

Exam Two, MTH 205 , Summer 2021

Ayman Badawi

(Stop working at 14:45 pm/ submit your solution by 15:00 pm / DO NOT SUBMIT BY EMAIL) ——— 36

QUESTION 1. (**6 points**)(**SHOW THE WORK**)

Find $y(t)$, where $y' - 7y = \int_0^t e^{2r+5t}\,dr - 12\int_0^t y(r)\,dr$, where $y(0) = y'(0) = 0$

QUESTION 2. (**SHOW THE WORK**)(**6 points**)

Solve for $x(t)$ and $y(t)$ where

$$x^{(2)}(t) + y'(t) = 0$$

$$x'(t) - y(t) = -4$$

, given $y(0) = 5, x(0) = 5, x'(0) = 1$

QUESTION 3. (**SHOW THE WORK**)(**6 points**) Find y_g, where $y^{(3)} - 4y^{(2)} = 2$.

QUESTION 4. (**SHOW THE WORK**)(**4 points**) Find y_g, $y^{(2)} - 4y' + 5y = 0$,

QUESTION 5. (**SHOW THE WORK**)(**4 points**) Find y_g, $t^2y^{(2)} - 3ty' + 5y = 0, t > 0$,

QUESTION 6. (**SHOW THE WORK**)(**6 points**) Find y_g, $\dfrac{y^{(2)}}{t^2} + 2\dfrac{y'}{t^3} = -\dfrac{ln(t)}{t^2}, t > 0$,

QUESTION 7. (**6 points**) ONLY Describe the general form of y_p for the following LDE (Hint: Laplace might be useful as explained in the class).

$$(1)\quad y^{(2)} + 6y' + 7y = sin(3t) + 2021e^{-t} + 3e^{6t}$$

$$(2)\quad y^{(2)} - y' = 2te^t + 5t + 6$$

QUESTION 8. (**6 points**) Find y_g

$$y^{(2)} + y = \frac{1}{sin(t)}$$

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Exam 2 - Hafsah Tahir

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Q1

$y' - 7y = \int_0^t e^{2r+5t}\,dr - 12\int_0^t y(r)\,dr$ given $y(0) = 0$

Apply Laplace

$sY(s) - y(0) - 7Y(s) = \underbrace{L\{\int_0^t e^{2r} \cdot e^{5t}\,dr\}}_{} - \frac{12}{s}Y(s)$

$L\{e^{5t} \cdot \int_0^t e^{2r}\,dr\} = \frac{1}{s-5} \cdot \frac{1}{(s-5)-2} = \frac{1}{(s-5)(s-7)}$

$sY(s) - 7Y(s) + \frac{12}{s}Y(s) = \frac{1}{(s-5)(s-7)}$

$Y(s)\left[s - 7 + \frac{12}{s}\right] = \frac{1}{(s-5)(s-7)}$

$Y(s)\left[\frac{s^2 - 7s + 12}{s}\right] = \frac{1}{(s-5)(s-7)}$

$Y(s) = \frac{s}{(s-5)(s-7)(s-4)(s-3)} = \frac{A}{s-5} + \frac{B}{s-7} + \frac{C}{s-4} + \frac{D}{s-3}$

using cover method $= \frac{-5}{4} \cdot \frac{1}{s-5} + \frac{7}{24} \cdot \frac{1}{s-7} + \frac{4}{3} \cdot \frac{1}{s-4} - \frac{3}{8} \cdot \frac{1}{s-3}$

apply laplace inverse:

$\boxed{y(t) = -\frac{5}{4}e^{5t} + \frac{7}{24}e^{7t} + \frac{4}{3}e^{4t} - \frac{3}{8}e^{3t}}$

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Q2

$x''(t) + y'(t) = 0$

$x'(t) - y(t) = -4$

Apply Laplace:

$s^2X(s) - sx(0) - x'(0) + sY(s) - y(0) = 0$

$sX(s) - x(0) - Y(s) = -\frac{4}{s}$

Simplify

$s^2X(s) + sY(s) = 5s + 6$

$sX(s) - Y(s) = \frac{5s-4}{s}$

Determinate

$$X(s) = \frac{\begin{vmatrix} 5s+6 & s \\ \frac{5s-4}{s} & -1 \end{vmatrix}}{\begin{vmatrix} s^2 & s \\ s & -1 \end{vmatrix}} = \frac{-5s-6-5s+4}{-s^2-s^2} = \frac{-10s-2}{-2s^2} = \frac{10s+2}{2s^2}$$

$$= \frac{10}{2s} + \frac{2}{2s^2} = \frac{5}{s} + \frac{1}{s^2} = X(s)$$

$\boxed{x(t) = 5+t}$

$x'(t) - y(t) = -4$

$1 - y(t) = -4$

$\boxed{y(t) = 5}$

6

Q3 ③

fing $y_g \rightarrow$ $y''' - 4y'' = 2$

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$y = e^{mt}$ (undeterminate method)

$m^3 - 4m^2 = 0$

$m^2(m-4) = 0$

$\left.\begin{array}{l} m_1 = 0 \\ m_2 = 0 \\ m_3 = 4 \end{array}\right\} y_h = C_1 + C_2 t + C_3 e^{4t}$

$y_p = A \times x^2$

$= Ax^2$

$\left.\begin{array}{l} y' = 2Ax \\ y'' = 2A \\ y''' = 0 \end{array}\right\}$ sub in $\rightarrow$ $0 - 4(2A) = 2$

$-8A = 2$

$\boxed{A = \frac{-1}{4}}$

$y_g = y_h + y_p$

$\boxed{y_g = C_1 + C_2 t + C_3 e^{4t} - \frac{1}{4} x^2}$

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Q4

find $y_g \rightarrow y'' - 4y' + 5y = 0$

$y = e^{mt}$ (homogeneous)

characteristic polynomial: $m^2 - 4m + 5 = 0$

$m = 2 \pm i$ (use 1 result bc imaginary)

$y_p = 0$

$$\boxed{y_g = e^{2t}(c_1 \cos t + c_2 \sin t)}$$

✓ 4

Q5

find $y_g \rightarrow t^2 y'' - 3ty' + 5y = 0$

$y = t^n$ (cauchy euler - homogenous)

characteristic polynomial (by staring): $m^2 - m - 3m + 5 = 0$

$m^2 - 4m + 5 = 0$

$m = 2 \pm i$

$$\boxed{y_g = t^2 \left(c_1 \cos(\ln(t)) + c_2 \sin(\ln(t)) \right) \quad \text{where } t > 0}$$

4

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Q6:

find $y_g \to \frac{y''}{t^2} + \frac{2y'}{t^3} = -\frac{\ln(t)}{t^2}$

$\to$ cauchy euler

$y_h \Rightarrow m^2 - m + 2m = 0$

$m^2 + m = 0$

$m(m+1) = 0$

$m = 0 \quad m = -1 \to C_1 + C_2 t^{-1} = y_h$

$y_p \Rightarrow$ variation

$v_1' + v_2' t^{-1} = 0$

$0 \; \cancel{+} - v_2' t^{-2} = \dfrac{-\frac{\ln(t)}{\cancel{t^2}}}{\frac{1}{\cancel{t^2}}}$

$\left\{ \begin{array}{l} v_1' + v_2' t^{-1} = 0 \\ 0 - v_2' t^{-2} = -\ln(t) \end{array} \right. \boxed{* \text{ eq } 1}$

used on next page

$\begin{vmatrix} 1 & t^{-1} \\ 0 & -t^{-2} \end{vmatrix} - t^{-2} - 0 = -t^{-2}$

$\begin{vmatrix} 0 & t^{-1} \\ -\ln(t) & -t^{-2} \end{vmatrix} \; 0 + t^{-1}\ln(t) = \frac{t^{-1}\ln(t)}{-t^{-2}} = -t\ln(t) = v_1'$

$v_1 = -\int t \ln(t)\,dt$

by parts: $u = \ln(t) \quad v = \frac{1}{2}t^2$
$du = \frac{1}{t}dt \quad dv = t\,dt$

$-\frac{1}{2}\ln(t)t^2 + \frac{1}{2}\int \frac{1}{\cancel{t}} t^{\cancel{2}}\,dt$

$\underline{V_1 = -\frac{1}{2}\ln(t)t^2 + \frac{1}{4}t^2}$

~~see *eq 1~~

$\cancel{\frac{-v_2'}{t^2} = -\ln(t) \to v_2' = t^2\ln(t)}$

see next page $\to$

see * eq 1 (on previous page)

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$$\frac{\cancel{t}V_2'}{t^2} = \cancel{t}\ln(t)$$

$$V_2' = t^2\ln(t)$$

$$\int V_2'\,dt = \int t^2\ln(t)\,dt$$

by parts where

$u = \ln(t)$ $\quad v = \frac{1}{3}t^3$

$du = \frac{1}{t}$ $\quad dv = t^2\,dt$

$$V_2 = \frac{1}{3}\ln(t)t^3 - \frac{1}{3}\int t^2\,dt$$

$$\underline{V_2 = \frac{1}{3}\ln(t)t^3 - \frac{1}{9}t^3}$$

$$y_p = V_1(1) + V_2(t^{-1})$$

$$y_p = -\frac{1}{2}\ln(t)t^2 + \frac{1}{4}t^2 + \frac{1}{3}\ln(t)t^2 - \frac{1}{9}t^2$$

$$y_p = \frac{5}{36}t^2 - \frac{1}{6}\ln(t)t^2$$

$$y_g = y_h + y_p$$

$$\boxed{y_g = C_1 + C_2t^{-1} + \frac{5}{36}t^2 - \frac{1}{6}\ln(t)t^2}$$

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Q7 (7)

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① $y'' + 6y' + 7y = \sin(3t) + 2021e^{-t} + 3e^{6t}$

$y_h \Rightarrow m^2 + 6m + 7 = 0$

$m = -3 + \sqrt{2}$

$y_h = e^{-3t}(C_1 \cos\sqrt{2}t + C_2 \sin\sqrt{2}t)$

$\boxed{y_p \Rightarrow A\sin(3t) + B\cos(3t) + Ce^{-t} + De^{6t}}$ general form of y_p

✓ 3

② $y'' - y' = 2te^{t} + 5t + 6$

$m^2 - m$

$m(m-1)$

$y_h = C_1 + C_2 e^{t}$

~~$y_p = (A + Bt)e^{t} + (C + Dt)$~~

$y_p = \underbrace{(At + B)e^{t}}_{\text{part of } y_h} + Ct + D$ so multiply by t

$\boxed{y_p = (At^2 + Bt)e^{t} + Ct^2 + Dt}$ general form of y_p

✓ 3

Q8 (8)

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$y'' + y = \frac{1}{\sin t}$

$y_h \Rightarrow m^2 + 1 = 0$

$m = \pm i$

$\hookrightarrow \; c_1 \cos t + c_2 \sin t$

$y_p \Rightarrow$ variation

$v_1' \cos t + v_2' \sin t = 0$

$-v_1' \sin t + v_2' \cos t = \frac{1}{\sin t}$

$\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} \quad \cos^2 t + \sin^2 t = 1$

$\begin{vmatrix} 0 & \sin t \\ \frac{1}{\sin t} & \cos t \end{vmatrix} \quad \frac{-1}{1} = -1 = v_1'$

$v_1 = \int -1 \, dt$

$\underline{v_1 = -t}$

$\begin{vmatrix} \cos t & 0 \\ -\sin t & \frac{1}{\sin t} \end{vmatrix} \quad \frac{\cos t}{\sin t} \div 1 = \frac{\cos t}{\sin t} = v_2'$

$v_2 = \int \frac{\cos t}{\sin t} \, dt \qquad u = \sin t, \; du = \cos t$

$v_2 = \ln(\sin t)$

$y_p = -t(\cos t) + \ln|\sin t|(\sin t)$

$\boxed{y_g = c_1 \cos t + c_2 \sin t - t \cos t + \sin t \cdot \ln|\sin t|}$

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