2}} = \frac{\cancel{2}}{\cancel{2}} \quad \boxed{A = 1}$

$y_g = c_1 e^{t} + c_2 t e^{t} + t^2 e^{t}$

5) $(t^2-9)y'' + \sqrt{t+1}\; y' + t^2 y = 5t+1$, $y(2)=4$
$y'(2)=-3$

$$\downarrow$$

$t^2 - 9 \neq 0$

$\sqrt{t^2} \neq \pm\sqrt{9}$

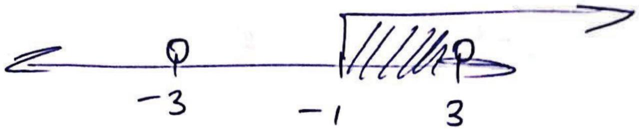
$\boxed{t \neq 3}$

$\boxed{t \neq -3}$

$\sqrt{t+1}$

$t+1 \geq 0$

$\boxed{t \geq -1}$



$I = [-1, 3)$

6) $y^{(3)} - 4y^{(2)} + 13y' = e^t + 8t \rightarrow$

$m^3 - 4m^2 + 13m = 0$

$m(m^2 - 4m + 13) = 0$

$m = 0$

$m = 2 \pm 3i$

$y_p = Ae^t + (Bt + C)t$

$y_p = Ae^t + 2Bt + C$

$y_p'' = Ae^t + 2B$

$y_p''' = Ae^t$

$y_h = C_1 + e^{2t}(C_2 \cos(3t) + C_3 \sin(3t))$

$Ae^t - 4(Ae^t + 2B) + 13(Ae^t + 2Bt + C) = e^t + 8t$

$Ae^t - 4Ae^t - 8B + 13Ae^t + 26Bt + 13C = e^t + 8t$

$10A = 1$

$A = 1/10$

$26B = 8$

$\frac{26B}{26} = \frac{8}{26}$

$B = 8/26$

$-8B + 13C = 0$

$C = \frac{32}{169}$

$y_p = \frac{1}{10}e^t + \frac{4}{13}t^2 + \frac{32}{169}t$

$y_g = c_1 + e^{2t}(c_2 \cos(3t) + c_3 \sin(3t)) + \frac{1}{10}e^t + \frac{4}{13}t^2 + \frac{32}{169}t$

$$7\ \mathcal{L}\{\ x'(t) - y(t) = 0 \qquad , \ x(0)=3$$
$$\mathcal{L}\{\ x(t) + y'(t) = 3 \qquad , \ y(0)=1$$

$$sX(s) - 3 - Y(s) = 0$$
$$X(s) + sY(s) - 1 = \frac{3}{s}$$

$$\begin{cases} sX(s) - Y(s) = 3 \\ X(s) + sY(s) = 1 + \frac{3}{s} \rightarrow \frac{s+3}{s} \end{cases}$$

$$X(s) = \frac{\begin{vmatrix} 3 & -1 \\ \frac{s+3}{s} & s \end{vmatrix}}{\begin{vmatrix} s & -1 \\ 1 & s \end{vmatrix}} = \frac{\frac{3s^2}{s} + \frac{s+3}{s}}{\frac{s^2+1}{1}} = \frac{3s^2+s+3}{s(s^2+1)}$$

$$\frac{3s^2+s+3}{s(s^2+1)} = \frac{a}{s} + \frac{bs+c}{s^2+1}$$
$$3s^2+s+3 = a(s^2+1) + (bs+c)s$$
$$3s^2+s+3 = as^2 + a + bs^2 + cs$$
$$a+b=3 \Rightarrow 3+b=3 \quad \boxed{b=0}.$$
$$\boxed{a=3} \qquad \boxed{c=1}$$

$$\mathcal{L}^{-1}\left[X(s) = \frac{3}{s} + \frac{1}{s^2+1}\right]$$

$$\boxed{x(t) = 3 + \sin t}$$

Easier calculations:
... . = 3(s^2 + 1)/(s(s^2 + 1) + s/(s(s^2 + 1)) = 3/s + 1/(s^2 + 1) .

Hence f(t) = 3 + sin(t)

(i) $\ell \left\{ \frac{}{(s+3)^2} \right\}$

$$\Rightarrow \left\{ \frac{s+3}{(s+3)^2} \; \frac{-3}{(s+3)^2} \right\}^{\ell^{-1}}$$

$$\left\{ \frac{1}{(s+3)} - \frac{3}{(s+3)^2} \right\}^{\ell^{-1}} \Rightarrow \boxed{e^{-3t} - 3 t e^{-3t}}$$

(ii) $\ell \left[\int_0^t e^{7t-5r} \cos(2r)\, dr \right]$ $\qquad$ $-5 = -7 + x$

$-5 + 7 = x$

$x = 2$

$$\Downarrow$$

$$\int_0^t e^{7(t-r)} \cos(2r)\, e^{2r}\, dr$$

$$\swarrow$$

$$\ell \left\{ e^{7t} * \cos(2t) e^{2t} \right\}$$

$$\swarrow \qquad \swarrow$$

$$\Rightarrow \frac{1}{s-7} \cdot \frac{(s-2)}{(s-2)^2+4}$$

(iii) $e^{-1} \left\{ \frac{s e^{-2s}}{(s+3)^2+9} \right\}$

$$\swarrow$$

$$e^{-1} \left\{ \frac{s}{(s+3)^2+9} \right\} = \left\{ \frac{(s+3)}{(s+3)^2+9} \; \frac{-3}{(s+3)^2+9} \right\}^{e^{-1}}$$

$$= e^{-3t} \cos 3t - e^{-3t} \sin 3t$$

$$\Rightarrow \left[e^{-3(t-2)} \cos 3(t-2) - e^{-3(t-2)} \sin 3(t-2) \right] U_2 t$$

9. $y = 3\sin(t)e^{t}$

$\mathcal{L}\{ay'' + by' + cy = 3\sin(t)e^{t}\}$ $\quad y(0) = 0$, $y'(0) = 3$

$a[s^2Y(s) - 3] + b[sY(s)] + cY(s) = \dfrac{3}{(s-1)^2 + 1}$

$Y(s)[as^2 + bs + c] = \left(\dfrac{3}{(s-1)^2+1} + 3a\right) \dfrac{1}{as^2+bs+c}$

$\dfrac{3}{(s-1)^2+1\,(as^2+bs+c)} + \dfrac{3a}{(as^2+bs+c)} = \dfrac{3}{(s-1)^2+1}$

$s^2 - 2s + 1 + 1$

$s^2 - 2s + 2$

$\dfrac{3}{(s-1)^2+}$ $\quad \dfrac{3 + 3a(s^2-2s+2)}{((s-1)^2+1)(as^2+bs+c)} = \dfrac{3}{(s^2-2s+2)}$

$3as^2 + 3bs + 3c = 3 + 3a(s^2 - 2s + 2)$

$3as^2 + 3bs + 3c = 3 + 3as^2 - 6as + 6a$

$3a = 3a$ $\quad$ $3b = -6a$ $\quad$ $3c = 3 + 6a$

$b = -2a$ $\quad$ $c = 1 + 2a$

infinite solution

a can be any real number

4.8 Solution for EXAM II

Leen Abu Zaid

84977

①

i) $t^2 y^{(2)} + 3t y' + 4y = 0$

~~the~~ Cauchy $\Rightarrow y = t^n \quad y' = nt^{n-1} \quad y^{(2)} = (n^2-n)t^{n-2}$

All have same degree n ,we can use cauchy

$n^2 - n + 3n + 4 = 0$

$n^2 + 2n + 4 = 0$

$n = -1 \pm \sqrt{3}\, i$

$y_h = \cancel{c}\; t^{-1}(c_1 \cos(\sqrt{3}\ln t) + c_2 \sin(\sqrt{3}\ln t))$

ii) $y^{(2)} - \left(\frac{1}{t} - 1\right) y' = \frac{e^{-t}}{t}$

$\underline{y_h}$: $y^{(2)} - \left(\frac{1}{t} - 1\right) y' = 0$

Reduction of order $\Rightarrow y_1 = 1$

$$y_2 = y_1 \int \frac{e^{-\int Q(t)dt}}{y_1^2}\, dt$$

$Q(t) = \left(-\frac{1}{t} + 1\right)$

$\Rightarrow e^{-\int Q(t)dt} = e^{\int \frac{1}{t} - 1\, dt} = e^{\ln t - t} = te^{-t}$

$y_2 = \int te^{-t}\, dt$

t	$+$	e^{-t}
1	$-$	$-e^{-t}$
0		e^{-t}

$y_2 = -te^{-t} - e^{-t}$

$y_h = c_1 + c_2(-te^{-t} - e^{-t})$

Another Solution
w = y^\, hence w^\ = y^\\/ so w^\ - (1/t - 1)w = e^{-t}/t
Q(t) = - (1/t - 1) = 1 - 1/t /   Hence I =   e^{Integral(1 - 1/t) dt} = e^t/t / Thus
w = Integral(I. e^{-t}/t)/ I = Integral (1/t^2)/I = (-1/t + c)/I = -e^{-t} + cte^{-t}
Since w = y^\ , Integral (w) = Integral (y^\) = y
Hence y = Integral(e^{-t}(ct - 1) dt) = -e^{-t}(ct - 1) - ce^{-t} + c_2. = e^{-t}(-ct - c) + c_2 + e^{-t}
Note y_p = e^{-t}
y_h = c e^{-t}(-t - 1) + c_2

$y_p = u_1 y_1 + u_2 y_2$

Subject

$u_1' y_1 + u_2' y_2 = 0$

$u_1' y_1' + u_2' y_2' = \frac{e^{-t}}{t}$

$u_1'(1) + u_2'(-te^{-t} - e^{-t}) = 0$

$u_1'(0) + u_2'(te^{-t}) = \frac{e^{-t}}{t}$

$$W = \begin{vmatrix} 1 & -te^{-t}-e^{-t} \\ 0 & te^{-t} \end{vmatrix} = te^{-t}$$

$$u_1' = \frac{\begin{vmatrix} 0 & -te^{-t}-e^{-t} \\ \frac{e^{-t}}{t} & te^{-t} \end{vmatrix}}{te^{-t}} = \frac{-\left(\frac{e^{-t}}{t}\right)(-te^{-t}-e^{-t})}{te^{-t}}$$

$$= \frac{te^{-t}+e^{-t}}{t^2} = \frac{e^{-t}}{t} + \frac{e^{-t}}{t^2} = \left(\frac{1}{t} + \frac{1}{t^2}\right)e^{-t}$$

$$u_2' = \frac{\begin{vmatrix} 1 & 0 \\ 0 & \frac{e^{-t}}{t} \end{vmatrix}}{W} = \frac{\frac{e^{-t}}{t}}{te^{-t}} = \frac{e^{-t}}{t^2 e^{-t}} = \frac{1}{t^2}$$

$u_2 = \int u_2' = \int \frac{1}{t^2} = \frac{-1}{t}$

$u_1 = \int u_1' = \int \left(\frac{1}{t} + \frac{1}{t^2}\right) \cancel{dt} e^{-t} \, dt = -\frac{1}{t} e^{-t}$

$y_p = \left(-\frac{1}{t} e^{-t}\right) + \left(-\frac{1}{t}\right)(-te^{-t} - e^{-t})$

$y_g = c_1 + c_2(-te^{-t} - e^{-t}) + -\frac{1}{t} e^{-t} + \left(-\frac{1}{t}\right)(-te^{-t} - e^{-t})$

iii) $t^2 y^{(2)} - 7t y' + 16y = 0$

$y = t^n$

$y' = nt^{n-1}$

$y^{(2)} = (n^2 - n)t^{n-2}$

Cauchy $\Rightarrow$ all have same degree n

$(n^2 - n) - 7n + 16 = 0$

$n^2 - 8n + 16 = 0$

$(n-4)(n-4)$

$n = 4 \qquad n = 4$

$y_1 = t^4 \qquad y_2 = t^4 \ln t$

$y_h = c_1 t^4 + c_2 t^4 \ln(t)$

iv) $ty' + 4y = 4t^2 e^t y^{\frac{3}{4}}$

Non-linear $\Rightarrow$ Bernoulli

$n = \frac{3}{4} \qquad 1 - n = \frac{1}{4} \qquad v = y^{1-n} = y^{1/4}$

$y' + \frac{4}{t} y = 4te^t y^{\frac{3}{4}}$

$\Downarrow$

$v' + \left(\frac{4}{t}\right)\left(\frac{1}{4}\right) v = 4te^t \left(\frac{1}{4}\right)$

$v' + \frac{1}{t} v = te^t$

$I = e^{\int \frac{1}{t} dt} = e^{\ln t} = t$

$v = \frac{\int I f(t)}{I}$

$$v = \frac{\int t(te^t)\,dt}{t} = \frac{\int t^2 e^t\,dt}{t}$$

polynomial		easy to integrate
t^2	$+$	e^t
$2t$	$-$	e^t
2	$+$	e^t
0		e^t

$$v = \frac{t^2e^t - 2te^t + 2e^t + C}{t}$$

$$v = te^t - 2e^t + \frac{2e^t}{t} + \frac{C}{t}$$

$$y = v^{\frac{1}{1-n}} = v^4$$

$$y = \left(te^t - 2e^t + \frac{2}{t}e^t + \frac{c}{t}\right)^4$$

v) $$\frac{dy}{dx} = \frac{2xy^3}{\sqrt{1+x^2}}$$

Seperable

$$\int \frac{dy}{y^3} = \int \frac{2x\,dx}{\sqrt{1+x^2}}$$

$$\int y^{-3} = \int \frac{2x}{\sqrt{1+x^2}}\,dx$$

$$-\frac{1}{2y^2} = 2\sqrt{1+x^2} + C$$

vi) $$\frac{dw}{dh} = \frac{1}{h+4w^3e^w}$$

$$\frac{dh}{dw} = h+4w^3e^w$$

$$h' - h = 4w^3e^w$$

$$\begin{cases} \text{Linear, first order} \\ \text{indep: } w \\ \text{dep: } h \end{cases}$$

$$I = e^{\int -1\,dw} = e^{-w}$$

$$h = \frac{\int I\,f(w)}{I} = \frac{\int e^{-w} \cdot 4w^3e^w\,dw}{e^{-w}}$$

$$= \frac{\int 4w^3\,dw}{e^{-w}} = \frac{w^4+C}{e^{-w}}$$

$$h = w^4e^w + Ce^w$$

vii) $y^{(2)} - \frac{4}{t} y' + \frac{4}{t^2} y = \frac{1}{t^2}$

$y^{(2)} - 4t^{-1} y' + 4t^{-2} y = \frac{1}{t^2}$

$\underline{y_n}$: $y^{(2)} - 4t^{-1} y' + 4t^{-2} y = 0$

$$\left.\begin{array}{l} y = t^n \\ y' = nt^{n-1} \\ y'' = (n^2-n)t^{n-2} \end{array}\right\} \text{ Cauchy since they have same degree } n-2$$

$n^2 - n - 4n + 4 = 0$

$n^2 - 5n + 4 = 0$

$(n-1)(n-4)$

$n = 1 \quad n = 4$

$y_1 = t \quad y_2 = t^4 \qquad\qquad y_n = c_1 t + c_2 t^4$

$\underline{y_p}$: $y_p = u_1 y_1 + u_2 y_2$

subject

$u_1' y_1 + u_2' y_2 = 0 \qquad\qquad u_1'(t) + u_2'(t^4) = 0$

$u_1' y_1' + u_2' y_2' = \frac{1}{t^2} \qquad\qquad u_1'(1) + u_2'(4t^3) = \frac{1}{t^2}$

$$W = \begin{vmatrix} t & t^4 \\ 1 & 4t^3 \end{vmatrix} = 4t^4 - t^4 = 3t^4$$

$$u_1' = \frac{\begin{vmatrix} 0 & t^4 \\ \frac{1}{t^2} & 4t^3 \end{vmatrix}}{W} = \frac{0 - t^2}{3t^4} = -\frac{1}{3t^2}$$

$$u_2' = \frac{\begin{vmatrix} t & 0 \\ 1 & 1/t^2 \end{vmatrix}}{W} = \frac{\frac{1}{t}}{3t^4} = \frac{1}{3t^5}$$

$u_1 = \int u_1' = \int -\frac{1}{3t^2} = \frac{1}{3t}$

$u_2 = \int u_2' = \int \frac{1}{3t^5} = -\frac{1}{12t^4}$

$y_p = \left(\frac{1}{3t}\right)(t) + \left(-\frac{1}{12t^4}\right)(t^4)$

$= \frac{1}{3} - \frac{1}{12} = \frac{1}{4}$

$y_g = y_h + y_p$

$= c_1 t + c_2 t^4 + \frac{1}{4}$

vii) $f_x = 2x + y^2 x + e^y + 2$

$f_y = x^2 y + x e^y + 4y^3 + 7$

$f_{xy} = 2xy + e^y$

$f_{yx} = 2yx + e^y$ $\quad f_{xy} = f_{yx} \Rightarrow$ DE is exact

$\int f_y \, dy = \underbrace{K(x,y) + h(x)}_{f(x,y)}$

$\int (x^2 y + x e^y + 4y^3 + 7)\, dy = \frac{x^2 y^2}{2} + x e^y + y^4 + 7y + h(x)$

To find $h(x)$:

$f_x(\text{given}) = f_x(x,y)$

$2x + y^2 x + e^y + 2 = y^2 x + e^y + h'(x)$

$h'(x) = 2x + 2$

$$h(x) = \int h'(x) = \int 2x + 2 = x^2 + 2x$$

$$f(x,y) = C$$

$$\frac{x^2y^2}{2} + xe^y + y^4 + 7y + x^2 + 2x = C$$

② $T = 180°C \quad t = 0$

$T_m = 23°C$

$T = 120°C \quad t = 2$

$$\frac{dT}{dt} = K(T - T_m)$$

$$\frac{dT}{dt} - KT = -KT_m$$

$$\frac{dT}{dt} - KT = -23K$$

$$I = e^{\int -K\,dt} = e^{-Kt}$$

$$T = \frac{\int I \cdot f(t)\,dt}{I} = \frac{\int e^{-Kt}(-23K)\,dt}{e^{-Kt}} = \frac{23\int e^{-Kt} \cdot -K\,dt}{e^{-Kt}}$$

$$= \frac{23e^{-Kt} + C}{e^{-Kt}}$$

$$T = 23 + Ce^{Kt}$$

$\underline{t=0}$

$180 = 23 + C \quad C = 157$

$\underline{t=2}$

$120 = 23 + 157e^{2K} \quad K = -0.24$

~~$T = 23 + C$~~

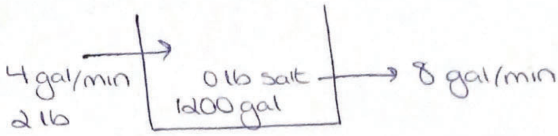
$$T = 23 + 157e^{-0.24t}$$

$$33 = 23 + 157e^{-0.24t}$$

$$-0.24t = -2.75$$

$$\boxed{t = 11.5 \text{ minutes}}$$

③



Volume = initial + (in - out)t

= 1200 + (4 - 8)t

= 1200 - 4t

$c(t) = \frac{A(t)}{1200-4t}$

$\frac{dA}{dt} = (4)(2) - 8\left(\frac{A(t)}{1200-4t}\right)$

$\frac{dA}{dt} + \frac{8A}{1200-4t} = 8$ → first order linear diff eqn.

$I = e^{\int Q(t)dt} = e^{\int \frac{8}{1200-4t} dt} = e^{-2\ln|1200-4t|}$

$\Downarrow$

$= (1200-4t)^{-2}$

$A = \frac{\int I \cdot f(t)}{I} = \frac{\int (1200-4t)^{-2}(8)\, dt}{(1200-4t)^{-2}}$

$= \cancel{8\int 1200-4t} \quad \frac{2(1200-4t)^{-1} + C}{(1200-4t)^{-2}}$

$A = 2(1200-4t) + C(1200 \cancel{+} -4t)^2$

$A(0) = 0$

$\Rightarrow 0 = 2(1200) + C(1200)^2$

$-2(1200) = C(1200)^2$

$-2 = 1200C \qquad C = \cancel{600} \ -\frac{1}{600}$

$$\boxed{A = 2(1200-4t) - \frac{1}{600}(1200-4t)^2}$$

Empty:
0 = 1200 - 4t. So t = 300 mint

4.9 Solution for Final Exam

Rahma Ali 79019

Differentials final exam

① Given

weight - 128 pounds

displacement = 2 ft

$x(0) = -0.5$ ft

$x'(0) = 2$ ft/sec

gravity = 32 ft/sec^2

$\beta \rightarrow 0$

$mx''(t) = -\overset{0}{\cancel{\beta x(t)}} - kx(t)$

$W = mg$

$\frac{128}{32} = \frac{m\,\cancel{32}}{\cancel{32}}$

$m = 4$ pounds

$W = kx \quad 128 = k(2)$

$k = 64$

i) $4x''(t) + 64x(t) = 0$

Undetermined =

$4m^2 + 64 = 0$

$\sqrt{m^2} = \sqrt{\frac{-64}{4}}$

$m = \pm 4$

$x(t) = c_1 \cos(4t) + c_2 \sin(4t)$

$x(0) = c_1 \cos(4(0)) + c_2 (\overset{0}{\cancel{\sin(4(0))}})$

$c_1 = -0.5 = -\frac{1}{2}$

$x'(t) = -4c_1 \sin(4t) + 4c_2 \cos(4t)$

$x'(0) = -4(-0.5)\sin(4(0)) + 4c_2 \cos(4(0))$

$\frac{2}{4} = \cancel{2(0)} + \frac{4c^2}{4} \qquad c_2 = \frac{1}{2}$

$x(t) = -\frac{1}{2} \cos 4t + \frac{1}{2} \sin 4t$

~~in~~ Phas

$\frac{1}{2}(\sin 4t - \cos 4t)$

$= \frac{\sqrt{2}}{2}\left(\sin 4t\left(\frac{1}{\sqrt{2}}\right) - \cos 4t\left(\frac{1}{\sqrt{2}}\right)\right)$

~~$\frac{1}{\sqrt{2}}(\sin 4t\left(\frac{1}{\sqrt{2}}\right) = co$~~

$\frac{1}{2}(\sin 4t + \cos\frac{\pi}{4} - \cos 4t \sin\frac{\pi}{4})$

$n\varphi 1 = \frac{1}{\sqrt{2}} \sin\left(4t - \frac{\pi}{4}\right)$ ✓

Phase angle $= \frac{\pi}{4}$

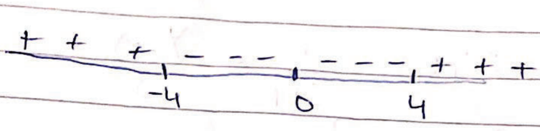
6/6

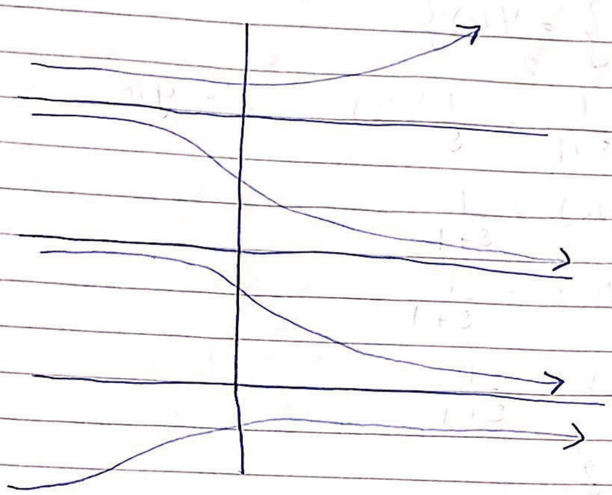
Q42) $y' = y^4 - 16y^2$

$m^4 - 16m^2 = 0$

$m^2(m^2 - 16) = 0$

$m = 0 \quad m = 4 \quad m = -4$





4 : unstable point

0 : semi stable point

-4 : stable point

4/4

Q3) $\frac{dy}{dx} = \frac{x}{2x(y+3)+y+3} = \frac{x}{(y+3)(2x+1)}$

$(y+3)\frac{dy}{dx} = \frac{x}{2x+1}$

$\int (y+3)\,dy = \int \frac{x}{2x+1}\,dx \longrightarrow \frac{y^2}{2} + 3y = \int \frac{x}{2x+1}\,dx$

$u = x \qquad du = \frac{1}{2x+1} \longrightarrow \frac{y^2}{2} + 3y = \cancel{\frac{1}{4}(2x+1}-$

$du = dx \qquad v = \ln(2x+1) \qquad \frac{1}{4}(2x+1-\ln(2x+1)+c_1$

4

Q4) $(x^2y + 4y^3)\,dx = (3xy^2 + x^3)\,dy$

$$\frac{dy}{dx} = \frac{x^2y + 4y^3}{3xy^2 + x^3}$$

$* \; y = vx \qquad v = \frac{y}{x}$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{v + 4v^3}{3v^2+1}$$

$$x\frac{dv}{dx} = \frac{v + 4v^3}{3v^2+1} - v$$

$$x\frac{dv}{dx} = \frac{v + 4v^3 - 3v^3 - v}{3v^2+1}$$

$$x\frac{dv}{dx} = \frac{v^3}{3v^2+1}$$

$$\frac{3v^2+1}{v^3}\,dv = \frac{1}{x}\cdot dx$$ ✓

$$\int\left(\frac{3}{v} + \frac{1}{v^3}\right)dv = \int \frac{1}{x}\cdot dx$$

$$3\ln v \; \phi - \frac{1}{2v^2} = \ln x + c$$

$$3\ln\left(\frac{y}{x}\right) - \frac{x^2}{2y^2} = \ln x + C$$ ✓

4

5) Find y_g

$$y'' + y' = \frac{1}{t} - \frac{1}{t^2} \quad , \quad t > 0$$

$$y'' + y' = \frac{1}{t} - \frac{1}{t^2} \left(\frac{t-1}{t^2} \right)$$

$y'' + y' = 0$

$D^2 + D = 0$ $\qquad D = 0 \qquad D = -1$

$D(D+1) = 0$

$y(x) = c_1 e^{0.x} + c_2 e^{-t}$

$y_1 = 1 \qquad y_2 = e^{-t}$

$y_1' = 0 \qquad y_2' = -e^{-t}$ ✓

$W = y_1 y_2' - y_2 y_1' = 1 x - e^{-\cancel{0}} - 0 = -e^{-t}$

$w = -e^{-\cancel{t} x} \qquad t(t) = \left(\frac{1}{t} - \frac{1}{t^2} \right)$

$$y_p = -y_1 \int \frac{y_2 f(t)}{w} dt + y_2 \int \frac{y_1 f(t)}{w} dt$$

$$= -1 \int \frac{e^{-t} \left(\frac{1}{t} - \frac{1}{t^2} \right)}{-e^{-t}} dt + e^{-t} \int \frac{1 (1/t - 1/t^2)}{-e^{-t}} dt$$

$$- \int \frac{1}{t^2} - \frac{1}{t} dt \mp e^{-t} \left[\int e^{t} \left(1/t - 1/t^2 \right) dt \right.$$

$$= - \left[-1/t - \ln t \right] \mp e^{-t} \left[e^{t} (1/t) \right\} dt$$

$$= \cancel{1/t} + \ln t \cancel{\mp \frac{1}{t}} = \ln t$$

$y_p = \ln t$ ✓ 4

⑥

Given DE : $y' = \dfrac{2x+y}{(4x+2y)^2+1} - 2$ — ①

$2x+y = v \longrightarrow 2v = 4x+2y$

$$2 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{dv}{dx} - 2$$

$$\frac{dv}{dx} - 2 = \frac{v}{(2(v))^2+1} - 2$$

$$\frac{dv}{dx} = \frac{v}{4v^2+1}$$ ✓

$$\left(\frac{4v^2+1}{v}\right) dv = dx$$ ✓

$\left(4v + \dfrac{1}{v}\right) \dfrac{d}{v} = dx$ ⟵ Integrate

$4\left(\dfrac{v^2}{2}\right) + \ln v = x + c$ ✓

$2v^2 + \ln v = x + c$

$2(2x+y)^2 + \ln(2x+y) = x + C$ ✓ 4

Q7) $y' + 2y = e^{-t} - \int_0^t y(t-r)\,dr$, $y(0) = 0$

$\ell\{y' + 2y\} = \ell\{e^{-t} - \int_0^t 1.y(t-r)\,dr\}$

$SY(s) - y(0) + 2Y(s) = \frac{1}{S+1} - \frac{Y(s)}{S}$

let $\ell\{y(t)\} = Y(s)\}$

$(S+2)Y(s) = \frac{1}{S+1} - \frac{1}{S}Y(s)$ ∵ $y(0) = 0$

$\left(S + 2 + \frac{1}{S}\right)Y(s) = \frac{1}{S+1}$

$\left(\frac{S^2+2S+1}{S}\right)Y(s) = \frac{1}{S+1}$

$\frac{(S+1)^2}{S}\,Y(s) = \frac{1}{S+1}$ ✓

$Y(s) = \frac{S}{(S+1)^3}$

$Y(t) = \ell^{-1}\left\{\frac{S}{(S+1)^3}\right\} \rightarrow y(t) = \ell^{-1}\left\{\frac{S+1-1}{(S+1)^3}\right\}$ ✓

$y(t) = \ell^{-1}\left\{\frac{S+1}{(S+1)^3} - \frac{1}{(S+1)^3}\right\}$ ✓

$y(t) = \ell^{-1}\left\{\frac{1}{(S+1)^2}\right\} - \ell^{-1}\left\{\frac{1}{(S+1)^3}\right\}$ 4/4

$y(t) = te^{-t} - \frac{1}{2}t^2e^{-t}$ ✓

$y(t) = e^{-t}\left(t - \frac{t^2}{2}\right)$ ✓

Q8) $y'' + 2y' + 5y = \delta_3(t)$ $y(0) = y'(0) = 0$

$S^2 Y(x) + 2s\,Y(s) + 5 = e^{-3s}$

$Y(s)\left[s^2+2s+5\right] = e^{-3s}$

$Y(s) = \frac{e^{-3s}}{s^2+2s+1-1+5} = \mathcal{L}^{-1}\left\{\frac{\boxed{e^{-3s}}}{(s+1)^2+4}\right\}$ $\to u_3(t)$

$t \to t-3$

$\mathcal{L}^{-1}\ \frac{1}{2}\left\{\frac{1(2)}{(s+1)^2+4}\right\} = \frac{1}{2}e^{-t}\sin(2t)$

$y(t) = u_3(t)$ ✓ $\left[\frac{1}{2}e^{-(t-3)}\sin(2(t-3))\right]$ ✓

4

9) $y^{(5)} + 6y^{(4)} + 9y^{(3)} = 4t^3 + t^2 e^{-3t}$

$m^5 + 6m^4 + 9m^3 = 0$

$m^3 (m^2 + 6m^2 + 9) = 0$

$m = 0, 0, 0, -3, -3$

$y_h = c_1 + c_2 t + c_3 t^2 + c_4 e^{-3t} + c_5 t e^{-3t}$

$y_p \to \left[a_1 + a_2 t + a_3 t^2 + a_4 t^3 + a_5 e^{-3t} + a_6 t e^{-3t} + a_7 t^2 e^{-3t}\right]$

$\times t^3$

$y_p = a_1 t^3 + a_2 t^4 + a_3 t^5 + a_4 t^6 + a_5 t^3 e^{-3t} + a_6 t^4 e^{-3t} + a_7 t^5 e^{-3t}$

✓

4/4

Q10) $y^3 + \frac{6.5}{t^2} y' = 0$

$t^{\cancel{5}3} y^{(3)} + 6.5 t y' = 0$

let $e^z = t$ $\quad z = \ln(t)$

$m(m-1)(m-2)y + 6.5\, my = 0$

$m(m^2 - 3m + 2)y + 6.5\, my = 0$

$(m^3 - 3m^2 + 2m + 6.5m)\, y = 0$

$m^3 - 3m^2 + 8.5m = 0$

$m = 0 \,,\; 1.5 \pm 2.5j$

$y = c_1 e^{(\cancel{0})z} + c_2 e^{1.5z} [c_2 \cos 2.5z + c_3 \sin 2.5z]$

$y = c_1 + t^{1.5} [c_2 \cos[2.5 \ln(t)] + c_3 \sin[2.5 \ln t]]$

4 ✓

Q11) $y_1 = 1$ $y_2 = \ln(t)$ $g = \frac{1}{t^2}$

$y'' + a_1(t)y' + a_0(t)y = \frac{1}{t^2}$

$y_p(t) = -y_1\int \frac{y_2 g}{W}\,dt + y_2\int \frac{y_1 g}{W}\,dt$

$W = \begin{vmatrix} 1 & \ln(t) \\ 0 & 1/t \end{vmatrix} = 1/t$

$y_p(t) = -\int \frac{\ln(t)}{t^2(\frac{1}{t})}\,dt + \ln(t)\int \frac{1}{t^2(1/t)}\,dt$

$y_p(t) = -\int \frac{1}{t}\ln(t)\,dt + \ln(t)\int \frac{1}{t}\,dt$

let $\ln(t) =$ ~~β~~ K

$\frac{1}{t}\,dt = dK$

$= -\int K \cdot dK + [\ln(t)]^2$

$y_p(t) = -\frac{K^2}{2} + [\ln(t)]^2$

$y_p(t) = \frac{-[\ln(t)]^2}{2} + [\ln(t)]^2 = \frac{(\ln(t))^2}{2}$ ✓

4/4

Q13) $\ell\{x'(t) - y(t) = 0\}$ $\quad x(0) = 0$

$\ell\{x(1) + y'(t) = t\}$, $y(0) = 2$

$sX(s) - \cancel{x(0)}^{0} - Y(s) = 0$

$x(s) + sY(s) - \underset{2}{y'(0)} = \frac{1}{s^2} + \frac{2s^2}{s^2}$

$sX(s) - Y(s) = 0$

$X(s) + sY(s) = \frac{1+2s^2}{s^2}$

~~Denominator~~ = Det. $\begin{vmatrix} s & -1 \\ 1 & s \end{vmatrix} \rightarrow s^2+1$

$x(s) = \begin{vmatrix} 0 & -1 \\ \frac{1+2s^2}{s^2} & 0 \end{vmatrix} = \frac{1+2s^2}{s^2}$

$e^{-1}\{x(s)\} = \ell^{-1}\left\{\frac{1+2s^2}{s^2} \div s^2+1\right\}$

$x(t) = \ell^{-1}\left\{\frac{\frac{1+2s^2}{s^2}}{s^2+1}\right\}$

$x(t) = t + \sin(t)$ ✓ 4

5 Section : Assessment Tools-Quizzes (unanswered)

5.1 Quiz I

Name————————————, **ID** ————————

Quiz One, MTH 205 , Fall 2020

Ayman Badawi

QUESTION 1. (i) $\ell^{-1}\{\frac{3}{2s+5}\}$

(ii) $\ell^{-1}\{\frac{3}{s^2+4} + \frac{7}{s^9}\}$

(iii) $\ell\{(t+2)^2\}$

QUESTION 2. find $y(t)$, where $y^{(2)} - 5y' + 6y = 1, y(0) = y'(0) = 0$

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5.2 Quiz II

Name——————————————, **ID** ——————

Quiz Two, MTH 205 , Fall 2020

Ayman Badawi

QUESTION 1. (i) $\ell^{-1}\{\frac{e^{-4s}}{s^2+9}\}$

(ii) $\ell^{-1}\{\frac{1}{(s-3)^2+4} + \frac{6e^{-3s}}{s^4}\}$

(iii) $\ell\{U_5(t)e^{t-5}cosh(t-5)\}$

QUESTION 2. find $y(t)$, where $y' - 2y = U_3e^{t-3}, y(0) = 0$

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5.3 Quiz III

Name——————————————, **ID** ——————————

Quiz Three, MTH 205 , Fall 2020

Ayman Badawi

QUESTION 1. Let $f(t) = \begin{cases} 1 & if \;\; 0 \leq t < 3 \\ 0 & if \;\; t \geq 3 \end{cases}$

(i) Write $f(t)$ in terms of unit-step functions

(ii) Find $y(t)$, where $y' - 4y = f(t).$

QUESTION 2. Find $y(t)$, where $y^{(2)} - 6y' - 5y = 0, y(0) = 0, y'(0) = 2.$

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5.4 Quiz IV

Name————————————, **ID** ——————

Quiz Four, MTH 205 , Fall 2020

Ayman Badawi

QUESTION 1. Find the general solution of the L.D.E : $y^{(3)} - 6y^{(2)} + 9y' = e^{-2t}$

QUESTION 2. Find the general solution of the L.D.E : $y' + 3y = cos(t)$

QUESTION 3. Find the general solution of the L.D.E : $y^{(3)} - 3y^{(2)} + 6.25y' = 25$

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5.5 Quiz V

Name——————————————, ID ———————

Quiz Five, MTH 205 , Fall 2020

Ayman Badawi

QUESTION 1. Consider $ty^{(2)} - 4y' = t^4$. Find y_g

QUESTION 2. Solve for y_g.
$ty' + y = tsin(t^2)$.

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5.6 Quiz VI

Name——————————————, **ID** ————————

Quiz Six, **MTH 205 , Fall 2020**

Ayman Badawi

QUESTION 1. (i) Given $\frac{dy}{dx} = \frac{-e^x y + 4x - 3y^2 + 2xy}{e^x - x^2 + 6yx + sin(y) - 7}$

a. Convince me that the given D.E is Exact (hint: rewrite it as $f_x dx + f_y dy = 0$ be careful with the sign)?SHOW THE WORK

b. Solve the D.E. (Show the work)

QUESTION 2. $\frac{dy}{dx} = y^3 - 6y^2 - 7y$. Classify each critical value as stable, semistable, or nonstable.

QUESTION 3. Imagine a company is making a fake-sweet drink (only water and sugar). The tank has capacity of 1200 liters. Initially, it contains 300 liters of brine (water and sugar) that contains 80 grams of sugar, i.e. $A(0) = 80$. A solution containing 2 grams of sugar per liter is pumped into the tank at rate 6 liter per min and the solution is pumped out at rate 3 liter per min.

(i) Let $c(t)$ be the concentration of the sugar in the tank at time t. Find $c(t)$.

(ii) Let $A(t)$ be the amount of sugar in the tank at time t. Find $A(t)$.

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6 Section : Assessment Tools-EXAMS (unanswered)

6.1 Exam I

Name————————————, **ID** ——————

Exam One, MTH 205 , Fall 2020

Ayman Badawi

QUESTION 1. (5 points) Use Laplace Transformation and find $y(t)$, where

$$y'' - 6y' + 5y = U_2(t)(e^{(t-2)}), \;\; y(0) = 0, \; y'(0) = 0$$

.

QUESTION 2. (5 points) Use Laplace Transformation and find $y(t)$, where

$$y'' - 4y' + 13y = 3\delta_0(t), \;\; y(0) = 0, \; y'(0) = 0$$

.

QUESTION 3. (5 points) Use Laplace Transformation and find $y(t)$, where

$$y' - 4y = U_2(t) \; - \; 4\int_0^t y(r)\, dr, \;\; y(0) = 0$$

.

QUESTION 4. (5 points) Find $y_g(t)$, where

$$y'' - 2y' + y = 2e^t$$

.

QUESTION 5. (5 points) Find the largest interval around $t = 2$, say I, so that the L. D. E:

$$(t^2 - 9)y'' + \sqrt{t+1}y' + t^2y = 5t + 1, \; y(2) = 4, y'(2) = -3$$

has unique solution over I. [hint: Use the Initial Value Fundamental Theorem]

QUESTION 6. (6 points) Find $y_g(t)$

$$y^{(3)} - 4y^{(2)} + 13y' = e^t + 8t$$

.

QUESTION 7. (5 points) Solve for $x(t)$ ONLY (do not find $y(t)$)

$$x'(t) - y(t) = 0, \;\; x(0) = 3$$

$$x(t) + y'(t) = 3, \;\; y(0) = 1$$

QUESTION 8. (9 points)

(i) Find $\ell^{-1}\{\frac{s}{(s+3)^2}\}$.

(ii) Find $\ell\{\int_0^t\ e^{(7t-5r)}cos(2r)\ dr\}$

(iii) Find $\ell^{-1}\{\frac{se^{-2s}}{(s+3)^2+9}\}$.

QUESTION 9. (5 points) Given $y = 3sin(t)e^t$ is the ONLY solution to the L.D.E

$$ay'' + by' + cy = 3sin(t)e^t$$

Find the values of a, b, c. [Hint: Personally, I will use Laplace, since $y = 3sin(t)e^t$, it is clear that $y(0) = 0$ and $y'(0) = 3$]

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6.2 Exam II

Name——————————————, **ID** ———————

Exam Two, MTH 205 , Fall 2020

Ayman Badawi

Score = $\frac{\qquad\qquad}{50}$

QUESTION 1. (i) **(4 points)** Find $y_h(t) : t^2y'' + 3ty' + 4y = 0, t > 0$

(ii) **(7 points)** Find $y_g(t) : y'' - (\frac{1}{t} - 1)y' = \frac{e^{-t}}{t}, t > 0$ [Hint: you might need $\int (aw(t) + w'(t))e^{at}\, dt = w(t)e^{at}$, where a is a real number!!, I gave you one version of this observation when $a = 1$]

(iii) **(4 points)** Find $y_h(t) : t^2y'' - 7ty' + 16y = 0, t > 0.$

(iv) **(6 points)** find $y(t) : ty' + 4y = 4t^2e^ty^{\frac{3}{4}}, t > 0$

(v) (**4 points**) Solve the nonlinear diff. equation: $\frac{dy}{dx} = \frac{2xy^3}{\sqrt{1+x^2}}, x \geq 0$

(vi) (**4 points**) Solve the nonlinear diff. equation: $\frac{dw}{dh} = \frac{1}{h+4w^3e^w}, w > 0$

(vii) **(6 points)** Find $y_g(t) : y'' - \frac{4}{t}y' + \frac{4}{t^2}y = \frac{1}{t^2}, t > 0$

(viii) (**5 points**) First convince me that the following D.E. is EXACT. Then solve it.

$$(2x + y^2x + e^y + 2)dx + (x^2y + xe^y + 4y^3 + 7)dy = 0$$

.

QUESTION 2. (**5 points**) Imagine: A cake is removed from an oven, its temperature is measured at 180°C. It is placed in a room temperature 23°C. Two minutes later its temperature is 120°C. How long will it take for the cake to reach 33°C?

QUESTION 3. (**5 points**) Imagine: A large tank is filled to capacity with 1200 gallons of pure water (i.e., $A(0) = 0$). Brine containing 2 pounds of salt per gallon is pumped into the tank at a rate of 4 gal/min. The well mixed solution is pumped out at rate 8 gal/min. Find the number $A(t)$ of pounds of salt in the tank at time t. When is the tank empty?

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6.3 Final Exam

Name———————————————, ID ——————————

Final-Exam, MTH 205 , Fall 2020

Ayman Badawi

$$\text{Score} = \frac{\qquad}{54}$$

QUESTION 1. (6 points) Imagine a steel ball weighing 128 pounds is attached to spring. The spring stretched 2 foot. The ball started in motion by displacing it in 0.5 foot above the equilibrium point with downward initial velocity 2 foot/second. (note gravity = $32ft/sec^2$)

i) Find the equation of the motion of the ball $x(t)$

ii) Rewrite $x(t)$ in terms of the phase angle Φ.

QUESTION 2. (4 points) Given $y' = y^4 - 16y^2$. Find all critical values. Then By drawing (as we did in class), classify each as stable, semi-stable or unstable.

QUESTION 3. (4 points) Solve the following $D.E$:

$$y' = \frac{x}{2xy + 6x + y + 3} \ , \quad x > 0$$

QUESTION 4. (4 points) Solve the following $D.E$:

$$(x^2y + 4y^3)dx + (-3xy^2 - x^3)dy = 0, \quad x > 0$$

QUESTION 5. (4 points) Find y_g

$$y'' + y' = \frac{1}{t} - \frac{1}{t^2} \ , \ \ t > 0$$

QUESTION 6. (4 points) Solve the following $D.E$:

$$y' = \frac{2x + y}{(4x + 2y)^2 + 1} - 2 \ , \quad x > 0$$

QUESTION 7. (4 points) Solve the following $D.E$:

$$y' + 2y = e^{-t} - \int_0^t y(t - r)\, dr, \ \ y(0) = 0$$

QUESTION 8. (4 points) Solve the following $D.E$:

$$y'' + 2y' + 5y = \delta_3(t), \quad y(0) = y'(0) = 0$$

QUESTION 9. (4 points) Write down the **general form** of y_p for the following $D.E$ (i.e., describe how y_p looks like),

but do not find it explicitly :

$$y^{(5)} + 6y^{(4)} + 9y^{(3)} = 4t^3 + t^2e^{-3t}$$

QUESTION 10. (4 points) Solve the following $D.E.$ [Note : $m(m-1)(m-2) = m^3 - 3m^2 + 2m$]

$$y^{(3)} + \frac{6.5}{t^2}y' = 0 \ , \quad t > 0$$

QUESTION 11. (4 points) Given $y_1 = 1$ and $y_2 = ln(t)$ $(t > 0)$ are solutions to

$$y'' + a_1(t)y' + a_0(t)y = 0$$

Use variation method to find y_p, when solving

$$y'' + a_1(t)y' + a_0(t)y = \frac{1}{t^2}$$

QUESTION 12. (4 points) Solve the following $D.E$:

$$y' = \frac{1}{(ln(y) + y^{-1})\sqrt[3]{t} - \frac{3}{2}t}$$

QUESTION 13. (4 points) (Note that $1 + 2b^2 = 1 + b^2 + b^2$) Solve for $x(t)$ ONLY:

$$x'(t) - y(t) = 0, \ \ x(0) = 0$$

$$x(t) + y'(t) = t, \ \ y(0) = 2$$

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Formula Sheet

1) $L\{\,f\,(t)\}\;=\;\int_{0}^{\infty} e^{-\,st}\;f\,(t)\,dt$ $\qquad$ $U_a(t) \equiv U(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & a \le t < \infty \end{cases}$

2) $L\{1\} = \dfrac{1}{s}$, $\qquad L\{e^{at}\} = \dfrac{1}{s-a}$ $\qquad L\{t^n\} = \dfrac{n!}{s^{n+1}}$ **, n is a positive integer**

3) $L\{\sin kt\} = \dfrac{k}{s^2 + k^2}$ $\qquad L\{\cos kt\} = \dfrac{s}{s^2 + k^2}$

4) $L\{\sinh kt\} = \dfrac{k}{s^2 - k^2}$ $\qquad L\{\cosh kt\} = \dfrac{s}{s^2 - k^2}$

5) $L\{e^{at} f(t)\} = F(s)\big|_{s \to s-a}$ $\qquad L^{-1}\left\{F(s)\big|_{s \to s-a}\right\} = e^{at} f(t)$

6) $L\{U(t-a)\} = \dfrac{e^{-as}}{s}$ $\qquad L^{-1}\{\dfrac{e^{-as}}{s}\} = U(t-a)$

7) $L\{g(t)U(t-a)\} = e^{-as} L\{g(t+a)\}$ $\qquad L^{-1}\{e^{-as} F(s)\} = f(t-a)U(t-a)$

8) $L\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots. - f^{(n-1)}(0)$

9) $L\{y'(t)\} = sY(s) - y(0)$ $\qquad L\{y''(t)\} = s^2 Y(s) - sy(0) - y'(0)$

10) $L\{t^n f(t)\}(s) = (-1)^n \dfrac{d^n F(s)}{ds^n}$ $\qquad L^{-1}\{\dfrac{d^n F(s)}{ds^n}\} = (-1)^n t^n f(t)$

11) $L\{f(t) * g(t)\;\} = F(s).G(s)$ $\qquad f(t) * g(t) = \int_{0}^{t} f(\tau) g(t-\tau) d\tau$

12) $L\{\int_{0}^{t} f(\tau) d\tau\} = \dfrac{F(s)}{s}$ $\qquad L^{-1}\{F(s).G(s)\} = f(t) * g(t)$

13) $L\{\delta(t)\} = 1$ $\qquad L\{\delta(t-a)\} = e^{-as}$

14) **If** $f(t)$ **is periodic with period** T **then** $L\{f(t)\} = \dfrac{1}{1-e^{-Ts}} \int_{0}^{T} e^{-st} f(t) dt$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$ $\qquad \sin A\;\; \cos B = \dfrac{1}{2}[\cos(A-B) - \cos(A+B)]$

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